a)
$$\int \frac{2\,dx}{\sqrt{3+4x^2}} = \int \frac{du}{\sqrt{a^2+u^2}} \qquad u = 2x, \quad du = 2\,dx, \quad a = \sqrt{3}$$
$$= \sinh^{-1}\left(\frac{u}{a}\right) + C \qquad \text{Formula from Table 7.11}$$
$$= \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C.$$

Therefore,
$$\int_0^1 \frac{2\,dx}{\sqrt{3+4x^2}} = \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right)\Bigg]_0^1 = \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - \sinh^{-1}(0)$$
$$= \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - 0 \approx 0.98665.$$

b) $\int_{1/5}^{3/13} \frac{dx}{x\sqrt{1-16x^2}} = \int_{4/5}^{12/13} \frac{du}{u\sqrt{a^2-u^2}}$, where u = 4x, du = 4 dx, a = 1
$= \left[-\operatorname{sech}^{-1} u\right]_{4/5}^{12/13} = -\operatorname{sech}^{-1} \frac{12}{13} + \operatorname{sech}^{-1} \frac{4}{5}$

c) $\int_1^e \frac{dx}{x\sqrt{1+(\ln x)^2}} = \int_0^1 \frac{du}{\sqrt{a^2+u^2}}$, where u = ln x, du = $\frac{1}{x}$ dx, a = 1
$= \left[\sinh^{-1} u\right]_0^1 = \sinh^{-1} 1 - \sinh^{-1} 0 = \sinh^{-1} 1$

.5 TECHNIQUES OF INTEGRATION

TABLE 8.1 Basic integration formulas

1. $\int k\,dx = kx + C \quad$ (any number k)
2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
3. $\int \frac{dx}{x} = \ln|x| + C$
4. $\int e^x\,dx = e^x + C$
5. $\int a^x\,dx = \frac{a^x}{\ln a} + C \quad (a > 0, a \neq 1)$
6. $\int \sin x\,dx = -\cos x + C$
7. $\int \cos x\,dx = \sin x + C$
8. $\int \sec^2 x\,dx = \tan x + C$
9. $\int \csc^2 x\,dx = -\cot x + C$
10. $\int \sec x \tan x\,dx = \sec x + C$
11. $\int \csc x \cot x\,dx = -\csc x + C$
12. $\int \tan x\,dx = \ln|\sec x| + C$
13. $\int \cot x\,dx = \ln|\sin x| + C$
14. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$
15. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$
16. $\int \sinh x\,dx = \cosh x + C$
17. $\int \cosh x\,dx = \sinh x + C$
18. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$
19. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
20. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$
21. $\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C \quad (a > 0)$
22. $\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C \quad (x > a > 0)$

5.1 Integration by Parts

Integration by parts is a technique for simplifying integrals of the form: $\int f(x)g(x)\,dx$.

$$\int u\,dv = uv - \int v\,du$$

EXAMPLE 1 Find

$$\int x \cos x\,dx.$$

Solution We use the formula $\int u\,dv = uv - \int v\,du$ with

$$u = x, \qquad dv = \cos x\,dx,$$
$$du = dx, \qquad v = \sin x. \qquad \text{Simplest antiderivative of } \cos x$$

Then

$$\int x \cos x\,dx = x \sin x - \int \sin x\,dx = x \sin x + \cos x + C.$$

EXAMPLE 2 Find

$$\int \ln x\,dx.$$

Solution Since $\int \ln x\,dx$ can be written as $\int \ln x \cdot 1\,dx$, we use the formula $\int u\,dv = uv - \int v\,du$ with

$$u = \ln x \qquad \text{Simplifies when differentiated} \qquad dv = dx \qquad \text{Easy to integrate}$$
$$du = \frac{1}{x}\,dx, \qquad v = x. \qquad \text{Simplest antiderivative}$$

Then

$$\int \ln x\,dx = x \ln x - \int x \cdot \frac{1}{x}\,dx = x \ln x - \int dx = x \ln x - x + C.$$

Remark: Sometimes we have to use integration by parts more than once as follows:

EXAMPLE 3 Evaluate

$$\int x^2 e^x \, dx.$$

Solution With $u = x^2$, $dv = e^x \, dx$, $du = 2x \, dx$, and $v = e^x$, we have

$$\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx.$$

The new integral is less complicated than the original because the exponent on x is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x$, $dv = e^x \, dx$. Then $du = dx$, $v = e^x$, and

$$\int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x + C.$$

Using this last evaluation, we then obtain

$$\begin{aligned} \int x^2 e^x \, dx &= x^2 e^x - 2 \int x e^x \, dx \\ &= x^2 e^x - 2x e^x + 2e^x + C. \end{aligned}$$

■

EXAMPLE 4 Evaluate

$$\int e^x \cos x \, dx.$$

Solution Let $u = e^x$ and $dv = \cos x \, dx$. Then $du = e^x \, dx$, $v = \sin x$, and

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

The second integral is like the first except that it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \qquad dv = \sin x \, dx, \qquad v = -\cos x, \qquad du = e^x \, dx.$$

Then

$$\begin{aligned} \int e^x \cos x \, dx &= e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x \, dx) \right) \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx. \end{aligned}$$

$$2\int e^{x}\cos x\,dx = e^{x}\sin x + e^{x}\cos x + C_1.$$

Dividing by 2 and renaming the constant of integration give

$$\int e^{x}\cos x\,dx = \frac{e^{x}\sin x + e^{x}\cos x}{2} + C.$$

<u>Evaluating Definite Integrals by Parts:</u>

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx$$

EXAMPLE 6 Find the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 4$.

Solution The region is shaded in Figure 8.1. Its area is

$$\int_0^4 xe^{-x}\,dx.$$

Let $u = x$, $dv = e^{-x}\,dx$, $v = -e^{-x}$, and $du = dx$. Then,

$$\begin{aligned}\int_0^4 xe^{-x}\,dx &= -xe^{-x}\Big]_0^4 - \int_0^4 (-e^{-x})\,dx \\ &= [-4e^{-4} - (0)] + \int_0^4 e^{-x}\,dx \\ &= -4e^{-4} - e^{-x}\Big]_0^4 \\ &= -4e^{-4} - e^{-4} - (-e^{0}) = 1 - 5e^{-4} \approx 0.91.\end{aligned}$$

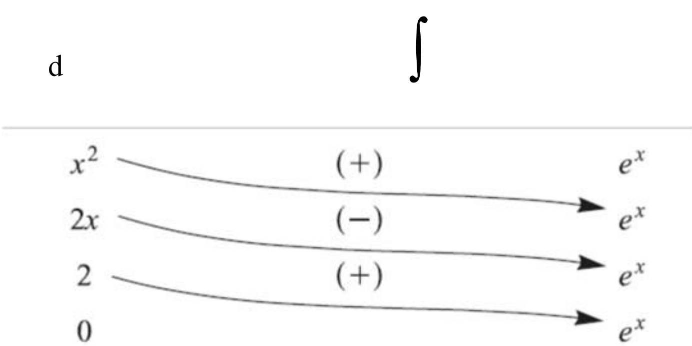
■

<u>Tabular Integration</u>

EXAMPLE 7 Evaluate

$$\int x^2e^{x}\,dx.$$

Solution With $f(x) = x^2$ and $g(x) = e^{x}$, we list:



Then

$$\int x^2 e^x \, dx = x^2 e^x - 2xe^x + 2e^x + C.$$

EXAMPLE 8 Evaluate

$$\int x^3 \sin x \, dx.$$

Solution With $f(x) = x^3$ and $g(x) = \sin x$, we list:

$f(x)$ **and its derivatives**		$g(x)$ **and its integrals**
x^3	$(+)$	$\sin x$
$3x^2$	$(-)$	$-\cos x$
$6x$	$(+)$	$-\sin x$
6	$(-)$	$\cos x$
0		$\sin x$

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C.$$

5.2 Trigonometric Integrals

$$\int \sec^2 x \, dx = \tan x + C.$$

Products of Powers of Sines and Cosines

We begin with integrals of the form:

$$\int \sin^m x \cos^n x \, dx,$$

where m and n are nonnegative integers (positive or zero). We can divide the appropriate substitution into three cases according to m and n being odd or even.