References:

1. **Maurice Weir, Joel Hass, George B. Thomas,** *Thomas Calculus*, 12th ed. (2012).
2. **G Stephenson** Mathematical Methods for Science Students (1983).
3. Anton Bivens Davis *Calculus* (2002).

.1 L'Hopital's Rule

THEOREM L'Hopital's Rule: Suppose that f(a) = g(a) = 0 or ∞, that f and g are differentiable on an open interval I containing a, and that g' (x) ≠0 on I if x ≠a. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

Indeterminate Form 0/0

EXAMPLE 1

$$\textbf{(a)}\ \lim_{x \to 0} \frac{3x - \sin x}{x} = \lim_{x \to 0} \frac{3 - \cos x}{1} = \left. \frac{3 - \cos x}{1} \right|_{x=0} = 2$$

$$\textbf{(b)}\ \lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x} = \lim_{x \to 0} \frac{\frac{1}{2\sqrt{1+x}}}{1} = \frac{1}{2}$$

$$\textbf{(c)}\ \lim_{x \to 0} \frac{\sqrt{1+x} - 1 - x/2}{x^2}$$

$$= \lim_{x \to 0} \frac{(1/2)(1+x)^{-1/2} - 1/2}{2x}$$

$$= \lim_{x \to 0} \frac{-(1/4)(1+x)^{-3/2}}{2} = -\frac{1}{8}$$

$$\textbf{(d)}\ \lim_{x \to 0} \frac{x - \sin x}{x^3}$$

$$= \lim_{x \to 0} \frac{1 - \cos x}{3x^2}$$

$$= \lim_{x \to 0} \frac{\sin x}{6x}$$

$$= \lim_{x \to 0} \frac{\cos x}{6} = \frac{1}{6}$$

EXAMPLE 2

$$\lim_{x \to 0} \frac{1 - \cos x}{x + x^2}$$

$$= \lim_{x \to 0} \frac{\sin x}{1 + 2x} = \frac{0}{1} = 0.$$

EXAMPLE 3

$$\textbf{(a)}\ \lim_{x \to 0^+} \frac{\sin x}{x^2}$$

$$= \lim_{x \to 0^+} \frac{\cos x}{2x} = \infty$$

$$\textbf{(b)}\ \lim_{x \to 0^-} \frac{\sin x}{x^2}$$

$$= \lim_{x \to 0^-} \frac{\cos x}{2x} = -\infty$$

Indeterminate Forms ∞\ ∞ , ∞.0 and ∞-∞

EXAMPLE 4: find the limit:

(a) $\lim_{x\to\pi/2} \dfrac{\sec x}{1+\tan x}$ (b) $\lim_{x\to\infty} \dfrac{\ln x}{2\sqrt{x}}$ (c) $\lim_{x\to\infty} \dfrac{e^x}{x^2}$.

Solution:

(a) $\lim_{x\to(\pi/2)^-} \dfrac{\sec x}{1+\tan x}$ $\quad \frac{\infty}{\infty}$ from the left

$$= \lim_{x\to(\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x\to(\pi/2)^-} \sin x = 1$$

(b) $\lim_{x\to\infty} \dfrac{\ln x}{2\sqrt{x}} = \lim_{x\to\infty} \dfrac{1/x}{1/\sqrt{x}} = \lim_{x\to\infty} \dfrac{1}{\sqrt{x}} = 0$ $\quad \dfrac{1/x}{1/\sqrt{x}} = \dfrac{\sqrt{x}}{x} = \dfrac{1}{\sqrt{x}}$

(c) $\lim_{x\to\infty} \dfrac{e^x}{x^2} = \lim_{x\to\infty} \dfrac{e^x}{2x} = \lim_{x\to\infty} \dfrac{e^x}{2} = \infty$

EXAMPLE 5: Find the limits of these $\infty \cdot 0$ forms:

(a) $\lim_{x\to\infty} \left(x \sin \frac{1}{x}\right)$ (b) $\lim_{x\to 0^+} \sqrt{x}\ln x$

(a) $\lim_{x\to\infty} \left(x \sin \frac{1}{x}\right) = \lim_{h\to 0^+} \left(\frac{1}{h} \sin h\right) = \lim_{h\to 0^+} \dfrac{\sin h}{h} = 1$ $\quad \infty \cdot 0$; Let $h = 1/x$.

(b) $\lim_{x\to 0^+} \sqrt{x}\ln x = \lim_{x\to 0^+} \dfrac{\ln x}{1/\sqrt{x}}$ $\quad \infty \cdot 0$ converted to ∞/∞

$$= \lim_{x\to 0^+} \frac{1/x}{-1/2x^{3/2}} \quad \text{l'Hôpital's Rule}$$

$$= \lim_{x\to 0^+} \left(-2\sqrt{x}\right) = 0$$

EXAMPLE 6 Find the limit of this $\infty - \infty$ form:

$$\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right).$$

Solution If $x \to 0^+$, then $\sin x \to 0^+$ and

$$\frac{1}{\sin x} - \frac{1}{x} \to \infty - \infty.$$

Similarly, if $x \to 0^-$, then $\sin x \to 0^-$ and

$$\frac{1}{\sin x} - \frac{1}{x} \to -\infty - (-\infty) = -\infty + \infty.$$

Solution If $x \to 0^+$, then $\sin x \to 0^+$ and

$$\frac{1}{\sin x} - \frac{1}{x} \to \infty - \infty.$$

Similarly, if $x \to 0^-$, then $\sin x \to 0^-$ and

$$\frac{1}{\sin x} - \frac{1}{x} \to -\infty - (-\infty) = -\infty + \infty.$$

Neither form reveals what happens in the limit. To find out, we first combine the fractions:

$$\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x} \qquad \text{Common denominator is } x \sin x.$$

Then we apply l'Hôpital's Rule to the result:

$$\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) = \lim_{x \to 0} \frac{x - \sin x}{x \sin x} \qquad \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{1 - \cos x}{\sin x + x \cos x} \qquad \text{Still } \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{\sin x}{2 \cos x - x \sin x} = \frac{0}{2} = 0. \qquad \blacksquare$$

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Indeterminate Powers

EXAMPLE 7 Apply l'Hôpital's Rule to show that $\lim_{x \to 0^+} (1 + x)^{1/x} = e$.

Solution The limit leads to the indeterminate form 1^∞. We let $f(x) = (1 + x)^{1/x}$ and find $\lim_{x \to 0^+} \ln f(x)$. Since

$$\ln f(x) = \ln (1 + x)^{1/x} = \frac{1}{x} \ln (1 + x),$$

l'Hôpital's Rule now applies to give

$$\lim_{x \to 0^+} \ln f(x) = \lim_{x \to 0^+} \frac{\ln (1 + x)}{x} \qquad \frac{0}{0}$$

$$= \lim_{x \to 0^+} \frac{\frac{1}{1 + x}}{1}$$

$$= \frac{1}{1} = 1.$$

If $\lim_{x \to a} \ln f(x) = L$, then

$$\lim_{x \to a} f(x) = \lim_{x \to a} e^{\ln f(x)} = e^L.$$

Here a may be either finite or infinite.

Therefore, $\lim_{x \to 0^+} (1 + x)^{1/x} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} e^{\ln f(x)} = e^1 = e.$

EXAMPLE 8 Find $\lim_{x \to \infty} x^{1/x}$.

Solution The limit leads to the indeterminate form ∞^0. We let $f(x) = x^{1/x}$ and find $\lim_{x \to \infty} \ln f(x)$. Since

$$\ln f(x) = \ln x^{1/x} = \frac{\ln x}{x},$$

l'Hôpital's Rule gives

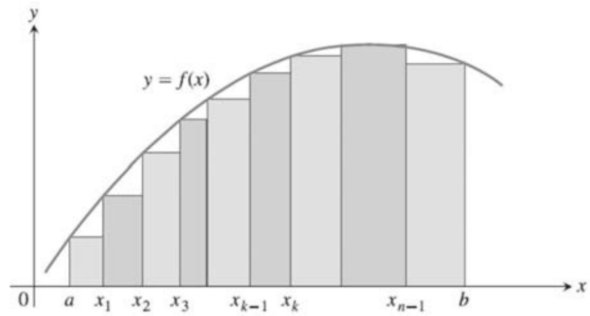
$$\lim_{x\to\infty} \ln f(x) = \lim_{x\to\infty} \frac{\ln x}{x} \qquad \frac{\infty}{\infty}$$

$$= \lim_{x\to\infty} \frac{1/x}{1}$$

$$= \frac{0}{1} = 0.$$

Therefore $\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln f(x)} = e^0 = 1.$ ■

.2 INTEGRATION:



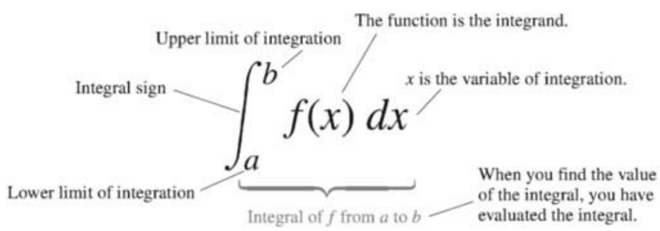
1) The Definite Integral

$$S_n = \sum_{k=1}^{n} f(c_k)\,\Delta x_k = \sum_{k=1}^{n} f(c_k)\left(\frac{b-a}{n}\right),$$

$$\Delta x_k = \Delta x = (b-a)/n \text{ for all } k$$

$$J = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\left(\frac{b-a}{n}\right) = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\,\Delta x. \qquad \Delta x = (b-a)/n$$

$$\Delta x = (b-a)/n$$



Rules satisfied by definite integrals

1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ — A Definition

2. *Zero Width Interval:* $\int_a^a f(x)\,dx = 0$ — A Definition when $f(a)$ exists

3. *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ — Any constant k

4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

6 $f(x) \geq g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

$f(x) \geq 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \geq 0$ (Special Case)