Chapter One: Complex Numbers

1- Sums and product

Definition: We define complex number as ordered pairs (x, y) of real number (two dimension number), which denoted by $z, w, \dots$ and the set of all complex number denoted by $\mathbb{C}$.

Definitions: Let $z = (x, y)$ be a complex number, x is called the real part of z and denoted by $Re(z)$ and y is called imaginary part of z and denoted by $Im(z)$, if $x = 0$ then $z = (0, y)$ is called a pure imaginary part.

Example: let $z = (-3,1)$ then $Re(z) = -3$ and $Im(z) = 1$.

Remark: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$, for all $x \in \mathbb{R}, x = (x, 0) \in \mathbb{C}$ thus $\mathbb{R} \subset \mathbb{C}$.

Definitions: Let $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ be two complex number then

1- $z_1 = z_2 \leftrightarrow x_1 = x_2$ and $y_1 = y_2$
2- $z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$
3- $z_1 . z_2 = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$

Example:

$$(2, -3)(-2,1) = (-4 - (-3), 2 + 6) = (-4 + 3,2 + 6) = (-1,8)$$

Definitions: We define the complex number $i = (0,1)$

Proposition: Let $i = (0,1)$ then

1- $i^2 = -1$
2- Every complex number $z = (x, y)$ can be written as $z = x + iy$

Proof:

1- $i^2 = (0,1)(0,1) = (0 - 1,0.1 + 1.0) = (-1,0) = -1$
2- Let $z = (x, y)$, so this lead to $x = (x, 0)$ and $y = (y, 0)$, $x, y \in \mathbb{R}$
$iy = (0,1)(y, 0) = (0, y)$ and so $x + iy = (x, 0) + (0, y) = (x, y) = z$

Example: Find $z_1 + z_2$ and $z_1 . z_2$, if $z_1 = 2 - i3$ and $z_2 = -2 + i$

Sol: $z_1 = 2 - i3 = (2,-3)$ and $z_2 = -2 + i \longrightarrow z_1 + z_2 = 0 - i2$

$z_1.z_2 = (2 - i3)(-2 + i) = -4 + i2 + 6i - i^2 3 = -4 + i2 + i6 + 3 =$ $-1 + i8 = (-1,8)$.

2- Algebraic Properties

Proposition: Let z_1, z_2 and z_3 be complex numbers then

1- $z_1 + z_2 = z_2 + z_1$ & $z_1.z_2 = z_2.z_1$
2- $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ & $z_1.(z_2.z_3) = (z_1.z_2).z_3$
3- $z_1.(z_2 + z_3) = z_1.z_2 + z_1.z_3$
4- $0 + z = z$, $1.z = z$
5- The additive inverse of $z = x + iy$ is $-z = -x - iy$
6- The multiplicative inverse of $0 \neq z = x + iy$ is $z^{-1} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$

Proof:

1- Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then

$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2) \\ &= (x_2 + x_1) + i(y_2 + y_1) = z_2 + z_1 (since (+) is\ commutative\ on\ R) \end{aligned}$$

(2,3,4,5) H.W

6- T.P $z^{-1} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$

$$\begin{aligned} z.z^{-1} &= (x + iy).\left(\frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}\right) \\ &= \frac{x^2}{x^2+y^2} - i\frac{xy}{x^2+y^2} + i\frac{xy}{x^2+y^2} - i^2\frac{y^2}{x^2+y^2} = 1 \end{aligned}$$

Example: Find z^{-1} if $z = i$

Sol: $z^{-1} = i^{-1} = \frac{1}{i}\left(\frac{-i}{-i}\right) = \frac{-i}{-i^2} = -i$

Example: Find z^{-1} if $z = 2 - i3$

Sol: $z^{-1} = \frac{1}{2-i3} \times \frac{2+i3}{2+i3} = \frac{2+i3}{4+i6-i6-i^2 9} = \frac{2+i3}{13} = \frac{2}{13} + i\frac{3}{13}$

Example: Find $\frac{-1+2i}{-3+4i}$

Sol:

$$\frac{-1+2i}{-3+4i} = \left(\frac{-1+2i}{-3+4i}\right).\left(\frac{-3-4i}{-3-4i}\right) = \frac{3+4i-6i-8i^2}{9+12i-12i-16i^2} = \frac{11-2i}{25}$$
$$= \frac{11}{25} - i\frac{2}{25}$$

3- Moduli and conjugate

Proposition: Let z_1 and z_2 be two complex numbers, if $z_1.z_2 = 0$ then either

$z_1 = 0$ or $z_2 = 0$

Proof: Assume $z_1.z_2 = 0$ and $z_1 \neq 0 \Longrightarrow z_1^{-1}$ exist $\Longrightarrow z_1^{-1}.z_1.z_2 = z_1^{-1}.0 = 0$

$$\Longrightarrow z_1^{-1}.z_1.z_2 = 0 \Longrightarrow 1.z_2 = 0 \Longrightarrow z_2 = 0$$

Example: Solve $z^2 + 3iz - 2 = 0$

Sol: $z^2 + 3iz - 2 = 0 \Longrightarrow (z+2i).(z+i) = 0$
either $z + 2i = 0 \Longrightarrow z = -2i$ or $z + i = 0 \Longrightarrow z = -i$

Example: Show that $z = -1 - i2$ satisfy the equation $z^2 + 2z + 5 = 0$

Sol:

$z^2 + 2z + 5 = (-1-i2)^2 + 2(-1-i2) + 5 = 1 + i4 - 4 - 2 - i4 + 5 = 0$

$\therefore z$ Solution for the equation

Example: Show that $Im(iz) = Re(z)$

Sol:

Let $z = x + iy \Longrightarrow Re(z) = x$ & $iz = ix - y = -y + ix \Longrightarrow Im(iz) = x$

$$\therefore Im(iz) = Re(z)$$

Exercise: H.W

Definitions: Modulus or absolute value of a complex number $z = x + iy$ is $|z| = \sqrt{x^2 + y^2}$.

Example: If $z = 3 - i4$ then $|z| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$

Properties of Modulus

1- For any complex number z, $|z| \in R$ and $|z| \geq 0$

2- $|z| = 0 \; iff \; z = 0$

3- Geometrically, $|z|$ is the length of the vector representing z

4- $z_1 < z_2$ is meaningless unless both z_1 and z_2 are real numbers.

5- $|z_1| < |z_2|$ Means that the point z_1 is closer to $(0,0)$ than the point z_2.

6- $|z_1 - z_2|$ gives the distance between the points z_1 and z_2

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

7- The equation of the circle with center $z_\circ$ and radius R is $|z_1 - z_2| = R$

Example: $|z - 2 + i3| = 3 \Rightarrow |z - (2 - i3)| = 3$ is circle with center at $(2, -3)$ and radius is $3(R = 3)$.

Proposition: Let z be a complex number, then

1- $Re(z) \leq |Re(z)| \leq |z|$

2- $Im(z) \leq |Im(z)| \leq |z|$

Proof: $|z| = \sqrt{x^2 + y^2}$ or $|z| = \sqrt{\left(Re(z)\right)^2 + \left(Im(z)\right)^2} \Rightarrow$

$$|z|^2 = \left(Re(z)\right)^2 + \left(Im(z)\right)^2$$

$$\left(Re(z)\right)^2 \leq |z|^2 \Rightarrow^{by\ square\ root} |Re(z)| \leq |z|$$

But $Re(z) \leq |Re(z)| \leq |z|$.

Definition: the conjugate of complex number $z = x + iy$ is $\bar{z} = x - iy$

Example: If $z = 2 - i3$ then $\bar{z} = 2 + i3$

Proposition: Let z be complex number then

1) $\bar{\bar{z}} = z$
2) $|\bar{z}| = |z|$
3) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$
4) $\overline{z_1 . z_2} = \bar{z}_1 . \bar{z}_2$
5) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$
6) $Re(z) = \frac{z+\bar{z}}{2}$
7) $Im(z) = \frac{z-\bar{z}}{2i}$
8) $z.\bar{z} = |z|^2$
9) $|z_1 . z_2| = |z_1| . |z_2|$
10) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

Proof: 4) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$$z_1 z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2) \Longrightarrow$$

$$\overline{z_1 z_2} = (x_1x_2 - y_1y_2) - i(x_1y_2 + y_1x_2) \qquad (1)$$

$$\bar{z}_1\bar{z}_2 = (x_1 - iy_1).(x_2 - iy_2)$$
$$= (x_1x_2 - y_1y_2) - i(x_1y_2 + y_1x_2) \qquad (2)$$

From (1) and (2) the proof is complete.

5) $\frac{z_1}{z_2} = \frac{z_1}{z_2} . \frac{\bar{z}_2}{\bar{z}_2} = \frac{z_1\bar{z}_2}{|z_2|^2} \Longrightarrow_{by\ conjucate\ for\ twe\ sides} \overline{\left(\frac{z_1}{z_2}\right)} = \overline{\left(\frac{z_1\bar{z}_2}{|z_2|^2}\right)} = \frac{\overline{z_1\bar{z}_2}}{|z_2|^2} = \frac{\bar{z}_1 z_2}{z_2\bar{z}_2} = \frac{\bar{z}_1}{\bar{z}_2}$

6) $z = x + iy$ & $\bar{z} = x - iy \Longrightarrow z + \bar{z} = 2x$ & $z - \bar{z} = i2y$

8) $z\bar{z} = (x + iy)(x - iy) = x^2 - ixy + iyx + y^2 = x^2 + y^2 = \overline{|z|^2}$