⑧ $z\bar{z} = (x+iy)(x-iy)$

$= x^2 \cancel{- ixy} \cancel{+ ixy} - i^2 y^2$

$= x^2 + y^2 = |z|^2$

Example: Find $\dfrac{1}{3-i4} + \dfrac{3-i2}{6+i8}$

Sol: $\left(\dfrac{1}{3-i4} \times \dfrac{3+i4}{3+i4}\right) + \left(\dfrac{3-i2}{6+i8} \times \dfrac{6-i8}{6-i8}\right)$

$= \left(\dfrac{3+i4}{9+16}\right) + \left(\dfrac{2-i36}{36+64}\right)$

$= \left(\dfrac{3+i4}{25}\right) + \left(\dfrac{2-i36}{100}\right)$

$= \dfrac{7}{50} - i\,\dfrac{10}{50}$.

~~**Example:** Show that the hyperbola~~
~~$y^2 - x^2 = 1$ can be written as~~
~~$z^2 + \bar{z}^2 = 2$, where $\bar{z}\bar{z}^2$~~
~~where $z = x+iy$.~~

Example: show that the Hyperbola $y^2-x^2=1$ can be written as $z^2+\bar{z}^2=-2$ where $z=x+iy$.

sol: $x=\mathrm{Re}(z)=\dfrac{z+\bar{z}}{2}$

$y=\mathrm{Im}(z)=\dfrac{z-\bar{z}}{i2}$

$$y^2-x^2=1 \Rightarrow \left(\frac{z-\bar{z}}{i2}\right)^2-\left(\frac{z+\bar{z}}{2}\right)^2=1$$

$$\Rightarrow \frac{(z-\bar{z})^2}{-4}-\frac{(z+\bar{z})^2}{4}=1$$

$$=\frac{-(z-\bar{z})^2}{4}-\frac{(z+\bar{z})^2}{4}=1$$

$$=\frac{-(z^2-2z\bar{z}+\bar{z}^2)-(z^2+2z\bar{z}+\bar{z}^2)}{4}=1$$

$$=\frac{-z^2+\cancel{2z\bar{z}}-\bar{z}^2-z^2-\cancel{2z\bar{z}}-\bar{z}^2}{4}=1$$

$$=\frac{-2z^2-2\bar{z}^2}{4}=-\frac{1}{2}z^2-\frac{1}{2}\bar{z}^2=1$$

$$\Longrightarrow \boxed{z^2+\bar{z}^2=-2}$$

4 Tringle Inequality

الموضوع : التاريخ

propositian: (Tringle inequality)

let z_1 & z_2 be complex numbers, then

① $|z_1 + z_2| \leq |z_1| + |z_2|$

② $|z_1 + z_2| \geq \big| |z_1| - |z_2| \big| \geq |z_1| - |z_2|$

proof ①:

$$|z_1+z_2|^2 = (z_1+z_2)\overline{(z_1+z_2)} = (z_1+z_2)(\bar{z}_1+\bar{z}_2)$$

$$= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2$$

$$= |z_1|^2 + z_1\bar{z}_2 + z_2\bar{z}_1 + |z_2|^2$$

$$= |z_1|^2 + z_1\bar{z}_2 + \overline{z_1\bar{z}_2} + |z_2|^2$$

$$= |z_1|^2 + 2\,Re(z_1\bar{z}_2) + |z_2|^2$$

where $w + \bar{w} = 2Re(w)$, w is complex

$$\leq |z_1|^2 + 2|z_1\bar{z}_2| + |z_2|^2$$

$$= |z_1|^2 + 2|z_1||\bar{z}_2| + |z_2|^2$$

$$= |z_1|^2 + 2|z_1||z_2| + |z_2|^2$$

$\Longrightarrow$

$$|z_1+z_2|^2 \le (|z_1|+|z_2|)^2 \quad \text{(بالجذر للطرفين)}$$

$$\Rightarrow |z_1+z_2| \le |z_1| + |z_2| \quad \blacksquare$$

<u>proof ②:</u>

$$|z_1| = |z_1+z_2-z_2|$$

$$\le |z_1+z_2| + |-z_2|$$

$$\Rightarrow |z_1| - |z_2| \le |z_1+z_2| \quad \text{— ①}$$

In the Same way, we show that

$$|z_1| - |z_2| \ge -|z_1+z_2| \quad \text{— ②}$$

$\therefore$ from ① & ② we get

$$\big||z_1| - |z_2|\big| \le |z_1+z_2|$$

or $$|z_1+z_2| \ge \big||z_1| - |z_2|\big| .$$

$||z_1|-|z_2|| \le |z_1 \mp z_2| \le |z_1|+|z_2|$

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Example: show that

$$|z_1 - z_2| \geq ||z_1| - |z_2||$$

Sol: $|z_1| = |z_1 - z_2 + z_2| \le |z_1 - z_2| + |z_2|$

$\Rightarrow |z_1| - |z_2| \le |z_1 - z_2| \rightarrow -|z_1 - z_2| \le |z_1| - |z_2|$

$\cancel{|z_1| - |z_2| \geq ||z_1| - |z_2||}$

$\Rightarrow -|z_1 - z_2| \le |z_1| - |z_2| \le |z_1| - |z_2|$

$\Rightarrow ||z_1| - |z_2|| \le |z_1 - z_2|$

$$|z_1 - z_2| = |z_1 + (-z_2)|$$

$$\geq ||z_1| - |z_2||.$$

Example: If z lies on the circle $|z| = 2$, then show that

$$3 \le |z^2 - 1| \le 5.$$

Sol: $|z^2 - 1| \le |z^2| + |-1|$

$= |z|^2 + 1 = 4 + 1 = 5$

$|z^2 - 1| \geq ||z|^2 - |1|| \geq |4 - 1| = 3$

$\therefore 3 \le |z^2 - 1| \le 5.$

$|z_1| - |z_2| \le |z_1 - z_2| \le |z_1 + z_2| \le |z_1| + |z_2|$

Example: Use the inequality

$$||z_1|-|z_2|| \le |z_1 - z_2|$$

show that if z lies on the circle $|z|=3$ then

$$|z^2-3z+2| \ge 2$$

Sol: $|z^2-3z+2| = |(z-2)(z-1)|$

$= |z-2|\,|z-1|$

$|z-2| \ge ||z|-2| = 1$

$|z-1| \ge ||z|-1| = 2$

$\because |z^2-3z+2| = |z-2|\,|z-1|$

$\ge 2 \cdot 1 = 2$

$\therefore |z^2-3z+2| \ge 2$ ■

H. W. ~~page~~ P. 13