4- $\int \csc x\, dx = \int \csc x \frac{(\csc x + \cot x)}{(\csc x + \cot x)} dx = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} dx$

$= \int \frac{-du}{u} = -\ln|u| + C = -\ln|\csc x + \cot x| + C$ $\quad$ $u = \csc x + \cot x$, $du = (-\csc x \cot x - \csc^2 x)\, dx$

Integrals of the tangent, cotangent, secant, and cosecant functions

$$\int \tan x\, dx = \ln|\sec x| + C \qquad \int \sec x\, dx = \ln|\sec x + \tan x| + C$$

$$\int \cot x\, dx = \ln|\sin x| + C \qquad \int \csc x\, dx = -\ln|\csc x + \cot x| + C$$

EXAMPLE:

$$\int_0^{\pi/6} \tan 2x\, dx = \int_0^{\pi/3} \tan u \cdot \frac{du}{2} = \frac{1}{2}\int_0^{\pi/3} \tan u\, du$$

$$= \frac{1}{2} \ln|\sec u| \Big]_0^{\pi/3} = \frac{1}{2}(\ln 2 - \ln 1) = \frac{1}{2}\ln 2$$

Substitute $u = 2x$, $dx = du/2$, $u(0) = 0$, $u(\pi/6) = \pi/3$

$$\int e^u\, du = e^u + C$$

EXAMPLE :

(a) $\int_0^{\ln 2} e^{3x}\, dx = \int_0^{\ln 8} e^u \cdot \frac{1}{3} du$ $\quad$ $u = 3x$, $\frac{1}{3}du = dx$, $u(0) = 0$, $u(\ln 2) = 3\ln 2 = \ln 2^3 = \ln 8$

$$= \frac{1}{3}\int_0^{\ln 8} e^u\, du$$

$$= \frac{1}{3} e^u \Big]_0^{\ln 8}$$

(b) $\int_0^{\pi/2} e^{\sin x} \cos x\, dx = e^{\sin x}\Big]_0^{\pi/2}$ $\quad$ Antiderivative from Example 2c

$$= e^1 - e^0 = e - 1$$

<u>The integral of a^u</u>

$$\int a^u\, du = \frac{a^u}{\ln a} + C.$$

EXAMPLE :

(a) $\frac{d}{dx}3^x = 3^x \ln 3$

(b) $\frac{d}{dx}3^{-x} = 3^{-x}(\ln 3)\frac{d}{dx}(-x) = -3^{-x}\ln 3$

(c) $\frac{d}{dx}3^{\sin x} = 3^{\sin x}(\ln 3)\frac{d}{dx}(\sin x) = 3^{\sin x}(\ln 3)\cos x$

(d) $\int 2^x\,dx = \frac{2^x}{\ln 2} + C$

(e) $\int 2^{\sin x}\cos x\,dx = \int 2^u\,du = \frac{2^u}{\ln 2} + C$

$$= \frac{2^{\sin x}}{\ln 2} + C$$

Example :

(a) $\frac{d}{dx}\log_{10}(3x+1) = \frac{1}{\ln 10}\cdot\frac{1}{3x+1}\frac{d}{dx}(3x+1) = \frac{3}{(\ln 10)(3x+1)}$

(b) $\int \frac{\log_2 x}{x}\,dx = \frac{1}{\ln 2}\int \frac{\ln x}{x}\,dx$ $\quad \log_2 x = \frac{\ln x}{\ln 2}$

$$= \frac{1}{\ln 2}\int u\,du \qquad u = \ln x,\quad du = \frac{1}{x}dx$$

$$= \frac{1}{\ln 2}\frac{u^2}{2} + C = \frac{1}{\ln 2}\frac{(\ln x)^2}{2} + C = \frac{(\ln x)^2}{2\ln 2} + C$$

<u>**Integration Formulas**</u>

1. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C \qquad$ (Valid for $u^2 < a^2$)

2. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C \qquad$ (Valid for all u)

3. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C \qquad$ (Valid for $|u| > a > 0$)

EXAMPLE

(a) $\int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x\Big]_{\sqrt{2}/2}^{\sqrt{3}/2} = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$

(b) $\int \frac{dx}{\sqrt{3 - 4x^2}} = \frac{1}{2}\int \frac{du}{\sqrt{a^2 - u^2}}$

$$= \frac{1}{2}\sin^{-1}\left(\frac{u}{a}\right) + C$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C$$

(c) $$\int \frac{dx}{\sqrt{e^{2x}-6}} = \int \frac{du/u}{\sqrt{u^2-a^2}}$$

$$= \int \frac{du}{u\sqrt{u^2-a^2}}$$

$$= \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$$

$$= \frac{1}{\sqrt{6}}\sec^{-1}\left(\frac{e^x}{\sqrt{6}}\right) + C$$

Example :

$$\int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x\,dx}{e^{2x}+1} \qquad \text{Multiply by } (e^x/e^x) = 1.$$

$$= \int \frac{du}{u^2+1} \qquad \text{Let } u = e^x,\ u^2 = e^{2x},\ du = e^x\,dx.$$

$$= \tan^{-1}u + C \qquad \text{Integrate with respect to } u.$$

$$= \tan^{-1}(e^x) + C \qquad \text{Replace } u \text{ by } e^x.$$

Example

(a) $\int \frac{dx}{\sqrt{4x-x^2}}$ **(b)** $\int \frac{dx}{4x^2+4x+2}$

Solution

(a) we first rewrite $4x - x^2$ by completing the square:

$$4x - x^2 = -(x^2-4x) = -(x^2-4x+4)+4 = 4-(x-2)^2.$$

$$\int \frac{dx}{\sqrt{4x-x^2}} = \int \frac{dx}{\sqrt{4-(x-2)^2}}$$

$$= \int \frac{du}{\sqrt{a^2-u^2}} \qquad = \sin^{-1}\left(\frac{u}{a}\right) + C$$

$$= \sin^{-1}\left(\frac{x-2}{2}\right) + C$$

(b) We complete the square on the binomial $4x^2 + 4x$:

$$4x^2+4x+2 = 4(x^2+x)+2 = 4\left(x^2+x+\frac{1}{4}\right)+2-\frac{4}{4}$$

$$= 4\left(x+\frac{1}{2}\right)^2 + 1 = (2x+1)^2+1.$$

T

$$\int \frac{dx}{4x^2+4x+2} = \int \frac{dx}{(2x+1)^2+1} = \frac{1}{2}\int \frac{du}{u^2+a^2}$$

$$= \frac{1}{2}\cdot\frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

$$= \frac{1}{2}\tan^{-1}(2x+1) + C$$

.4 HYPERBOLIC FUNCTIONS

The hyperbolic sine and hyperbolic cosine functions are defined by:

Hyperbolic sine:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Hyperbolic tangent:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cotangent:

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Hyperbolic secant:

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

Hyperbolic cosecant:

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$

$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

$$\tanh^2 x = 1 - \operatorname{sech}^2 x \qquad \left(\frac{e^x + e^{-x}}{2}\right)$$

$$\coth^2 x = 1 + \operatorname{csch}^2 x$$

$$= \frac{e^{2x} - e^{-2x}}{2}$$

$$= \sinh 2x.$$

<u>Derivatives and Integrals of Hyperbolic Functions</u>

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$

$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$

$$\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$$

proof :

1- $$\frac{d}{dx}(\sinh u) = \frac{d}{dx}\left(\frac{e^u - e^{-u}}{2}\right)$$

$$= \frac{e^u\, du/dx + e^{-u}\, du/dx}{2}$$

$$= \cosh u \frac{du}{dx}$$

2- $$\frac{d}{dx}(\operatorname{csch} u) = \frac{d}{dx}\left(\frac{1}{\sinh u}\right)$$
$$= -\frac{\cosh u}{\sinh^2 u}\frac{du}{dx}$$
$$= -\frac{1}{\sinh u}\frac{\cosh u}{\sinh u}\frac{du}{dx}$$
$$= -\operatorname{csch} u \coth u \frac{du}{dx}$$

Integrals

$$\int \sinh u\, du = \cosh u + C$$
$$\int \cosh u\, du = \sinh u + C$$
$$\int \operatorname{sech}^2 u\, du = \tanh u + C$$
$$\int \operatorname{csch}^2 u\, du = -\coth u + C$$
$$\int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$$
$$\int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C$$

Example

(a) $$\frac{d}{dt}\left(\tanh\sqrt{1+t^2}\right) = \operatorname{sech}^2\sqrt{1+t^2}\cdot\frac{d}{dt}\left(\sqrt{1+t^2}\right)$$
$$= \frac{t}{\sqrt{1+t^2}}\operatorname{sech}^2\sqrt{1+t^2}$$

(b) $$\int \coth 5x\, dx = \int \frac{\cosh 5x}{\sinh 5x}dx = \frac{1}{5}\int\frac{du}{u}$$
$u = \sinh 5x$, $du = 5\cosh 5x\, dx$
$$= \frac{1}{5}\ln|u| + C = \frac{1}{5}\ln|\sinh 5x| + C$$

(c) $$\int_0^1 \sinh^2 x\, dx = \int_0^1 \frac{\cosh 2x - 1}{2}dx$$
Table 7.6
$$= \frac{1}{2}\int_0^1 (\cosh 2x - 1)\, dx = \frac{1}{2}\left[\frac{\sinh 2x}{2} - x\right]_0^1$$
$$= \frac{\sinh 2}{4} - \frac{1}{2} \approx 0.40672$$
Evaluate with a calculator.

(d) $$\int_0^{\ln 2} 4e^x \sinh x\, dx = \int_0^{\ln 2} 4e^x\,\frac{e^x - e^{-x}}{2}dx = \int_0^{\ln 2}(2e^{2x} - 2)\, dx$$
$$= \left[e^{2x} - 2x\right]_0^{\ln 2} = (e^{2\ln 2} - 2\ln 2) - (1 - 0)$$
$$= 4 - 2\ln 2 - 1 \approx 1.6137$$
■

Inverse Hyperbolic Functions

Derevatives

$$\frac{d(\sinh^{-1} u)}{dx} = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$$

$$\frac{d(\cosh^{-1} u)}{dx} = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}, \qquad u > 1$$

$$\frac{d(\tanh^{-1} u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx}, \qquad |u| < 1$$

$$\frac{d(\coth^{-1} u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx}, \qquad |u| > 1$$

$$\frac{d(\operatorname{sech}^{-1} u)}{dx} = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}, \quad 0 < u < 1$$

$$\frac{d(\operatorname{csch}^{-1} u)}{dx} = -\frac{1}{|u|\sqrt{1+u^2}}\frac{du}{dx}, \quad u \neq 0$$

Integrals

1. $$\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C, \qquad a > 0$$

2. $$\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C, \qquad u > a > 0$$

3. $$\int \frac{du}{a^2-u^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) + C, & u^2 < a^2 \\ \frac{1}{a}\coth^{-1}\left(\frac{u}{a}\right) + C, & u^2 > a^2 \end{cases}$$

4. $$\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + C, \qquad 0 < u < a$$

5. $$\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right| + C, \qquad u \neq 0 \text{ and } a > 0$$

EXAMPLE : find the derivative of y

a) $y = \cosh^{-1} 2\sqrt{x+1}$ b) $y = \operatorname{csch}^{-1}\left(\frac{1}{2}\right)^{\theta}$ c) $y = \sinh^{-1}(\tan x)$

sol:

a) $y = \cosh^{-1} 2\sqrt{x+1} = \cosh^{-1}\left(2(x+1)^{1/2}\right) \Rightarrow \frac{dy}{dx} = \frac{(2)\left(\frac{1}{2}\right)(x+1)^{-1/2}}{\sqrt{[2(x+1)^{1/2}]^2 - 1}} = \frac{1}{\sqrt{x+1}\sqrt{4x+3}} = \frac{1}{\sqrt{4x^2+7x+3}}$

b) $y = \operatorname{csch}^{-1}\left(\frac{1}{2}\right)^{\theta} \Rightarrow \frac{dy}{d\theta} = -\frac{\left[\ln\left(\frac{1}{2}\right)\right]\left(\frac{1}{2}\right)^{\theta}}{\left(\frac{1}{2}\right)^{\theta}\sqrt{1+\left[\left(\frac{1}{2}\right)^{\theta}\right]^2}} = -\frac{\ln(1) - \ln(2)}{\sqrt{1+\left(\frac{1}{2}\right)^{2\theta}}} = \frac{\ln 2}{\sqrt{1+\left(\frac{1}{2}\right)^{2\theta}}}$

c) $y = \sinh^{-1}(\tan x) \Rightarrow \frac{dy}{dx} = \frac{\sec^2 x}{\sqrt{1+(\tan x)^2}} = \frac{\sec^2 x}{\sqrt{\sec^2 x}} = \frac{\sec^2 x}{|\sec x|} = \frac{|\sec x|\,|\sec x|}{|\sec x|} = |\sec x|$

EXAMPLE : Evaluate

a) $\int_0^1 \frac{2\,dx}{\sqrt{3+4x^2}}$. b) $\int_{1/5}^{3/13} \frac{dx}{x\sqrt{1-16x^2}}$ c) $\int_1^e \frac{dx}{x\sqrt{1+(\ln x)^2}}$

Sol: