6- Product and quotient in exponential form

Proposition: Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then

1) $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$
2) $\frac{1}{z_1} = \frac{1}{r_1} e^{-i\theta_1}$

Proof: 1) $z_1 = r_1 e^{i\theta_1} = r_1(\cos\theta_1 + i\sin\theta_1)$ &$z_2 = r_2 e^{i\theta_2} = r_2(\cos\theta_2 + i\sin\theta_2)$

$$z_1 z_2 = r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - \sin\theta_1\cos\theta_2\ )$$

$= r_1 r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\cos\theta_2 + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)\ )$

$= r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\ ) = r_1 r_2 e^{i(\theta_1+\theta_2)}$.

2) $\frac{1}{z_1} = \frac{1}{r_1 e^{i\theta_1}}\frac{e^{-i\theta_1}}{e^{-i\theta_1}} = \frac{1}{r_1} e^{-i\theta_1}$

Example: If $z_1 = -1 - i$ & $z_2 = \sqrt{3} - i$ find the exponential form of $z_1 z_2$ &$\frac{z_1}{z_2}$

Sol: $r_1 = |z_1| = \sqrt{2}, \theta_1 = Arg\ (z_1) = -\pi + tan^{-1}\left(\frac{y}{x}\right) = -\pi + \frac{\pi}{4} = -\frac{3\pi}{4}$

$\Longrightarrow z_1 = \sqrt{2} e^{-i\frac{3\pi}{4}}$

$r_2 = |z_2| = \sqrt{4} = 2, \theta_2 = Arg\ (z_2) = tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

$\Longrightarrow z_2 = 2e^{-i\frac{\pi}{6}}$

$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)} = 2\sqrt{2} e^{-i\frac{11\pi}{12}}$ & $\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)} = \frac{1}{\sqrt{2}} e^{-i\frac{7\pi}{12}}$

Proposition: I f $z = re^{i\theta}$ then $z^n = r^n e^{in\theta}$ for any $n \in \mathbb{Z}$

Proof: by mathematical induction

1) When $n = 0 \Longrightarrow z^0 = r^0 e^0 = 1$
2) When $n = 1 \Longrightarrow z = re^{i\theta}$

3) Assume it is true for k, this mean $z^k = r^k e^{ik\theta}$ then $z^{k+1} = z^k z = \left(r^k e^{ik\theta}\right)\left(re^{i\theta}\right) = r^{k+1} e^{i(k+1)\theta}$

Therefore $z^n = r^n e^{in\theta}$ is the true for all $n \geq 0$

<u>For $n < 0$</u>

$z^n = \left(\frac{1}{z}\right)^{-n} = \left(\frac{1}{r} e^{-i\theta}\right)^{-n} \Longrightarrow n < 0 \Longrightarrow -n > 0 \Longrightarrow z^n = \left(\frac{1}{r}\right)^{-n} e^{-i(-n)\theta} = r^n e^{in\theta}$.

Remark: If $r = 1$ the above formula becomes $\left(e^{i\theta}\right)^n = e^{in\theta}$, this gives the useful de Moivers formula:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Example: write $\left(\sqrt{3} + i\right)^5$ in the form $x + iy$

Sol: since $r = \sqrt{\left(\sqrt{3}\right)^2 + 1} = \sqrt{3+1} = \sqrt{4} = 2$

$$\theta = Arg\left(\sqrt{3} + i\right) = tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{6}$$

$$\Longrightarrow \sqrt{3} + i = re^{i\theta} = 2e^{i\frac{\pi}{6}}$$

$$\therefore \left(\sqrt{3} + i\right)^5 = \left(2e^{i\frac{\pi}{6}}\right)^5 = 32e^{i\frac{5\pi}{6}} = 32\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$

$= 32\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = 32\left(\cos\left(\pi - \frac{\pi}{6}\right) + i\sin\left(\pi - \frac{\pi}{6}\right)\right) = -16\left(\sqrt{3} - i\right)$.

Example: Use de Moivers formula to derive the trigonometric identities

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta \ \& \ \sin(2\theta) = 2\sin\theta\cos\theta$$

Sol: since $(\cos\theta + i\sin\theta)^2 = \cos(2\theta) + i\sin(2\theta) \Longrightarrow$

$$\cos^2\theta - \sin^2\theta + i2\sin\theta\cos\theta = \cos(2\theta) + i\sin(2\theta)$$

$$\Longrightarrow \cos(2\theta) = \cos^2\theta - \sin^2\theta \quad \& \quad \sin(2\theta) = 2\sin\theta\cos\theta$$

Proposition: Let z_1 and z_2 be two complex numbers then

1) $arg(z_1 z_2) = arg(z_1) + arg(z_2)$
2) $arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2)$

Proof: 1) $arg(z_1) = Arg(z_1) + 2n_1\pi$ & $arg(z_2) = Arg(z_2) + 2n_2\pi$

$$\Longrightarrow z_1 z_2 = r_1 r_2 e^{i(Arg(z_1)+Arg(z_2))}$$

Then we have

$$z_1 z_2 = |z_1 z_2| e^{iArg(z_1 z_2)} \Longrightarrow r_1 r_2 e^{i(Arg(z_1)+Arg(z_2))} = r_1 r_2 e^{iArg(z_1 z_2)}$$

$$\Longrightarrow Arg(z_1 z_2) = Arg(z_1) + Arg(z_2) + 2n\pi + 2k\pi$$

But $Arg(z_1) = arg(z_1) - 2n_1\pi$ & $Arg(z_2) = arg(z_2) - 2n_2\pi$

$\Longrightarrow arg(z_1 z_2) = arg(z_1) - 2n_1\pi + arg(z_2) - 2n_2\pi + 2n\pi + 2k\pi$

$\Longrightarrow arg(z_1 z_2) = arg(z_1) + arg(z_2) + 2\pi(k + n - n_1 - n_2)$ zero

$$\therefore arg(z_1 z_2) = arg(z_1) + arg(z_2)$$

Remark:

1) The equation in the above proposition means that if particular values are assigned to any two of the three terms, then there is a value of the third term, so the equality hold.
2) The right and left hand sides of the equation in the above proposition coincide as set.

Example: Let $z_1 = -1$ and $z_2 = i$, show that:

$$arg(z_1 z_2) = arg(z_1) + arg(z_2) \;\&\; arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2)$$

Sol: since $z_1 = -1 \;\&\; z_2 = i \Rightarrow z_1 z_2 = -i \;\&\; \frac{z_1}{z_2} = -\frac{1}{i} = i$

$$Arg(z_1) = Arg(-1) = \pi \Rightarrow arg(z_1) = \pi + 2n_1\pi, n_1 \in \mathbb{Z}$$

$$Arg(z_2) = Arg(i) = \frac{\pi}{2} \Rightarrow arg(z_2) = \frac{\pi}{2} + 2n_2\pi, n_2 \in \mathbb{Z}$$

$$Arg(z_1 z_2) = Arg(-i) = -\frac{\pi}{2} \Rightarrow arg(z_1 z_2) = -\frac{\pi}{2} + 2n\pi, n \in \mathbb{Z}$$

$$Arg\left(\frac{z_1}{z_2}\right) = Arg(i) = \frac{\pi}{2} \Rightarrow arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2} + 2m\pi, m \in \mathbb{Z}$$

Now T.P. $arg(z_1 z_2) = arg(z_1) + arg(z_2)$

$$\frac{3\pi}{2} = -\frac{\pi}{2} + 2\pi = arg(z_1 z_2) = arg(z_1) + arg(z_2) = \pi + \frac{\pi}{2} = \frac{3\pi}{2}$$

Where, we choose $n_1 = n_2 = 0 \;\&\; n = 1$

$$\frac{\pi}{2} = arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2) = \pi - \frac{\pi}{2} = \frac{\pi}{2}$$

Where, we choose $n_1 = n_2 = m = 0$

Example: given that $z_1 z_2 \neq 0$, use the exponential forms of z_1 &z_2, prove $Im(z_1\bar{z_2}) = |z_1||z_2| \leftrightarrow arg(z_1) - arg(z_2) = \frac{\pi}{2} + 2m\pi, m \in \mathbb{Z}$

Sol:

$z_1\bar{z_2} = |z_1||z_2|e^{i(Arg(z_1) - Arg(z_2))}$ since

$$\left(z_1 = |z_1|e^{i(Arg(z_1))} \& \bar{z_2} = |z_2|e^{-i(Arg(z_2))}\right)$$

$$Im(z_1\bar{z_2}) = |z_1||z_2|\sin\big(Arg(z_1) - Arg(z_2)\big) \leftrightarrow \sin\big(Arg(z_1) - Arg(z_2)\big) = 1$$

$$\leftrightarrow Arg(z_1) - Arg(z_2) = \frac{\pi}{2} + 2n\pi, n \in \mathbb{Z}$$

$$Arg(z_1) = arg(z_1) - 2n_1\pi \;\&\; Arg(z_2) = arg(z_2) - 2n_2\pi$$

$$\leftrightarrow Arg(z_1) - Arg(z_2) = arg(z_1) - 2n_1\pi - (arg(z_2) - 2n_2\pi) = \frac{\pi}{2} + 2n\pi$$

$$\leftrightarrow arg(z_1) - arg(z_2) = \frac{\pi}{2} + 2\pi(n + n_1 - n_2) = \frac{\pi}{2} + 2m\pi, m = (n + n_1 - n_2).$$

Exercise: H.W

7- Roots of complex number

In this section, we use the fact $z^n = |z|^n e^{in\, arg(z)}, n \in \mathbb{Z}$, to find the $n-th$ roots of any nonzero complex number.

Definition: Let $z_\circ$ be nonzero complex number. An $n-th$ root of $z_\circ$ is a nonzero complex number z such that $z^n = z_\circ$.

ملاحظة : في الاعداد الحقيقية تكون الجذور للاعداد الموجبة فقط، بمعنى في الاعداد الحقيقية تكون الجذور للاعداد قيمة واحدة فقط اما في الاعداد العقدية تكون جذور الاعداد مجموعة من القيم.

Remark: How can we find all the $n-th$ root of $z_\circ \neq 0$?
Since $z_\circ \neq 0$ it has a polar form $z_\circ = r_\circ e^{i\theta_\circ}$, similarity any n-th root z of $z_\circ$ is nonzero and hence it has a polar form $z = re^{i\theta}$

$$z_\circ \neq 0 \Longrightarrow z_\circ = r_\circ e^{i\theta_\circ} \Longrightarrow z^n = z_\circ \& z \neq 0 \Longrightarrow z = re^{i\theta}$$ هذا المطلوب

$$r^n e^{in\theta} = r_\circ e^{i\theta_\circ} \Longrightarrow r^n = r_\circ \& n\theta = \theta_\circ + 2k\pi, k \in Z \Longrightarrow r = \sqrt[n]{r_\circ}, \theta = \frac{\theta_\circ + 2k\pi}{n}$$

$$\Longrightarrow z = re^{i\theta} \Longrightarrow z = \sqrt[n]{r_\circ} e^{i\left(\frac{\theta_\circ + 2k\pi}{n}\right)}$$

Therefore any n-th root of $z_\circ$ has the form $z = \sqrt[n]{r_\circ} e^{i\left(\frac{\theta_\circ + 2k\pi}{n}\right)}, k \in \mathbb{Z}$
Note that $|z| = \sqrt[n]{r_\circ}$ for any n-th root z of $z_\circ$. This means that the roots lie on the circle $|z| = \sqrt[n]{r_\circ}$. Moreover they are equally spaced every $\frac{2\pi}{n}$ radians. Thus there are only n distinct roots.

ملاحظة: جميع حلول (الجذور) يكون لهم نفس ال module

These distinct roots can be obtained when $k = 0,1, \ldots, n-1$ and they denoted by C_k, therefore the n-th roots of $z_\circ = r_\circ e^{i\theta_\circ}$ are given by

$$C_k = \sqrt[n]{r_\circ} e^{i\left(\frac{\theta_\circ + 2k\pi}{n}\right)}, k = 0,1, \ldots, n-1$$