$\boxed{3}$ Moduli and conjucate.

<u>Def:</u> Modulus or absolute value of a complex number $Z = x + iy$ is $\sqrt{x^2+y^2} = |Z|$

<u>Exmple:</u> If $Z = 3 - i4$ then

$$|Z| = \sqrt{x^2+y^2} = \sqrt{9+16} = \sqrt{25} = 5.$$

<u>properties of Modulus</u>

① for any Complex number Z, $|Z| \in \mathbb{R}$ and $|Z| \geq 0$

② $|Z| = 0$ iff $Z = 0$

③ geometrically, $|Z|$ is the length of the vector representing Z.

④ $Z_1 < Z_2$ is meaningless unless both Z_1 & Z_2 are real numbers.

ملاحظة: المقارنة بين Z_1, Z_2 عقدياً غير موجودة ولكن يمكن المقارنة حقيقياً من خلال قربها او بعدها من الصفر (القيمة المطلقة).

⑤ $|Z_1| < |Z_2|$ means that the point Z_1 is closer to $(0,0)$ than the point Z_2.

⑥ $|Z_1 - Z_2|$ gives the distance between the points Z_1 & Z_2.

$$|Z_1 - Z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

⑦ The eq. of the circle with center Z_0 and radius R is

$$|Z - Z_0| = R.$$

Example: $|Z - 2 + i3| = 3$

$= |Z - (2 - i3)| = 3$

is circle with center at $(2, -3)$ and radius is 3 $(R = 3)$

Proposition: Let Z be a complex number, then

$\boxed{1}$ $Re(Z) \leq |Re(Z)| \leq |Z|$

$\boxed{2}$ $Im(Z) \leq |Im(Z)| \leq |Z|$

proof: $|Z| = \sqrt{x^2 + y^2}$

or $|Z| = \sqrt{(Re(Z))^2 + (Im(Z))^2}$

$|Z|^2 = (Re(Z))^2 + (Im(Z))^2$

$\therefore (Re(Z))^2 \leq |Z|^2$ (باخذ الجذر التربيعي)

$\Rightarrow |Re(Z)| \leq |Z|$

But $Re(Z) \leq |Re(Z)|$

$\therefore Re(Z) \leq |Re(Z)| \leq |Z|$.

$\boxed{2}$ H.W.

Def: The conjugate of Complex number $z = x + iy$ is $\bar{z} = x - iy$.

Example: If $z = 2 - i3$ then $\bar{z} = 2 + i3$.

proposition: Let z be complex number then:

① $\bar{\bar{z}} = z$ ② $|\bar{z}| = |z|$

③ $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2 \quad \forall z_1, z_2 \in \mathbb{C}$

④ $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

⑤ $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

⑥ $Re(z) = \frac{z + \bar{z}}{2}$ ⑦ $Im(z) = \frac{z - \bar{z}}{i2}$

⑧ $z\bar{z} = |z|^2$ ⑨ $|z_1 z_2| = |z_1||z_2|$

⑩ $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

proof: (4) let $z_1 = x_1 + i y_1$

$z_2 = x_2 + i y_2$

$$z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$$

$$\overline{z_1 \cdot z_2} = (x_1 x_2 - y_1 y_2) - i(x_1 y_2 + y_1 x_2) \quad \text{— (1)}$$

$$\bar{z}_1 \bar{z}_2 = (x_1 - i y_1)(x_2 - i y_2)$$
$$= x_1 x_2 - y_1 y_2 - i(x_1 y_2 + y_1 x_2) \quad \text{— (2)}$$

from (1) + (2) we get (4).

(5) $$\frac{z_1}{z_2} = \frac{z_1}{z_2} \cdot \frac{\bar{z}_2}{\bar{z}_2} = \frac{z_1 \bar{z}_2}{|z_2|^2}$$

ناخد conjucate للطرفين

$$\overline{\left(\frac{z_1}{z_2}\right)} = \overline{\left(\frac{z_1 \bar{z}_2}{|z_2|^2}\right)} = \frac{\bar{z}_1 \bar{\bar{z}}_2}{|\bar{z}_2|^2} = \frac{\bar{z}_1 \cancel{z_2}}{\cancel{z_2} \bar{z}_2}$$

$$\therefore \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$$

(6) $z = x + iy$ + $\bar{z} = x - iy$

$\Rightarrow z + \bar{z} = x + iy + x - iy = 2x = 2\,Re(z)$

$\therefore Re(z) = \frac{z + \bar{z}}{2}$ + $z - \bar{z} = i2y$.