Case 1 If **m is odd**, we write m as $2k + 1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x. \qquad (1)$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x \, dx$ equal to $-d(\cos x)$.

Case 2 If **m is even and n is odd** in $\int \sin^m x \cos^n x \, dx$, we write n as $2k + 1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$$

We then combine the single $\cos x$ with dx and set $\cos x \, dx$ equal to $d(\sin x)$.

Case 3 If **both m and n are even** in $\int \sin^m x \cos^n x \, dx$, we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2} \qquad (2)$$

to reduce the integrand to one in lower powers of $\cos 2x$.

EXAMPLE 1 Evaluate

$$\int \sin^3 x \cos^2 x \, dx.$$

Solution This is an example of Case 1.

$$
\begin{aligned}
\int \sin^3 x \cos^2 x \, dx &= \int \sin^2 x \cos^2 x \sin x \, dx && \textit{m is odd.} \\
&= \int (1 - \cos^2 x) \cos^2 x \, (-d(\cos x)) && \textit{sin x dx} = -d(\cos x) \\
&= \int (1 - u^2)(u^2)(-du) && u = \cos x \\
&= \int (u^4 - u^2) \, du && \textit{Multiply terms.} \\
&= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C. && \blacksquare
\end{aligned}
$$

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EXAMPLE 2   Evaluate

$$\int \cos^5 x \, dx.$$

Solution   This is an example of Case 2, where $m = 0$ is even and $n = 5$ is odd.

$$\int \cos^5 x \, dx = \int \cos^4 x \cos x \, dx = \int (1 - \sin^2 x)^2 \, d(\sin x) \qquad \cos x \, dx = d(\sin x)$$

$$= \int (1 - u^2)^2 \, du \qquad\qquad u = \sin x$$

$$= \int (1 - 2u^2 + u^4) \, du \qquad\qquad \text{Square } 1 - u^2.$$

$$= u - \frac{2}{3} u^3 + \frac{1}{5} u^5 + C = \sin x - \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x + C. \qquad \blacksquare$$

EXAMPLE 3   Evaluate

$$\int \sin^2 x \cos^4 x \, dx.$$

Solution   This is an example of Case 3.

$$\int \sin^2 x \cos^4 x \, dx = \int \left(\frac{1 - \cos 2x}{2} \right) \left(\frac{1 + \cos 2x}{2} \right)^2 dx \qquad m \text{ and } n \text{ both even}$$

$$= \frac{1}{8} \int (1 - \cos 2x)(1 + 2 \cos 2x + \cos^2 2x) \, dx$$

$$= \frac{1}{8} \int (1 + \cos 2x - \cos^2 2x - \cos^3 2x) \, dx$$

$$= \frac{1}{8} \left[x + \frac{1}{2} \sin 2x - \int (\cos^2 2x + \cos^3 2x) \, dx \right].$$

For the term involving $\cos^2 2x$, we use

$$\int \cos^2 2x \, dx = \frac{1}{2} \int (1 + \cos 4x) \, dx$$

$$= \frac{1}{2} \left(x + \frac{1}{4} \sin 4x \right). \qquad \text{Omitting the constant of integration until the final result}$$

For the $\cos^3 2x$ term, we have

$$\int \cos^3 2x\, dx = \int (1 - \sin^2 2x) \cos 2x\, dx \qquad \begin{array}{l} u = \sin 2x, \\ du = 2\cos 2x\, dx \end{array}$$

$$= \frac{1}{2}\int (1 - u^2)\, du = \frac{1}{2}\left(\sin 2x - \frac{1}{3}\sin^3 2x\right). \qquad \begin{array}{l} \text{Again} \\ \text{omitting } C \end{array}$$

Combining everything and simplifying, we get

$$\int \sin^2 x \cos^4 x\, dx = \frac{1}{16}\left(x - \frac{1}{4}\sin 4x + \frac{1}{3}\sin^3 2x\right) + C. \qquad \blacksquare$$

Eliminating Square Roots

In the next example, we use the identity $\cos^2\theta = (1 + \cos 2\theta)/2$ to eliminate a square root.

EXAMPLE 4 Evaluate

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx.$$

Solution To eliminate the square root, we use the identity

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2} \qquad \text{or} \qquad 1 + \cos 2\theta = 2\cos^2\theta.$$

With $\theta = 2x$, this becomes

$$1 + \cos 4x = 2\cos^2 2x.$$

Therefore,

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx = \int_0^{\pi/4} \sqrt{2\cos^2 2x}\, dx = \int_0^{\pi/4} \sqrt{2}\sqrt{\cos^2 2x}\, dx$$

$$= \sqrt{2}\int_0^{\pi/4} |\cos 2x|\, dx = \sqrt{2}\int_0^{\pi/4} \cos 2x\, dx \qquad \begin{array}{l} \cos 2x \ge 0 \\ \text{on } [0,\, \pi/4] \end{array}$$

$$= \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/4} = \frac{\sqrt{2}}{2}[1 - 0] = \frac{\sqrt{2}}{2}. \qquad \blacksquare$$

Integrals of Powers of tan x and sec x

We use $\tan^2 x = \sec^2 x - 1$ and $\sec^2 x = \tan^2 x + 1$

EXAMPLE 5   Evaluate

$$\int \tan^4 x \, dx.$$

Solution

$$\int \tan^4 x \, dx = \int \tan^2 x \cdot \tan^2 x \, dx = \int \tan^2 x \cdot (\sec^2 x - 1) \, dx$$

$$= \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx$$

$$= \int \tan^2 x \sec^2 x \, dx - \int (\sec^2 x - 1) \, dx$$

$$= \int \tan^2 x \sec^2 x \, dx - \int \sec^2 x \, dx + \int dx.$$

In the first integral, we let

$$u = \tan x, \qquad du = \sec^2 x \, dx$$

and have

$$\int u^2 \, du = \frac{1}{3} u^3 + C_1.$$

The remaining integrals are standard forms, so

$$\int \tan^4 x \, dx = \frac{1}{3} \tan^3 x - \tan x + x + C.$$

EXAMPLE 6   Evaluate

$$\int \sec^3 x \, dx.$$

Solution   We integrate by parts using

$$u = \sec x, \qquad dv = \sec^2 x \, dx, \qquad v = \tan x, \qquad du = \sec x \tan x \, dx.$$

Then

$$\int \sec^3 x \, dx = \sec x \tan x - \int (\tan x)(\sec x \tan x \, dx)$$

$$= \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx \qquad \tan^2 x = \sec^2 x - 1$$

$$= \sec x \tan x + \int \sec x \, dx - \int \sec^3 x \, dx.$$

Combining the two secant-cubed integrals gives

$$2 \int \sec^3 x \, dx = \sec x \tan x + \int \sec x \, dx$$

and

$$\int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C.$$

Products of Sines and Cosines

The integrals

$$\int \sin mx \sin nx \, dx, \qquad \int \sin mx \cos nx \, dx, \qquad \text{and} \qquad \int \cos mx \cos nx \, dx$$

$$\sin mx \sin nx = \frac{1}{2} [\cos (m - n)x - \cos (m + n)x],$$

$$\sin mx \cos nx = \frac{1}{2} [\sin (m - n)x + \sin (m + n)x],$$

$$\cos mx \cos nx = \frac{1}{2} [\cos (m - n)x + \cos (m + n)x].$$

EXAMPLE 7　　Evaluate

$$\int \sin 3x \cos 5x \, dx.$$

Solution　　From Equation (4) with $m = 3$ and $n = 5$, we get

$$\int \sin 3x \cos 5x \, dx = \frac{1}{2} \int [\sin (-2x) + \sin 8x] \, dx$$

$$= \frac{1}{2} \int (\sin 8x - \sin 2x) \, dx$$

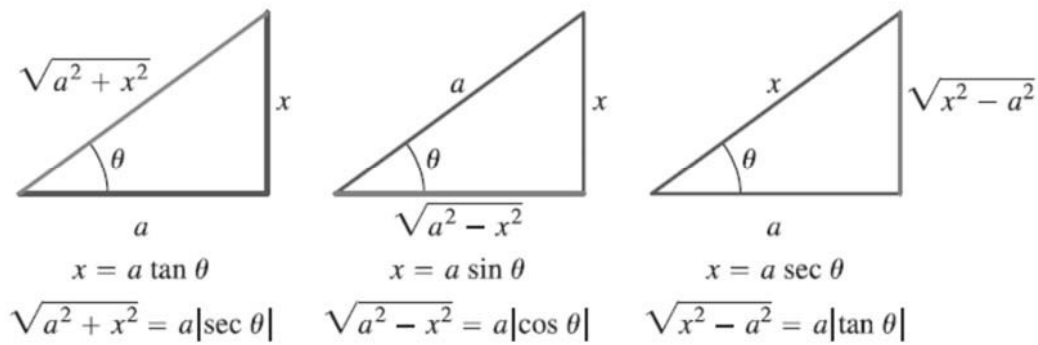$$= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C.$$

5.3　　Trigonometric Substitutions

Trigonometric substitutions occur when we replace the variable of integration by a trigonometric function. The most common substitutions are:

If $\sqrt{a^2 + x^2}$　then we use　$x = a \tan \theta$, 　$a^2 + x^2 = a^2 + a^2 \tan^2 \theta = a^2(1 + \tan^2 \theta) = a^2 \sec^2 \theta$.

If $\sqrt{a^2 - x^2}$,　then we use　$x = a \sin \theta$　$a^2 - x^2 = a^2 - a^2 \sin^2 \theta = a^2(1 - \sin^2 \theta) = a^2 \cos^2 \theta$

If $\sqrt{x^2 - a^2}$　then we use　$x = a \sec \theta$　$x^2 - a^2 = a^2 \sec^2 \theta - a^2 = a^2(\sec^2 \theta - 1) = a^2 \tan^2 \theta$

$$\sqrt{a^2 + x^2} \qquad x$$
$$a \qquad x$$
$$x \qquad \sqrt{x^2 - a^2}$$
$$\theta \qquad\qquad \theta \qquad\qquad \theta$$
$$a \qquad\qquad \sqrt{a^2 - x^2} \qquad\qquad a$$

$$x = a \tan \theta \qquad\qquad x = a \sin \theta \qquad\qquad x = a \sec \theta$$

$$\sqrt{a^2 + x^2} = a|\sec \theta| \qquad \sqrt{a^2 - x^2} = a|\cos \theta| \qquad \sqrt{x^2 - a^2} = a|\tan \theta|$$

Remark : In order to get θ we use the invers of trigonometric functions then we suppose that:

$$x = a \tan \theta, \quad \text{with} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$x = a \sin \theta \quad \text{with} \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},$$

$$x = a \sec \theta \quad \text{with} \quad \begin{cases} 0 \le \theta < \frac{\pi}{2} & \text{if } \frac{x}{a} \ge 1, \\[2mm] \frac{\pi}{2} < \theta \le \pi & \text{if } \frac{x}{a} \le -1. \end{cases}$$

EXAMPLE 1 ~~Evaluate~~

$$\int \frac{dx}{\sqrt{4 + x^2}}.$$

Solution We set

$$x = 2 \tan \theta, \qquad dx = 2 \sec^2 \theta \, d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$4 + x^2 = 4 + 4 \tan^2 \theta = 4(1 + \tan^2 \theta) = 4 \sec^2 \theta.$$

Then

$$\int \frac{dx}{\sqrt{4 + x^2}} = \int \frac{2 \sec^2 \theta \, d\theta}{\sqrt{4 \sec^2 \theta}} = \int \frac{\sec^2 \theta \, d\theta}{|\sec \theta|} \qquad \sqrt{\sec^2 \theta} = |\sec \theta|$$

$$= \int \sec \theta \, d\theta \qquad\qquad \sec \theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$= \ln |\sec \theta + \tan \theta| + C$$

$$= \ln \left| \frac{\sqrt{4 + x^2}}{2} + \frac{x}{2} \right| + C. \qquad\qquad \text{From Fig. 8.4}$$