Example: Find the 3-rd roots of $z_\circ = 1$

Sol:

$$|z_\circ| = r_\circ = 1 \Longrightarrow \theta_\circ = Arg(z_\circ) = 0 \Longrightarrow z_\circ = e^{i0} = 1$$

The third root of $z_\circ = 1$ are

$$C_k = \sqrt[3]{r_\circ} e^{i\left(\frac{0+2k\pi}{3}\right)}, k = 0,1,2$$

$$C_k = \sqrt[3]{1} e^{i\left(\frac{2k\pi}{3}\right)}, k = 0,1,2$$

$$C_0 = e^{i(0)} = 1, C_1 = e^{i\left(\frac{2\pi}{3}\right)} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, C_2 = e^{i\left(\frac{4\pi}{3}\right)} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

Remark:

1) For any complex number $z_\circ \neq 0, z_\circ^{\frac{1}{n}} = \{z: z^n = z_\circ\}$.
2) If $r_\circ$ is the positive real number, then $r_\circ^{\frac{1}{n}}$ denotes the set of all $n-th$ roots of $r_\circ, r_\circ^{\frac{1}{n}} = \{z: z^n = r_\circ\}$ but $\sqrt[n]{r_\circ}$ denote the unique positive real $n-th$ root of $r_\circ$.
3) **If** $\theta_\circ = Arg(z_\circ)$ then $C_\circ = \sqrt{r_\circ} e^{i\left(\frac{\theta_\circ}{n}\right)}$ is called the principle root of $z_\circ$.

<u>The $n-th$ roots of unity:</u>

The $n-th$ roots of 1 are given by

$$C_k = e^{i\left(\frac{2k\pi}{n}\right)}, k = 0,1,\dots,n-1$$

Let $w_n = C_1 = e^{i\left(\frac{2\pi}{n}\right)}$ then $C_2 = e^{i\left(\frac{4\pi}{n}\right)} = w_n^2, \dots, C_{n-1} = e^{i\left(\frac{2(n-1)\pi}{n}\right)} = w_n^{n-1}$

Therefore, the $n-th$ roots of unity are $1, w_n, w_n^2, \dots, w_n^{n-1}$, where $w_n = e^{i\left(\frac{2\pi}{n}\right)}$

$$z^n = 1 \Longrightarrow (1)^{\frac{1}{n}} = z \Longrightarrow z_\circ = 1, C_k = \sqrt[n]{|z_n|} e^{i\left(\frac{Arg(z_\circ)+2k\pi}{n}\right)}, k = 0,1,\dots,n-1$$

$$|z_\circ| = 1, Arg(z_\circ) = 0, C_k = e^{i\left(\frac{2k\pi}{n}\right)}$$

$$C_0 = 1, C_1 = e^{i\left(\frac{2\pi}{n}\right)} = w_n, C_2 = e^{i\left(\frac{4\pi}{n}\right)} = w_n^2, C_3 = e^{i\left(\frac{6\pi}{n}\right)} = w_n^3, \dots, C_{n-1} = e^{i\left(\frac{2(n-1)\pi}{n}\right)} = w_n^{n-1}$$

Therefore the $n-th$ roots of units are $1, w_n, w_n^2, \dots, w_n^{n-1}$, where $w_n = e^{i\left(\frac{2\pi}{n}\right)}$.

The $n-th$ roots of $z_\circ$:

Let $\theta_\circ = Arg(z_\circ)$ then the $n-th$ roots of $z_\circ = r_\circ e^{i\theta_\circ}$ are given by:

$$C_k = \sqrt[n]{r_\circ} e^{i\left(\frac{\theta_\circ + 2k\pi}{n}\right)}, k = 0,1,\dots,n-1$$

They can be written as:

$$C_k = \sqrt[n]{r_\circ} e^{i\left(\frac{\theta_\circ}{n}\right)} e^{i\left(\frac{2k}{n}\right)} = C_\circ w_n^k, k = 0,1,\dots,n-1$$

Note:

$$z_\circ \neq 0, C_k = \sqrt[n]{|z_\circ|} e^{i\left(\frac{Arg(z_\circ)+2k\pi}{n}\right)}, k = 0,1,\dots,n-1$$

$$\Longrightarrow C_k = \sqrt[n]{|z_\circ|} e^{i\left(\frac{\theta_\circ}{n}\right)} e^{i\left(\frac{2k\pi}{n}\right)} = C_\circ w_n^k, k = 0,1,\dots,n-1$$

where $C_\circ = \sqrt[n]{r_\circ} e^{i\left(\frac{\theta_\circ}{n}\right)}$ is the principle root of $z_\circ$ and $w_n = e^{i\left(\frac{2\pi}{n}\right)}$.

Example: Find the 2-nd roots of i

Sol:

$$C_k = C_\circ w_n^k, k = 0,1 \Longrightarrow C_\circ = \sqrt{|i|} e^{i\left(\frac{Arg(i)}{2}\right)}, w_2 = e^{i\left(\frac{2\pi}{2}\right)} \Longrightarrow$$

$$C_\circ = \sqrt{|i|} e^{i\left(\frac{\frac{\pi}{2}}{2}\right)} = e^{i\left(\frac{\pi}{4}\right)} = \frac{1+i}{\sqrt{2}} \ \& w_2 = e^{i(\pi)} = -1$$

$\Longrightarrow$ The roots of $z^2 = i$ are:

$$C_\circ = \frac{1+i}{\sqrt{2}}, C_1 = C_\circ w_2 = -C_\circ = \frac{-(1+i)}{\sqrt{2}}$$

Example: Find all values of $\left(2 + i2\sqrt{3}\right)^{\frac{1}{4}}$

Sol:

$z_\circ = 2 + i2\sqrt{3}$, the root are $C_k = C_\circ w_n^k, k = 0,1,2,3$

$$\Longrightarrow C_\circ = \sqrt[4]{|z_\circ|} e^{i\left(\frac{Arg(z_\circ)}{4}\right)} = \sqrt[4]{4} e^{i\left(\frac{\frac{\pi}{3}}{4}\right)} = \sqrt[2]{2} e^{i\left(\frac{\pi}{12}\right)} \& \ w_4^k = ?$$

$\Longrightarrow$ The roots are:

$$C_\circ, iC_\circ, -C_\circ, -iC_\circ$$

Example: Find all values of $\left(-4\sqrt{3}+i4\right)^{\frac{1}{3}}$

Sol:

$z_\circ = -4\sqrt{3}+i4$, the root are $C_k = C_\circ w_n^k, k = 0,1,2$

$$C_\circ = \sqrt[3]{r_\circ}\, e^{i\left(\frac{Arg(z_\circ)}{3}\right)} \Longrightarrow r_\circ = |z_\circ| = \sqrt{48+16} = \sqrt{64} = 8 \Longrightarrow Arg(z_\circ) = \frac{5\pi}{6}$$

$$\Longrightarrow C_\circ = \sqrt[3]{8}\, e^{i\left(\frac{5\pi}{18}\right)} = 2e^{i\left(\frac{5\pi}{18}\right)} \;\&\; w_3 = e^{i\left(\frac{2\pi}{3}\right)} = \frac{-1+i\sqrt{3}}{2}$$

$\Longrightarrow$ The roots are:

$$C_\circ, C_1 = C_\circ w_3, C_2 = C_\circ\, w_3^2$$

Note: If $z^n = z_\circ \,\&\, n < 0 \Longrightarrow z_\circ^{\frac{1}{n}} = \left(\frac{1}{z_\circ}\right)^{-\frac{1}{n}}$

Example: $(2)^{-\frac{1}{2}} = \left(\frac{1}{2}\right)^{\frac{1}{2}}$

Exercise: H.W

8- Regions in complex plane

In this section, we will study sets of points in complex $z - plane$.

Definition:

1) An $\epsilon-$ neighborhood of a point $z_\circ$ is the set

$$N_\epsilon(z_\circ) = \{z \in \mathbb{C}: |z - z_\circ| < \epsilon\}$$

consisting of all points lying inside the circle $|z - z_\circ| < \epsilon$

2) A deleted $\epsilon-$neighborhood of a point $z_\circ$ is an $\epsilon-$neighborhood of $z_\circ$ except $z_\circ$ itself is an $\epsilon-$neighborhood of $z_\circ$ itself. That is, it is the set

$$\{z \in \mathbb{C}: 0 < |z - z_\circ| < \epsilon\}$$

3) A point $z_\circ$ is said to be interior point of a set S of complex numbers if there is an $\epsilon-$neighborhood $N_\epsilon(z_\circ)$ of $z_\circ$ such that $N_\epsilon(z_\circ) \subset S$.

4) A point $z_\circ$ is an exterior point of a set S of complex numbers if there is an ε −neighborhood $N_\epsilon(z_\circ)$ of $z_\circ$ such that $N_\epsilon(z_\circ) \cap S = \emptyset$.
5) A point $z_\circ$ is a boundary point of a set S if it is neither an interior point nor an exterior point.
6) The set of all boundary points of a set S is called the boundary of S.

Example: Let $S = \{z \in \mathbb{C}: |z| < 1\}$ then the boundary of S is the set $S = \{z \in \mathbb{C}: |z| = 1\}$, $z_1 = \frac{1+i}{2}$ is an interior point of S but $z_2 = 1 + i$ is an exterior point.

Definition: Let S be a set of complex numbers

1) S is open if it contains none of the boundary points
2) S is closed if it its boundary points
3) The closure of S is the closed set consisting of all points in S together with the boundary of S.

Example: Let $S = \{z \in \mathbb{C}: Im\, z < 1\}$ then S is open. The closure of S is the set

$\{z \in \mathbb{C}: Im\, z \leq 1\}$.

Definitions:

1) An open set is said to be connected if each pair of points z_1 & z_2 in S can be joined by a polygonal line, consisting of a finite number of line segments jointed end to end, that lies entirely in S.
2) A set S is called a domain if it is open and connected.
3) A region is a domain together with some, none, or all of its boundary.

Example: Sketch $S = \{z \in \mathbb{C}: |z + 3| > 2\}$ and determine whether it is a domain or not.

Definitions:

1) A set S is said to be bounded if there is a real number $R > 0$ such that $|z| < R$, $\forall\, z \in S$.
2) A point $z_\circ$ is called an accumulation point of a set S if each deleted neighborhood $z_\circ$ contains at least one point of S. i.e. $N_\epsilon(z_\circ)/\{z_\circ\} \cap S \neq \emptyset$

Example: Let $S = \{\frac{i^n}{n} : n = 1,2, \dots\}$, find the accumulation points of S.

ملاحظة: اي جوار يجب ان ينجح بمعنى ، هل ان الجوار ينفصل عن المجموعة ام لا اذا تمكنا من فصل الجوار فان المجموعة لا تمتلك نقطة تجمع واذا لم نتمكن من فصله هذا يعني ان المجموعة تمتلك نقطة تجمع.

$$\left|\frac{i}{n}\right| = \lim_{n \to \infty} \frac{1}{n} = 0$$

الصفر نقطة التجمع الوحيدة للمثال .

Exercise: H.W