

$$\begin{aligned}
 |R_N(z)| &\leq \frac{r^N}{2\pi} \frac{2\mu r_1 \pi}{(r_1 - r) r_1^N} \\
 &= \frac{\mu r_1}{r_1 - r} \left(\frac{r}{r_1}\right)^N, \quad \frac{r}{r_1} < 1
 \end{aligned}$$

So, when  $N \rightarrow \infty$ , we have  $R_N(z) \rightarrow 0$ . Therefore, for each point  $z$  inside  $C_0$ , the limit of the sum for the first  $N$  terms on the right in Eq.(2) as  $N \rightarrow \infty$ , is  $f(z)$ . That is, if  $f$  is analytic inside a circle centered at  $z_0$  with radius  $r_0$ , then  $f(z)$  is represented by a Taylor series

$$f(z) = f(z_0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n, \text{ where } |z - z_0| < r_0. \blacksquare$$

### **Important Note:**

The special case in which  $z_0 = 0$ ; i.e.:

$$\begin{aligned}
 f(z) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n \\
 &= f(0) + f'(0)z + \frac{f''(0)z^2}{2!} + \dots + \frac{f^{(n)}(0)z^n}{n!} + \dots
 \end{aligned}$$

is called a Maclaurin series.

**Example:** Find the Maclaurin series expansion for the following:

$$\sin z, \cos z, \sinh z, \cosh z \text{ and } e^z$$

**Solution:**

❖ Let  $f(z) = \sin z$ , then

$$f(0) = \sin 0 = 0$$

$$f'(z) = \cos z \rightarrow f'(0) = 1$$

$$f''(z) = -\sin z \rightarrow f''(0) = 0$$

$$f^{(3)}(z) = -\cos z \rightarrow f^{(3)}(0) = -1$$

$$f^{(4)}(z) = \sin z \rightarrow f^{(4)}(0) = 0$$

$\vdots$

$$\begin{aligned}
f(z) = \sin z &= f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \dots + \frac{f^{(n)}(0)}{n!}z^n + \dots \\
&= 0 + z + 0 - \frac{z^3}{3!} + 0 + \frac{z^5}{5!} + \dots \\
&= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots
\end{aligned}$$

i.e.:

$$\sin z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}, \quad |z| < \infty \quad \dots (1)$$

❖ To find the series of  $\cos z$ :

Differentiating both sides of (1) with respect to  $z$ , we get:

$$\cos z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}, \quad |z| < \infty \quad \dots (2)$$

❖ To find the series of  $\sinh z$ :

Since  $\sinh z = -i \sin iz$ , it follows from (1), that

$$\begin{aligned}
\sinh z &= -i \sum_{k=0}^{\infty} (-1)^k \frac{(iz)^{2k+1}}{(2k+1)!} \\
&= -i \sum_{k=0}^{\infty} (-1)^k (i)^{2k+1} \frac{z^{2k+1}}{(2k+1)!} \\
&= \sum_{k=0}^{\infty} (-1)^k (-i)(i)(i^2)^k \frac{z^{2k+1}}{(2k+1)!} \\
&= \sum_{k=0}^{\infty} (-1)^k (-1)^k \frac{z^{2k+1}}{(2k+1)!} \quad [((-1)^2)^k = 1]
\end{aligned}$$

$$\sinh z = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}, \quad |z| < \infty \quad \dots (3)$$

❖ To find the series of  $\cosh z$ :

Differentiating both sides of (3) with respect to  $z$ , we get:

$$\cosh z = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}, \quad |z| < \infty \quad \dots (4)$$

❖ To find the series of  $e^z$ :

When  $f(z) = e^z$ , then  $f^{(n)}(z) = e^z$

$f^{(n)}(0) = 1$ , since  $e^z$  is analytic for all  $z$ , so:

$$\begin{aligned} e^z &= e^0 + e^0 z + \frac{e^0}{2!} z^2 + \cdots + \frac{e^0}{n!} z^n + \cdots \\ &= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots \\ &= \sum_{k=0}^{\infty} \frac{z^k}{k!} \quad \dots (5) \end{aligned}$$

**Example:** Expand  $\cos z$  into a Taylor series about the point  $z = \frac{\pi}{2}$ .

**Solution:** let  $f(z) = \cos z$ , then

$$f(z) = \cos z = f\left(\frac{\pi}{2}\right) + \frac{f'\left(\frac{\pi}{2}\right)\left(z-\frac{\pi}{2}\right)}{1!} + \frac{f''\left(\frac{\pi}{2}\right)\left(z-\frac{\pi}{2}\right)^2}{2!} + \cdots + \frac{f^{(n)}\left(\frac{\pi}{2}\right)\left(z-\frac{\pi}{2}\right)^n}{n!} + \cdots$$

Now,

$$f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$$

$$f'(z) = -\sin z \rightarrow f'\left(\frac{\pi}{2}\right) = -1$$

$$f''(z) = -\cos z \rightarrow f''\left(\frac{\pi}{2}\right) = 0$$

$$f^{(3)}(z) = \sin z \rightarrow f^{(3)}\left(\frac{\pi}{2}\right) = 1$$

$$f^{(4)}(z) = \cos z \rightarrow f^{(4)}\left(\frac{\pi}{2}\right) = 0$$

$\vdots$

$$\begin{aligned} \rightarrow \cos z &= 0 - \frac{\left(z-\frac{\pi}{2}\right)}{1!} + 0 + \frac{\left(z-\frac{\pi}{2}\right)^3}{3!} + 0 - \frac{\left(z-\frac{\pi}{2}\right)^5}{5!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{z^{2n+1}}{(2n+1)!} \end{aligned}$$

**Example:** Show that

$$\frac{1}{z^2} = \sum_{n=0}^{\infty} (n+1)(z+1)^n$$

where  $|z+1| < 1$ .

Solution:

Since  $|z + 1| < 1 \rightarrow z_0 = -1$  and,

$$f(z) = \frac{1}{z^2} \rightarrow f(-1) = 1$$

$$f'(z) = \frac{-2}{z^3} \rightarrow f'(-1) = 2$$

$$f''(z) = \frac{2 \cdot 3}{z^4} \rightarrow f''(-1) = 3!$$

$$f^{(3)}(z) = \frac{-2 \cdot 3 \cdot 4}{z^5} \rightarrow f^{(3)}(-1) = 4!$$

$\vdots$

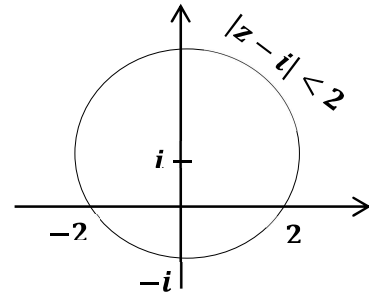
$$f^{(n)}(z) = \frac{(-1)^n \cdot 2 \cdot 3 \cdots (n+1)}{z^{n+2}} \rightarrow f^{(n)}(-1) = (n+1)!$$

$$\begin{aligned} \frac{1}{z^2} &= \sum_{n=0}^{\infty} \frac{(n+1)!}{n!} (z+1)^n \\ &= \sum_{n=0}^{\infty} (n+1)(z+1)^n \end{aligned}$$

**Example:** Expand  $f(z) = \frac{3}{z+i}$  into a Taylor series about  $|z-i| < 2$ .

Solution:

Note that  $-i$  is a singular point located on the perimeter. The largest size circle that can be found is the one that the function is not analytic



at it, which is  $-i$ . The distance between  $i$  and  $-i$  represents the radius of convergence which is 2, and that's why we have the circle  $|z-i| < 2$ . And if we have  $|z-i| < 3$  then the Taylor series cannot be applied, since the function will not be analytic and one of its conditions is that the function must be analytic inside  $C$ .

$$\begin{aligned} \frac{3}{z+i} &= \frac{3}{z+2i-i} \\ &= \frac{3}{2i+(z-i)} \end{aligned}$$

$$= \frac{3}{2i \left(1 + \frac{z-i}{2i}\right)}$$

$$= \frac{3}{2i} \left[ \frac{1}{1 + \frac{z-i}{2i}} \right] \text{ (Geometric series } a = 1, r = \frac{z-i}{2i} \text{)}$$

$$= \frac{3}{2i} \left[ \sum_{n=0}^{\infty} (-1)^n \left( \frac{z-i}{2i} \right)^n \right]$$

**Note:**  $\left| \frac{z-i}{2i} \right| < 1 \rightarrow |z-i| < 2.$

**Example:**

1. Expand  $f(z) = \frac{1}{1+z}$  about  $z = 0$ .

**Solution:**

$$f(z) = \frac{1}{1-(-z)}$$

$$= 1 - z + z^2 - z^3 + \dots + (-1)^n z^n + \dots, |z| < 1$$

$$= \sum_{n=0}^{\infty} (-1)^n z^n$$

2. Expand  $f(z) = \frac{1}{1-z^2}$  about  $z = 0$ .

**Solution:**

$$f(z) = \sum_{n=0}^{\infty} (z^2)^n = \sum_{n=0}^{\infty} z^{2n}, |z| < 1$$

**Note:** to find the radius of convergence  $= \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|}$ , such as in the previous example,

$$\left. \begin{array}{l} a_{n+1} = -1 \rightarrow |a_{n+1}| = 1 \\ a_n = 1 \rightarrow |a_n| = 1 \end{array} \right\} \rightarrow \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \rightarrow \infty} \frac{1}{1} = 1$$

$$\therefore r_0 = 1$$

$$|z - 0| < r_0 \rightarrow |z| < 1.$$

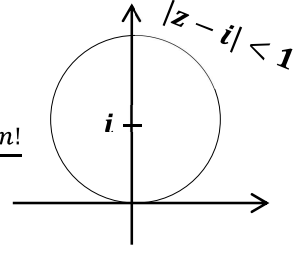
**Example:** Write  $f(z) = \frac{1}{z}$  into a Taylor series about  $z = i, r_0 = 1$ .

**Solution:** from Taylor's theorem  $|z - z_0| < r_0 \rightarrow |z - i| < 1$

$$\frac{1}{z} = \sum_{n=0}^{\infty} a_n (z-i)^n, \quad |z-i| < 1$$

$$f(i) = \frac{0!}{i}, f'(i) = \frac{-1!}{i^2}, f''(i) = \frac{2!}{i^3}, \dots, f^{(n)}(i) = \frac{(-1)^n n!}{i^{n+1}}$$

$$\therefore \frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n \frac{(z-i)^n}{i^{n+1}}$$



Or:

$$\begin{aligned} \frac{1}{z} &= \frac{1}{z-i+i} = \frac{1}{i+(z-i)} \\ &= \frac{1}{i\left(1+\frac{z-i}{i}\right)} \\ &= \frac{1}{i} \left[ \frac{1}{1+\frac{z-i}{i}} \right], \text{ since } |z-i| < 1 \\ &= \frac{1}{i} \sum_{n=0}^{\infty} (-1)^n \frac{(z-i)^n}{i^n} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{(z-i)^n}{i^{n+1}} \end{aligned}$$

**Example:** Write  $f(z) = \frac{1}{z}$  into a power series for  $(z-1)$ .

Solution:

$$\begin{aligned} \frac{1}{z} &= \frac{1}{z-1+1} \\ &= \frac{1}{1+(z-1)} \quad (\text{Geometric series } a = 1, r = (z-1)) \\ &= \sum_{n=0}^{\infty} (-1)^n (z-1)^n, \quad |z-1| < 1 \end{aligned}$$

**Example:** Represent the function

$$f(z) = \frac{z}{(z-3)(z-1)}$$

into a series of negative power of  $(z-1)$ , which converges to  $f(z)$  where  $0 < |z-1| < 2$

Solution:

$$f(z) = \frac{z}{(z-3)(z-1)} = \frac{A}{z-1} + \frac{B}{z-3}$$