

Example: Find $\frac{1}{3-i4} + \frac{3-i2}{6+i8}$

Sol:

$$\begin{aligned} \left(\frac{1}{3-i4} \times \frac{3+i4}{3+i4} \right) + \left(\frac{3-i2}{6+i8} \times \frac{6-i8}{6-i8} \right) &= \left(\frac{3+i4}{9+16} \right) + \left(\frac{18-i24-i12-16}{36+64} \right) \\ &= \frac{3+i4}{25} + \frac{2-i36}{100} = \frac{7}{50} - i\frac{10}{50} \end{aligned}$$

Example: show that the hyperbola $y^2 - x^2 = 1$ can be written as $z^2 + \bar{z}^2 = 2$ where $z = x + iy$

Sol:

Since $x = \operatorname{Re}(z) = \frac{z+\bar{z}}{2}$ & $y = \operatorname{Im}(z) = \frac{z-\bar{z}}{i2}$

$$\begin{aligned} \Rightarrow y^2 - x^2 = 1 &\Rightarrow \left(\frac{z-\bar{z}}{i2} \right)^2 - \left(\frac{z+\bar{z}}{2} \right)^2 = 1 \Rightarrow \frac{-(z-\bar{z})^2}{4} - \frac{(z+\bar{z})^2}{4} = 1 \\ &\frac{-(z^2 - 2z\bar{z} + \bar{z}^2)}{4} - \frac{(z^2 + 2z\bar{z} + \bar{z}^2)}{4} = 1 \\ &-\frac{1}{2}z^2 - \frac{1}{2}\bar{z}^2 = 1 \Rightarrow z^2 + \bar{z}^2 = -2 \end{aligned}$$

4- Triangle Inequality

Proposition (Triangle Inequality): Let z_1 and z_2 be complex numbers then

$$1) |z_1 + z_2| \leq |z_1| + |z_2|$$

Proof: $|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)} = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) =$

$$z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2 = |z_1|^2 + z_1\bar{z}_2 + \overline{z_1\bar{z}_2} + |z_2|^2 =$$

$|z_1|^2 + |z_2|^2 + 2\operatorname{Re}(z_1\bar{z}_2)$ where $w + \bar{w} = 2\operatorname{Re}(w)$, w is complex

$$\Rightarrow \text{by absolut v. } |z_1 + z_2|^2 \leq |z_1|^2 + |z_2|^2 + 2|z_1\bar{z}_2| \leq |z_1|^2 + 2|z_1||\bar{z}_2| + |z_2|^2 =$$

$$(|z_1| + |z_2|)^2$$

$$\Rightarrow |z_1 + z_2| \leq |z_1| + |z_2|$$

$$2) |z_1 + z_2| \geq ||z_1| - |z_2|| \geq |z_1| - |z_2|$$

$$\text{Proof: } |z_1| = |z_1 + z_2 - z_2| \leq |z_1 + z_2| + |-z_2|$$

$$\Rightarrow |z_1| - |z_2| \leq |z_1 + z_2| \quad (1)$$

In the same way, we show that

$$|z_2| - |z_1| \leq |z_1 + z_2|$$

$$\Rightarrow |z_1| - |z_2| \geq -|z_1 + z_2| \quad (2)$$

\therefore from (1)& (2), we get

$$||z_1| - |z_2|| \leq |z_1 + z_2| \text{ or } |z_1 + z_2| \geq ||z_1| - |z_2||$$

Example: show that $|z_1 - z_2| \geq ||z_1| - |z_2||$

$$\text{Sol: } |z_1 - z_2| = |z_1 + (-z_2)| \geq ||z_1| - |z_2||$$

Example: If z lies on the circle $|z| = 2$ then show that $3 \leq |z^2 - 1| \leq 5$

$$\text{Sol: } |z^2 - 1| \leq |z|^2 + |-1| = |z|^2 + 1 = 4 + 1 = 5$$

$$|z^2 - 1| \geq ||z|^2 - |1|| \geq |4 - 1| = 3$$

$$\therefore 3 \leq |z^2 - 1| \leq 5$$

Example: Use the inequality $||z_1| - |z_2|| \leq |z_1 - z_2|$, show that if z lies on the circle $|z| = 3$ then $|z^2 - 3z + 2| \geq 2$

$$\text{Sol: } |z^2 - 3z + 2| = |(z - 2)(z - 1)| = |z - 2||z - 1|$$

$$\text{Since } |z - 2| \geq ||z| - 2| = 1 \text{ \& } |z - 1| \geq ||z| - 1| = 2$$

$$\Rightarrow |z^2 - 3z + 2| \geq 2$$

Exercise: H.W

5- polar coordinates and Euler formula

Let r and θ be the polar coordinates of the point (x, y) that corresponding to the nonzero complex number $z = x + iy$ since $x = r\cos\theta$ and $y = r\sin\theta$, z can be written in polar form as $z = r(\cos\theta + i\sin\theta)$.

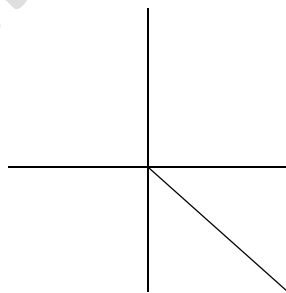
Remark:

- 1) If $z = 0$ then θ is not defined and hence we cannot write $z = 0$ in polar form.
- 2) In complex analysis we assume $r \geq 0$ and $r = \sqrt{x^2 + y^2}$
- 3) The real number θ represent the angle, measured in radius that z makes with the positive real axis.
- 4) θ has infinite number of values, both positive and negative, that differ by integer multiplication 2π .
- 5) The value of θ can be determined by specifying the quadrant containing $z = x + iy$ and by using one of the following three equation

$$\tan\theta = \frac{y}{x} \quad \& \quad \cos\theta = \frac{x}{|z|} \quad \& \quad \sin\theta = \frac{y}{|z|}$$

Example: write $z = 1 - i$ in polar form

Sol: $r = |z| = \sqrt{1 + 1} = \sqrt{2}$ & $\tan\theta = \frac{y}{x} = \frac{-1}{1} = -1 \Rightarrow \theta = -\frac{\pi}{4}$
 $\Rightarrow z = r(\cos\theta + i\sin\theta) = \sqrt{2} \left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right) \right)$



Definition: Let z be nonzero complex number

- 1) Each value of θ such that $z = x + iy = r(\cos \theta + i \sin \theta)$ is called an argument of z
- 2) The set of all argument of z is denoted by $arg z$
- 3) The principle value of $arg z$ is the unique value θ such that $-\pi < \theta \leq \pi$ and is denoted by $Arg z$.

$$argz = \{\theta: z = r(\cos \theta + i \sin \theta)\}, Argz \in (-\pi, \pi]$$

$$argz = \{Argz + 2n\pi: n \in \mathbb{Z}\}$$

Example: Find $Arg(5)$, $Arg(-5)$, $Arg(i2)$ and $Arg(-i3)$

Sol: $Arg(5) = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}(0) = 0$

$$Arg(-5) = \pi + \tan^{-1}(0) = \pi$$

$$Arg(i2) = \frac{\pi}{2} \quad \& \quad Arg(-i3) = -\frac{\pi}{2}$$

Example: Find $arg(z)$, if $z = 1 + i$

Sol: $arg(z) = Arg(z) + 2n\pi, n \in \mathbb{Z}$

$$\Rightarrow Arg(z) = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

$$\therefore arg(z) = \frac{\pi}{4} + 2n\pi, n \in \mathbb{Z}$$

Example: Find $Arg(z)$, if $z = \frac{1+i\sqrt{3}}{1-i\sqrt{3}}$

Sol: $z = \frac{1+i\sqrt{3}}{1-i\sqrt{3}} \times \frac{1+i\sqrt{3}}{1+i\sqrt{3}} = \frac{1-3+i2\sqrt{3}}{4} = \frac{-1+i\sqrt{3}}{2}$

$$\begin{aligned} Arg(z) &= \pi + \tan^{-1}\left(\frac{y}{x}\right) = \pi + \tan^{-1}\left(\frac{\sqrt{3}/2}{-1/2}\right) = \pi + \tan^{-1}(-\sqrt{3}) \\ &= \pi - \tan^{-1}(\sqrt{3}) = \pi - \frac{\pi}{3} = \frac{2\pi}{3} \end{aligned}$$

ملاحظة: إذا كان $z = 0$ فإن θ غير معروفة لأن $\frac{y}{x} = 0$ و $\theta = \tan^{-1}\left(\frac{y}{x}\right) = 0$ و هذا مقدار غير معروف و عليه لا يمكن تمثيل $z = 0$ بالاحداثيات القطبية. 8

Euler formula: For any real number θ , we will use the notation

$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$\Rightarrow z = r(\cos \theta + i \sin \theta) = r e^{i\theta} = r \exp(i\theta)$$

Proposition (without proof): Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ then $z_1 = z_2$ if and only if $r_1 = r_2$ and $\theta_1 = \theta_2 + 2n\pi, n \in \mathbb{Z}$.

Remark: $z = z_0 + R e^{i\theta}, 0 \leq \theta \leq 2\pi$ is a parametric representation of the circle $|z - z_0| = R$.

Example: Write $z_1 = -1, z_2 = 1 + i$ in polar form

Sol: $r_1 = |z_1| = |-1| = 1, \theta_1 = \text{Arg}(z_1) = \pi \Rightarrow z_1 = e^{i\pi}$

$$r_2 = |z_2| = |\sqrt{2}| = \sqrt{2}, \theta_2 = \text{Arg}(z_2) = \frac{\pi}{4} \Rightarrow z_2 = \sqrt{2} e^{i\frac{\pi}{4}}$$

Example: Express $z = 3e^{i\frac{\pi}{3}}$ in the form $a + ib$

Sol: $z = 3e^{i\frac{\pi}{3}} = 3 \left[\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right] = 3 \left[\frac{1}{2} + i \frac{\sqrt{3}}{2} \right]$

Remark: $z = z_0 + R e^{i\theta}, 0 \leq \theta \leq 2\pi$ is a parametric representation of the circle.

i.e. $|z - z_0| = R \equiv z = z_0 + R e^{i\theta}$.