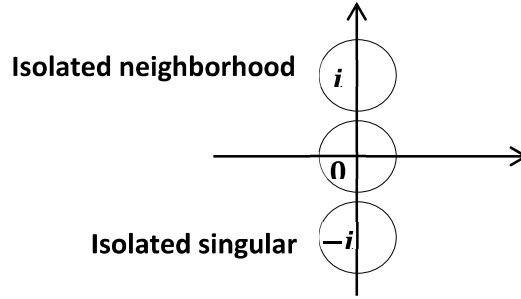


Note: not every singular point is isolated, as in example 3, 5.

6. $f(z) = \frac{1}{z(z^2+1)}$, has singular and isolated points at $z = 0, i, -i$.



Let z_0 be any isolated singular point of f , then f is analytic at each point z , when $0 < |z - z_0| < R$, so $f(z)$ can be represented by a Laurent series

$$f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \quad \dots (1)$$

$$\text{where } a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

$$\text{and } b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} dz$$

Hence:

$$b_1 = \frac{1}{2\pi i} \int_C f(z) dz \quad \dots (2)$$

Or

$$\int_C f(z) dz = 2\pi i b_1$$

where C is any simple closed contour around z_0 described in positive sense. The coefficient b_1 of $\frac{1}{z - z_0}$ in expansion (1) is called the residue of f at the isolated singular point z_0 . Formula (2) gives us a powerful method for evaluating certain integrals around simple closed contours and it is denoted by

$$b_1 = \text{Res}[f, z_0]$$

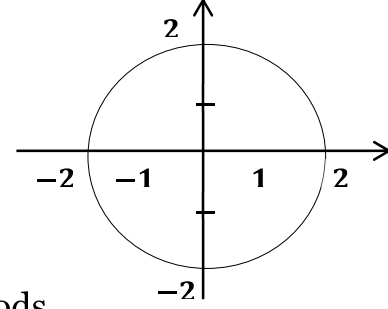
Example: Evaluate

$$\oint_C \frac{e^{-z}}{(z-1)^2} dz$$

such that $C : |z| = 2$.

Solution:

Note: we can solve this integral by two methods.



i. By Cauchy integral formula

$$\oint_C \frac{e^{-z}}{(z-1)^2} dz = 2\pi i f'(z_0)$$

$$f(z) = e^{-z} \rightarrow f'(z) = -e^{-z} \rightarrow f'(1) = -e^{-1}$$

$$\oint_C \frac{e^{-z}}{(z-1)^2} dz = 2\pi i f'(1)$$

$$= 2\pi i (-e^{-1})$$

$$= -\frac{2\pi i}{e}$$

ii. Note that $f(z) = \frac{e^{-z}}{(z-1)^2}$ is analytic over C except $z_0 = 1$, so by Laurent theorem

$$\begin{aligned} \frac{e^{-z}}{(z-1)^2} &= \frac{e^{-1}e^{-(z-1)}}{(z-1)^2} \\ &= \frac{e^{-1}}{(z-1)^2} \sum_{n=0}^{\infty} (-1)^n \frac{(z-1)^n}{n!}, |z-1| < \infty \\ &= \frac{1}{e} \sum_{n=0}^{\infty} (-1)^n \frac{(z-1)^{n-2}}{n!} \\ &= \frac{1}{e} \left[\frac{1}{(z-1)^2} - \frac{1}{z-1} + \frac{1}{2!} - \frac{z-1}{3!} + \dots \right] \end{aligned}$$

where the coefficient of $(z - z_0)^{-1} = (z - 1)^{-1}$ is $\frac{-1}{e} = b_1$, so:

$$\oint_C \frac{e^{-z}}{(z-1)^2} dz = 2\pi i (b_1)$$

$$= \frac{-2\pi i}{e}$$

Note: if z_0 is an isolated point, then we can find the integral by Laurent and then we find the residue of the function at $-z$.

Example: Evaluate

$$\oint_C \frac{e^z}{z} dz$$

such that $C : |z| = 1$.

Solution: Note $z_0 = 0$ is a singular point of f .

$$\begin{aligned} \frac{1}{z} e^z &= \frac{1}{z} \left[1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots \right] \\ &= \frac{1}{z} + 1 + \frac{z}{2!} + \frac{z^2}{3!} + \dots \end{aligned}$$

Note that b_1 is the coefficient of $\frac{1}{z}$, then $b_1 = 1$ and

$$\begin{aligned} \oint_C \frac{e^z}{z} dz &= 2\pi i b_1 \\ &= 2\pi i \end{aligned}$$

Example: Evaluate

$$\oint_C e^{1/z^2} dz$$

such that $C : |z| = 2$.

Solution: Note that there is no fraction so we cannot solve by the two previous methods that is Cauchy integral formula cannot be applied here, so we will solve by residue.

$z_0 = 0$ is a singular and isolated point of f .

$$\begin{aligned} e^z &= \sum_{n=0}^{\infty} \frac{z^n}{n!}, |z| < \infty, \text{ so} \\ e^{1/z^2} &= \sum_{n=0}^{\infty} \frac{(1/z^2)^n}{n!}, \left| \frac{1}{z^2} \right| < \infty \\ &= \sum_{n=0}^{\infty} \frac{1}{n! z^{2n}} \rightarrow \left| \frac{1}{z} \right| < \infty \\ &= 1 + \frac{1}{z^2} + \frac{1}{2! z^4} + \dots, 0 < |z| < \infty \end{aligned}$$

The coefficient of $(z - z_0)^{-1} = (z - 0)^{-1}$ is 0, then $b_1 = 0$ so that:

$$\oint_C e^{1/z^2} dz = 2\pi i b_1 = 0$$

And this is clear, since f is analytic on C and so by Cauchy $\oint_C f(z) dz = 0$.

Example: Evaluate the following integral by using residues:

$$\oint_C z^3 \cos\left(\frac{1}{z}\right) dz ; C : |z + 1 + i| = 4$$

Solution:

The point $z_0 = 0$ is an isolated singularity of $\cos\left(\frac{1}{z}\right)$ and lies in the given contour of integration; we want a Laurent series expansion of $z^3 \cos\left(\frac{1}{z}\right)$ about this point (i.e. $z_0 = 0$), since:

$$\cos z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}, \text{ we have}$$

$$\cos\left(\frac{1}{z}\right) = 1 - \frac{\left(\frac{1}{z}\right)^2}{2!} + \frac{\left(\frac{1}{z}\right)^4}{4!} - \frac{\left(\frac{1}{z}\right)^6}{6!} + \dots$$

$$\rightarrow z^3 \cos\left(\frac{1}{z}\right) = z^3 - \frac{1}{2!} z + \frac{1}{4!} \frac{1}{z} - \frac{1}{6!} \frac{1}{z^3} + \dots$$

$$\rightarrow b_1 = \frac{1}{4!}$$

$$\begin{aligned} \oint_C z^3 \cos\left(\frac{1}{z}\right) dz &= 2\pi i b_1 = \frac{2\pi i}{4!} \\ &= \frac{\pi i}{12} \end{aligned}$$

Example: Let C be a positively oriented unit circle $z_0 = 0$. Evaluate

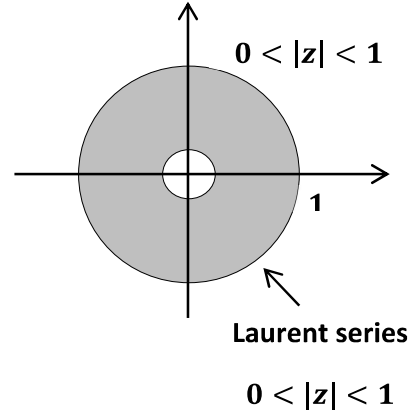
$$\oint_C \frac{dz}{z^3 + z^2}$$

Solution: The isolated singular points are $z = 0$ and $z = -1, -1 \notin 0 < |z| < 1$

$$\begin{aligned}
 f(z) &= \frac{1}{z^3+z^2} = \frac{1}{z^2(z+1)} \\
 &= \frac{1}{z^2} \left(\frac{1}{1+z} \right) \\
 &= \frac{1}{z^2} (1 - z + z^2 - z^3 + \dots) \\
 &= \frac{1}{z^2} - \frac{1}{z} + 1 - z + z^2 - \dots
 \end{aligned}$$

$\rightarrow b_1 = -1$, so

$$\begin{aligned}
 \oint_C f(z) dz &= \oint_C \frac{1}{z^3+z^2} dz \\
 &= 2\pi i b_1 \\
 &= -2\pi i
 \end{aligned}$$



Example: Evaluate

$$\oint_C \frac{e^{-z}}{(z-1)^2} dz$$

where C is the circle $|z| = 2$, described in the positive sense.

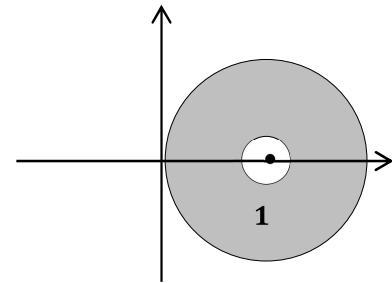
Solution:

$f(z) = \frac{e^{-z}}{(z-1)^2}$ is analytic on C and its interior except at the isolated singular point $z = 1$, now

$$\begin{aligned}
 e^{-z} &= e^{-z+1-1} \\
 &= e^{-1} e^{1-z} \\
 &= e^{-1} \sum_{n=0}^{\infty} \frac{(1-z)^n}{n!} \\
 &= e^{-1} \sum_{n=0}^{\infty} (-1)^n \frac{(z-1)^n}{n!} \\
 &= e^{-1} \left[1 - (z-1) + \sum_{n=2}^{\infty} (-1)^n \frac{(z-1)^n}{n!} \right]
 \end{aligned}$$

$$\therefore e^{-z} = e^{-1} - e^{-1}(z-1) + e^{-1} \sum_{n=2}^{\infty} (-1)^n \frac{(z-1)^n}{n!}$$

Since $|z-1| > 0$, we can divide both sides by $(z-1)^2$



$$\therefore \frac{e^{-z}}{(z-1)^2} = \frac{e^{-1}}{(z-1)^2} - \frac{e^{-1}}{(z-1)} + e^{-1} \sum_{n=2}^{\infty} (-1)^n \frac{(z-1)^{n-2}}{n!}$$

$\therefore b_1$ at $z = 1$ is equal to $-e^{-1}$, so

$$\begin{aligned} \oint_C \frac{e^{-z}}{(z-1)^2} dz &= 2\pi i b_1 \\ &= -\frac{2\pi i}{e} \end{aligned}$$

Example: Evaluate

$$\oint_C \frac{dz}{z(z-1)}$$

where C is the circle $|z-1|=1$ (i.e.: or described in the positive sense as shown in the following figure).

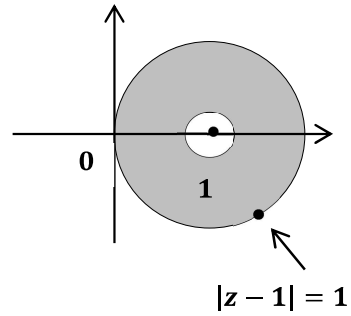
Solution:

$f(z) = \frac{1}{z(z-1)}$, which is analytic on C and at all points inside C except at $z = 1$, which is an isolated singular point. The Laurent series expansion of $f(z)$ that converges in the annular region centered at $z = 1$, is

$$\begin{aligned} \frac{1}{z(z-1)} &= -\frac{1}{z} + \frac{1}{z-1} \\ &= \frac{1}{z-1} - \frac{1}{(z-1)+1} \\ &= (z-1)^{-1} - \sum_{n=0}^{\infty} (-1)^n (z-1)^n \\ &= (z-1)^{-1} - 1 + (z-1) - (z-1)^2 + \dots \end{aligned}$$

$\therefore b_1 = 1$, so

$$\oint_C \frac{dz}{z(z-1)} = 2\pi i b_1 = 2\pi i$$



Example: Evaluate $\oint_C \frac{\sin z}{z \sinh z} dz$, around $|z| = 1$.

Solution:

$z = 0$ is the only isolated singular point inside $|z| = 1$, recall that: