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.1 L'Hopital's Rule

THEOREM L'Hopital's Rule: Suppose that $f(a) = g(a) = 0$ or ∞ , that f and g are differentiable on an open interval I containing a , and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)},$$

Indeterminate Form 0/0

EXAMPLE 1

$$(a) \lim_{x \rightarrow 0} \frac{3x - \sin x}{x} = \lim_{x \rightarrow 0} \frac{3 - \cos x}{1} = \frac{3 - \cos x}{1} \Big|_{x=0} = 2$$

$$(b) \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x} = \lim_{x \rightarrow 0} \frac{\frac{1}{2\sqrt{1+x}}}{1} = \frac{1}{2}$$

$$(c) \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1 - x/2}{x^2} \\ = \lim_{x \rightarrow 0} \frac{(1/2)(1+x)^{-1/2} - 1/2}{2x} \\ = \lim_{x \rightarrow 0} \frac{-(1/4)(1+x)^{-3/2}}{2} = -\frac{1}{8}$$

$$(d) \lim_{x \rightarrow 0} \frac{x - \sin x}{x^3} \\ = \lim_{x \rightarrow 0} \frac{1 - \cos x}{3x^2} \\ = \lim_{x \rightarrow 0} \frac{\sin x}{6x} \\ = \lim_{x \rightarrow 0} \frac{\cos x}{6} = \frac{1}{6}$$

EXAMPLE 2

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x + x^2} \\ = \lim_{x \rightarrow 0} \frac{\sin x}{1 + 2x} = \frac{0}{1} = 0.$$

EXAMPLE 3

$$(a) \lim_{x \rightarrow 0^+} \frac{\sin x}{x^2} \\ = \lim_{x \rightarrow 0^+} \frac{\cos x}{2x} = \infty$$

$$(b) \lim_{x \rightarrow 0^-} \frac{\sin x}{x^2} \\ = \lim_{x \rightarrow 0^-} \frac{\cos x}{2x} = -\infty$$

Indeterminate Forms ∞/∞ , $\infty \cdot 0$ and $\infty - \infty$

EXAMPLE 4: find the limit:

$$(a) \lim_{x \rightarrow \pi/2} \frac{\sec x}{1 + \tan x} \qquad (b) \lim_{x \rightarrow \infty} \frac{\ln x}{2\sqrt{x}} \qquad (c) \lim_{x \rightarrow \infty} \frac{e^x}{x^2}.$$

Solution:

$$(a) \lim_{x \rightarrow (\pi/2)^-} \frac{\sec x}{1 + \tan x} \quad \frac{\infty}{\infty} \text{ from the left}$$

$$= \lim_{x \rightarrow (\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x \rightarrow (\pi/2)^-} \sin x = 1$$

$$(b) \lim_{x \rightarrow \infty} \frac{\ln x}{2\sqrt{x}} = \lim_{x \rightarrow \infty} \frac{1/x}{1/\sqrt{x}} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x}} = 0 \quad \frac{1/x}{1/\sqrt{x}} = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}$$

$$(c) \lim_{x \rightarrow \infty} \frac{e^x}{x^2} = \lim_{x \rightarrow \infty} \frac{e^x}{2x} = \lim_{x \rightarrow \infty} \frac{e^x}{2} = \infty$$

EXAMPLE 5: Find the limits of these $\infty \cdot 0$ forms:

$$(a) \lim_{x \rightarrow \infty} \left(x \sin \frac{1}{x} \right) \qquad (b) \lim_{x \rightarrow 0^+} \sqrt{x} \ln x$$

$$(a) \lim_{x \rightarrow \infty} \left(x \sin \frac{1}{x} \right) = \lim_{h \rightarrow 0^+} \left(\frac{1}{h} \sin h \right) = \lim_{h \rightarrow 0^+} \frac{\sin h}{h} = 1 \quad \infty \cdot 0; \text{ Let } h = 1/x.$$

$$(b) \lim_{x \rightarrow 0^+} \sqrt{x} \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{1/\sqrt{x}} \quad \infty \cdot 0 \text{ converted to } \infty/\infty$$

$$= \lim_{x \rightarrow 0^+} \frac{1/x}{-1/2x^{3/2}} \quad \text{l'Hôpital's Rule}$$

$$= \lim_{x \rightarrow 0^+} (-2\sqrt{x}) = 0$$

EXAMPLE 6 Find the limit of this $\infty - \infty$ form:

$$\lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right).$$

Solution If $x \rightarrow 0^+$, then $\sin x \rightarrow 0^+$ and

$$\frac{1}{\sin x} - \frac{1}{x} \rightarrow \infty - \infty.$$

Similarly, if $x \rightarrow 0^-$, then $\sin x \rightarrow 0^-$ and

$$\frac{1}{\sin x} - \frac{1}{x} \rightarrow -\infty - (-\infty) = -\infty + \infty.$$

Solution If $x \rightarrow 0^+$, then $\sin x \rightarrow 0^+$ and

$$\frac{1}{\sin x} - \frac{1}{x} \rightarrow \infty - \infty.$$

Similarly, if $x \rightarrow 0^-$, then $\sin x \rightarrow 0^-$ and

$$\frac{1}{\sin x} - \frac{1}{x} \rightarrow -\infty - (-\infty) = -\infty + \infty.$$

Neither form reveals what happens in the limit. To find out, we first combine the fractions:

$$\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x} \quad \text{Common denominator is } x \sin x.$$

Then we apply l'Hôpital's Rule to the result:

$$\begin{aligned} \lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) &= \lim_{x \rightarrow 0} \frac{x - \sin x}{x \sin x} && \frac{0}{0} \\ &= \lim_{x \rightarrow 0} \frac{1 - \cos x}{\sin x + x \cos x} && \text{Still } \frac{0}{0} \\ &= \lim_{x \rightarrow 0} \frac{\sin x}{2 \cos x - x \sin x} = \frac{0}{2} = 0. && \blacksquare \end{aligned}$$

Indeterminate Powers

EXAMPLE 7 Apply l'Hôpital's Rule to show that $\lim_{x \rightarrow 0^+} (1 + x)^{1/x} = e$.

Solution The limit leads to the indeterminate form 1^∞ . We let $f(x) = (1 + x)^{1/x}$ and find $\lim_{x \rightarrow 0^+} \ln f(x)$. Since

$$\ln f(x) = \ln (1 + x)^{1/x} = \frac{1}{x} \ln (1 + x),$$

l'Hôpital's Rule now applies to give

$$\begin{aligned} \lim_{x \rightarrow 0^+} \ln f(x) &= \lim_{x \rightarrow 0^+} \frac{\ln (1 + x)}{x} && \frac{0}{0} \\ &= \lim_{x \rightarrow 0^+} \frac{1}{1 + x} \\ &= \frac{1}{1} = 1. \end{aligned}$$

If $\lim_{x \rightarrow a} \ln f(x) = L$, then

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} e^{\ln f(x)} = e^L.$$

Here a may be either finite or infinite.

Therefore, $\lim_{x \rightarrow 0^+} (1 + x)^{1/x} = \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} e^{\ln f(x)} = e^1 = e$.

EXAMPLE 8 Find $\lim_{x \rightarrow \infty} x^{1/x}$.

Solution The limit leads to the indeterminate form ∞^0 . We let $f(x) = x^{1/x}$ and find $\lim_{x \rightarrow \infty} \ln f(x)$. Since

$$\ln f(x) = \ln x^{1/x} = \frac{\ln x}{x},$$

l'Hôpital's Rule gives

$$\begin{aligned}\lim_{x \rightarrow \infty} \ln f(x) &= \lim_{x \rightarrow \infty} \frac{\ln x}{x} \quad \frac{\infty}{\infty} \\ &= \lim_{x \rightarrow \infty} \frac{1/x}{1} \\ &= \frac{0}{1} = 0.\end{aligned}$$

Therefore $\lim_{x \rightarrow \infty} x^{1/x} = \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} e^{\ln f(x)} = e^0 = 1.$ ■

.2 INTEGRATION:

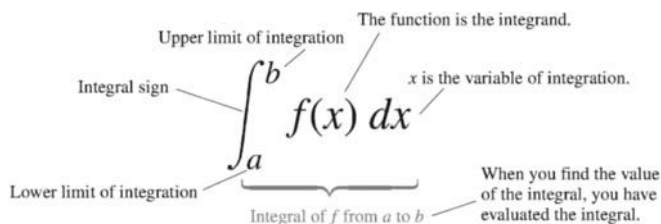
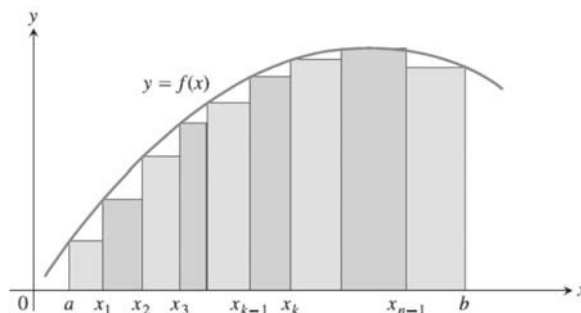
1) The Definite Integral

$$S_n = \sum_{k=1}^n f(c_k) \Delta x_k = \sum_{k=1}^n f(c_k) \left(\frac{b-a}{n} \right),$$

$$\Delta x_k = \Delta x = (b-a)/n \text{ for all } k$$

$$J = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \left(\frac{b-a}{n} \right) = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(c_k) \Delta x. \quad \Delta x = (b-a)/n$$

$$\Delta x = (b-a)/n$$



Rules satisfied by definite integrals

1. *Order of Integration:* $\int_b^a f(x) dx = -\int_a^b f(x) dx$ A Definition
2. *Zero Width Interval:* $\int_a^a f(x) dx = 0$ A Definition
when $f(a)$ exists
3. *Constant Multiple:* $\int_a^b kf(x) dx = k \int_a^b f(x) dx$ Any constant k
4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x)) dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$
5. *Additivity:* $\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$
6. $f(x) \geq g(x)$ on $[a, b] \Rightarrow \int_a^b f(x) dx \geq \int_a^b g(x) dx$
 $f(x) \geq 0$ on $[a, b] \Rightarrow \int_a^b f(x) dx \geq 0$ (Special Case)