2 Algebraic properties.

<u>proposition</u>: Let z_1, z_2 and z_3 be complex number then:

① $z_1 + z_2 = z_2 + z_1$ & $z_1 \cdot z_2 = z_2 z_1$

② $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

$z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$

③ $z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3$

④ $0 + z = z$ & $1 \cdot z = z$

⑤ The addition inverse of $z = x + iy$ is $-z = -x - iy$.

⑥ The multiplacation inverse of $0 \neq z = x + iy$ is

$$z^{-1} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$ (Prove that H.W.)

<u>Note</u>: The multiplcation inverse z^{-1} for $z \neq 0$ satisf $z \cdot z^{-1} = z^{-1} z = 1$.

proof: ① $z_1 + z_2 = z_2 + z_1$

let $z_1 = x_1 + i y_1$ & $z_2 = x_2 + i y_2$

then

$$z_1 + z_2 = (x_1 + i y_1) + (x_2 + i y_2)$$

$$= (x_1 + x_2) + i (y_1 + y_2)$$

$$= (x_2 + x_1) + i (y_2 + y_1)$$

(since + is comutative on R)

$$= (x_2 + i y_2) + (x_1 + i y_1)$$

$$= z_2 + z_1 .$$

⑥ $z^{-1} = \frac{x}{x^2+y^2} - i \frac{y}{x^2+y^2}$

$$z z^{-1} = (x + i y)\left(\frac{x}{x^2+y^2} - i \frac{y}{x^2+y^2}\right)$$

$$= \frac{x^2}{x^2+y^2} - i \cancel{\frac{xy}{x^2+y^2}} + i \cancel{\frac{yx}{x^2+y^2}}$$

$$- i^2 \frac{y^2}{x^2+y^2} = \frac{x^2+y^2}{x^2+y^2} = 1$$

Example [1]: Find z^{-1} if $z = i$

Sol: $z^{-1} = i^{-1} = \frac{1}{i} = \frac{1}{i}\left(\frac{-i}{-i}\right)$

$$= \frac{-i}{-i^2} = -i$$

Example [2]: Find z^{-1} if $z = 2 - i3$

Sol: $z^{-1} = \frac{1}{2-i3} \times \frac{2+i3}{2+i3}$

$$= \frac{2+i3}{4+\cancel{i6}-i6-i^2 9} = \frac{2+i3}{4+9}$$

$$\therefore z^{-1} = \frac{2+i3}{13} = \boxed{\frac{2}{13} + i\frac{3}{13}}$$

Example [3]: Find $\frac{-1+i2}{-3+i4}$

Sol: $\frac{-1+i2}{-3+i4} = \frac{-1+i2}{-3+i4} \cdot \left(\frac{-3-i4}{-3-i4}\right)$

$$= \frac{3+i4-i6-i^2 8}{9+i12-i12-i^2 16}$$

$$= \frac{11-i2}{9+16} = \frac{11}{25} - i\frac{2}{25}$$

Moduli and conjugates

proposition: Let Z_1 & Z_2

be two complex numbers,

if $Z_1 \cdot Z_2 = 0$ then either

$Z_1 = 0$ or $Z_2 = 0$

proof: assume $Z_1 \cdot Z_2 = 0$ &

assume $Z_1 \neq 0 \Rightarrow Z_1^{-1}$ exist.

$\Rightarrow Z_1^{-1} \cdot Z_1 \cdot Z_2 = Z_1^{-1} \cdot 0 = 0$

$\Rightarrow 1 \cdot Z_2 = 0 \Rightarrow Z_2 = 0$

Example: Solve $Z^2 + i3Z - 2 = 0$

Sol: $Z^2 + i3Z \overset{+i^2}{-} 2 = 0$

$(Z + i2)(Z + i) = 0$

either $Z + i2 = 0 \Rightarrow Z = -i2$

or $Z + i = 0 \Rightarrow Z = -i$

Example: Show that

$z = -1 - i2$ satisfy the eq.

$$z^2 + 2z + 5 = 0$$

Sol: $z^2 + 2z + 5 = \cancel{(-1-i2)^2}$

$$(-1-i2)^2 + 2(-1-i2) + 5 =$$

$$\cancel{1} + \cancel{i4} \cancel{-4} \cancel{-2} \cancel{-i4} \cancel{+5} = 0$$

$\therefore$ z sol. for the eq.

Example: Show that

$$Im(iz) = Re(z)$$

Sol: Let $z = x + iy$

$$\Rightarrow \quad Re(z) = x$$

$$iz = ix - y = -y + ix$$

$$\therefore \quad Im(iz) = x$$

$$\therefore \quad Re(z) = Im(iz).$$

$\boxed{\text{(H.W. p.5)}}$