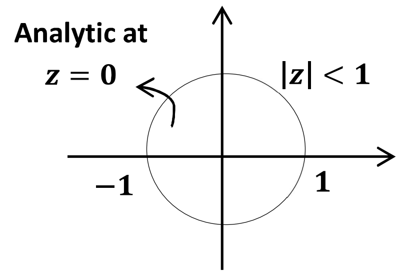
$$f(z) = \frac{\sin 1/2}{z-1} + \frac{(\sin z/(z+1))'\big|_{z=1}/1!\,(z-1)}{z-1} + \frac{(\sin z/(z+1))''\big|_{z=1}/2!\,(z-1)^2}{z-1} + \cdots$$

Example: Let $f(z) = \frac{z+1}{z-1}$, find:

1. Maclaurin series (Taylor about $z_0 = 0$).
2. Laurent series about $z_0 = 0$.

Solution:



1. $f(z) = \frac{z+1}{z-1} = \frac{z-1+2}{z-1}$

$$= 1 - \frac{1}{1-z}$$

$$= 1 - 2\left(\frac{1}{1-z}\right)$$

$$= 1 - 2(1 + z + z^2 + \cdots),\ |z| < 1$$

$$= -1 - 2z - 2z^2 - \cdots$$

2. $f(z) = \frac{z+1}{z-1} = 1 + \frac{2}{z-1}$

$$= 1 + \frac{2}{z\left(1-\frac{1}{z}\right)}$$

$$= 1 + \frac{2}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right]$$

$$= 1 + \frac{1}{z} + \frac{2}{z^2} + \frac{2}{z^3} + \cdots$$

Example: Let $f(z) = \frac{z-1}{z^2}$, calculate:

1. Taylor series expansion about $z = 1$.
2. Laurent series expansion about $z = 1$.

Solution:

Since $z = 1$ then the series is of power $(z-1)$:

"Inside the circle Taylor means positive powers for $(z-1)$"

"Outside the circle Laurent means negative powers for $(z-1)$"

1. $f(z) = \frac{z-1}{z^2} = (z-1)\frac{1}{z^2}$

$$= (z-1)\left(\frac{1}{(z-1)+1}\right)^2$$

$$= (z-1)\left(\frac{1}{1+(z-1)}\right)^2$$

$$= (z-1)\left(\sum_{n=0}^{\infty}(-1)^n(z-1)^n\right)^2$$

$$= (z-1)(1-(z-1)+(z-1)^2-\cdots)^2$$

$$= (z-1)[1-2(z-1)+3(z-1)^2-\cdots],\ |z-1|<1$$

$$= (z-1)-2(z-1)^2+3(z-1)^3-\cdots$$

2. To find Laurent series of $f(z)$:

$$f(z) = \frac{z-1}{z^2} = (z-1)\left[\frac{1}{(z-1)\left(1+\frac{1}{z-1}\right)}\right]^2$$

$$= (z-1)\frac{1}{(z-1)^2}\left[\frac{1}{1+\frac{1}{z-1}}\right]^2$$

$$= \frac{1}{z-1}\left[1-\frac{1}{z-1}+\frac{1}{(z-1)^2}-\cdots\right]^2$$

$$= \frac{1}{z-1}\left[1-\frac{2}{z-1}+\frac{3}{(z-1)^2}-\cdots\right]$$

$$= \frac{1}{z-1}-\frac{2}{(z-1)^2}+\frac{3}{(z-1)^3}-\cdots,\ \left(\left|\frac{1}{z-1}\right|<1 \to |z-1|>1\right)$$

[3] Integration and Differentiation of Power Series

Theorem:

Let C be any contour interior to the circle of convergence of $S(z)=\sum_{n=0}^{\infty} a_n z^n$ and let $g(z)$ be any continuous function on C, then

$$\int g(z)S(z)\,dz = \sum_{n=0}^{\infty} a_n \int_C g(z)z^n\,dz$$

Example: Expand the function $f(z) = \frac{1}{z}$ into a power series of $z - 1$; then obtain by differentiation the expansion of $\frac{1}{z^2}$ in powers of $z - 1$.

Solution:

$$\frac{1}{z} = \frac{1}{1-(1-z)} = \sum_{n=0}^{\infty}(1-z)^n$$

$$= \sum_{n=0}^{\infty}(-1)^n(z-1)^n$$

$$\to \frac{d}{dz}\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty}(-1)^n \frac{d}{dz}(z-1)^n$$

$$= \sum_{n=0}^{\infty}(-1)^n\, n(z-1)^{n-1}$$

$$\to \frac{-1}{z^2} = \sum_{n=0}^{\infty}(-1)^n\, n(z-1)^{n-1}$$

$$\to \frac{1}{z^2} = \sum_{n=0}^{\infty} n(-1)^{n+1}(z-1)^{n-1}$$

$$= \sum_{n=1}^{\infty} n(-1)^{n+1}(z-1)^{n-1}$$

Example: Expand the function $f(z) = \frac{1}{z}$ in a Laurent series in powers of $z - 1$; then obtain by differentiation the Laurent series of $\frac{z-1}{z^2}$ in powers of $z - 1$.

Solution:

$$\frac{1}{z} = \frac{1}{1-(1-z)} = \frac{1}{(1-z)\left(\frac{1}{1-z}-1\right)}$$

$$= \frac{1}{(z-1)\left(1-\frac{1}{1-z}\right)}$$

$$= \frac{1}{z-1}\sum_{n=0}^{\infty}\left(\frac{1}{1-z}\right)^n, \left|\frac{1}{1-z}\right| < 1 \to |1-z| > 1$$

$$= \frac{1}{z-1}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(z-1)^n}$$

$$\therefore \frac{1}{z} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{(z-1)^{n+1}}$$

Now, differentiating both sides with respect to z, we get:

$\frac{-1}{z^2} = \sum_{n=0}^{\infty}(-1)^n - (n+1)(z-1)^{-(n+2)}$

$\frac{-1}{z^2} = \sum_{n=0}^{\infty}(-1)^n \frac{-(n+1)}{(z-1)^{n+2}}$

Or $\frac{1}{z^2} = \sum_{n=0}^{\infty}(-1)^n \frac{(n+1)}{(z-1)^{n+2}}$

$\rightarrow \frac{z-1}{z^2} = \sum_{n=0}^{\infty}(-1)^n \frac{(n+1)}{(z-1)^{n+1}}$

$= \sum_{n=1}^{\infty}(-1)^{n-1} \frac{n}{(z-1)^n}$

Which is a Laurent series for $f(z) = \frac{z-1}{z^2}$ in powers of $z - 1$.

Example: Suppose that f and g are analytic functions at z_0 and $f(z_0) = g(z_0)$, while $g(z_0) \neq 0$, prove that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$$

Solution:

$\lim_{z \to z_0} \frac{f(z)}{z - z_0} = f'(z_0)$, and

$\lim_{z \to z_0} \frac{g(z)}{z - z_0} = g'(z_0)$

Then,

$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f(z)/(z-z_0)}{g(z)/(z-z_0)}$

$= \frac{\lim_{z \to z_0} f(z)/(z-z_0)}{\lim_{z \to z_0} g(z)/(z-z_0)}$

$= \frac{f'(z_0)}{g'(z_0)}$

Chapter Six

Residues and Poles

Definition 1:

A point z_0 is called a singular of f if the function f fails to be analytic at z_0 but it is analytic at some point in every neighborhood of z_0.

Definition 2:

A singular point z_0 is said to be isolated, if in addition, there is some neighborhood of z_0 for which f is analytic except at z_0.

Example:

1. $f(z) = \frac{1}{z}$, this function has a singular point at $z = 0$, which is an isolated singular point of f.
2. $f(z) = \frac{1}{z^2(z-1)(z^2+1)}$, this function has four isolated singular points $z = 0, 1, \pm i$.
3. $f(z) = \operatorname{Log} z$, this function has a singular point at $z = 0$, but this point is not isolated, because each neighborhood of $z = 0$ contains points on the negative real axis and $\operatorname{Log} z$ fails to be analytic at each of these points.
4. $f(z) = e^z$, has no singular points.
5. $f(z) = \frac{1}{\sin\frac{\pi}{z}}$, has the singular points $z = 0$ and $z = \frac{1}{n}$, $n = \pm 1, \pm 2, \ldots$, each singular point $z = \frac{1}{n}$ is isolated but $z = 0$ is not isolated singular of f, since when $z = 0$ every neighborhood of $z = 0$ contains other singular points of f. For example, take $z = \frac{1}{N}$, N large enough, then

$$\frac{1}{N} \to 0 \Longrightarrow \sin\frac{\pi}{z} = \sin\frac{\pi}{\frac{1}{1/N}} = \sin N\pi = 0$$