Chapter Two: Analytic Functions

In this chapter, we consider function of a complex variable and study continuity and differentiability of such functions, the main goal of this chapter is to introduce analytic functions.

9- Function of a complex variable

Definitions: Let S be a set of complex numbers. A function f defined on S is a value that assigns to each $z \in S$ a single complex number w. In such case we write $w = f(z)$.

Remark:

a. If f is a function defined on S, then S is the domain of f.
b. If the domain is not given explicitly, then we agree that it's the largest possible set.

Example(1): Let f be given by $f(z) = \frac{1}{z-1}$, then f is a function of complex variables with domain $S = \mathbb{C}/\{1\}$.

Example(2): Let f be given by $f(z) = \frac{1}{z^2+1}$, then domain of f is $S = \mathbb{C}/\{i, -i\}$.

10- Real and Imaginary parts of function

Let $w = f(z), z = x + iy$ be a function and let $w = u + iv$ then we have $u + iv = f(z + iy)$ both of the real numbers u and v depends on x and y thus $f(z) = u(x, y) + iv(x, y)$. Similarly, if $z = re^{i\theta}$ then $f(z) = u(r, \theta) + iv(r, \theta)$.

Example(1): Write $f(z) = z^2 + 2z$ in the forms $f(z) = u(x, y) + iv(x, y)$ and $f(z) = u(r, \theta) + iv(r, \theta)$

Solution: $f(z) = \left(\frac{x^2-y^2+2xy}{u(x,y)}\right) + i\left(\frac{2xy+y}{v(x,y)}\right)$ and

$$f(z) = \left(re^{i\theta}\right)^2 + 2\left(re^{i\theta}\right) = r^2e^{i2\theta} + 2re^{i\theta}$$

$$f(z) = \left(\frac{r^2 \cos 2\theta + 2r\, cos\theta}{u(r, \theta)}\right) + i\left(\frac{r^2 \sin 2\theta + 2r\, sin\theta}{v(r, \theta)}\right)$$

Remark: If $v(x,y)=0$ for all z then $f(z)=u(x,y)$ is called a real valued function of a complex variable and the following example.

Example(1): $f(z)=|z|=\sqrt{x^2+y^2}$ is a real valued function of a complex variable.

11- Polynomial and rational functions

a. If n is non-negative integer, and if $a_\circ, a_1, \dots, a_n$ are complex number such that $a_n \neq 0$ then the function $f(z)=a_\circ + a_1 z + \cdots + a_n z^n$ is a polynomial of degree n. The domain of a polynomial function is the entire $z-plane$.

b. If $p(z)$ and $Q(z)$ are polynomials, then $f(z)=\frac{p(z)}{Q(z)}$ is a rational function, the domain of a rational function $f(z)=\frac{p(z)}{Q(z)}$ is the $\{z \in \mathbb{C}: Q(z)\neq 0\}$.

Examples:

a. $f(z)=iz^4-(1+i)z^3+2z-5-i3$ is a polynomial of degree n, whose domain is $\mathbb{C}$.

b. $f(z)=\frac{2z^2+i3z+4}{z^3-(1+i)z^2+iz}$ is a rational function whose domain is $\mathbb{C}/\{0,1,i\}$.

c. Let $f(z)=x^2+y^2+i2(x+y)$, use $x=\frac{z+\bar{z}}{2}, y=\frac{z-\bar{z}}{i2}$ to write $f(z)$ in terms of z.

Solution: $f(z)=\left(\frac{z+\bar{z}}{2}\right)^2+\left(\frac{z-\bar{z}}{i2}\right)^2+i2\left(\left(\frac{z+\bar{z}}{2}\right)+\left(\frac{z-\bar{z}}{i2}\right)\right)$

$$=\frac{1}{4}(z^2+2z\bar{z}+z^2)-\frac{1}{4}(z^2+2z\bar{z}+z^2)+i(z+\bar{z})+z-\bar{z}$$

$$=z\bar{z}+z-\bar{z}+i(z+\bar{z})$$

$$=|z|^2+(1+i)z-(1-i)\bar{z}$$

12- Multivalued function

A multivalued function is a rule that assigns more than value to a point z in the domain of definition.

Example(1): $f(z) = z^2 \Longrightarrow w = z^2$ (لايجاد المعكوس)$\Longrightarrow w^2 = z \Rightarrow w = z^{\frac{1}{2}} \Longrightarrow$

$$z = re^{i\theta} = \begin{cases} C_\circ \\ C_\circ w_2 \end{cases} = \pm C_\circ \pm \sqrt{r}e^{i\frac{\theta}{2}}$$

ملاحظات:

- هذا النوع من الدوال يسمى (multivalued function)اذا اخذنا واحدة من القيم و المطلوب تكون الدالة (continuous)بمعنى نختار فرع من الفروع و تسمى (Branch)
- المعكوس لمتعددة القيم حتى يكون دالة ناخذ احد الحلول.

Remark:

a. Multivalued function occurs in the theory of functions of a complex variables just as. They do in the case of real variables. They are naturally introduced as the inverse of single-valued function.
b. When, we study multivalued functions, we choose only one of the possible values in a systematic manner, and we defined a continuous single valued function.

Example(2): Let $z = re^{i\theta}, z \neq 0$ then $z^{\frac{1}{2}} = \pm\sqrt{r}e^{i\frac{\theta}{2}}$ where
$-\pi < \theta = Arg(z) \leq \pi$ $(does\ not\ continouse)$
If we choose only one of these two values, say

$$z^{\frac{1}{2}} = \sqrt{r}e^{i\frac{\theta}{2}} - \pi < \theta = Arg(z) < \pi$$

ملاحظة: حتى تكون الدالة مستمرة تم حذف المساواة من الπ

Then f is a single valued function, whose domain is the entire complex plane except for the ray $\theta = \pi$((the negative real axis)).

13- Limits

Definition (Limit of a function): Let f is a function defined in some deleted neighborhood of a point $z_\circ \in \mathbb{C}$. We say that the limit of $f(z)$ as z approaches $z_\circ$ is a number L or $\lim_{z\to z_\circ} f(z) = L$.

If for each ϵ there is $\delta > 0$ such that $|f(z) - L| < \epsilon$ whenever $0 < |z - z_\circ| < \delta$.

Example(1): Prove that $\lim_{z\to z_\circ} z = z_\circ$

Solution: Let $f(z) = z$, $L = z_\circ$Given $\epsilon > 0$ and let $\delta = \epsilon$ then

$0 < |z - z_\circ| < \delta \Rightarrow |f(z) - L| = |z - z_\circ| < \delta = \epsilon$

Example(2): Show that $\lim_{z\to i}(i3z + 2) = -1$

Solution: $f(z) = i3z + 2, z_\circ = i, L = -1$

Let $\epsilon > 0$ be given and let $\delta = \frac{\epsilon}{3}$

Now$0 < |z - z_\circ| < \delta \Rightarrow 3|i(z - i + i) + 1| \Rightarrow |f(z) - L| = |i3z + 2 + 1| = 3|iz + 1| = 3|i(z - i)| = 3|i||z - i| = 3|z - i| < 3\delta = \epsilon$.

Remark:

a. The definition of limit requires f to be defined in some deleted of a point$z_\circ$. Such a deleted neighborhood always exists when $z_\circ$ is an interior point of the domain of f.
b. We can extend definition of f limit to the case in which $z_\circ$ is a boundary point of the domain of f.

In this case we require $|f(z) - L| < \epsilon$ only for z in both the domain of f and $0 < |z - z_\circ| < \delta$

c. If $\lim_{z\to z_\circ} z = L$ Exists, then its unique.
d. The symbol $z \to z_\circ$ means that z is allowed to approach $z_\circ$ in arbitrary manner, thus if the limit approach different values from different directions, then the limit does not exist.

ملاحظات

1. الغاية في الحقيقي كان فيها يمين و يسار
2. الغاية في العقدي يوجد لدينا جوار لذلك يوجد بها عدة اتجاهات
3. ممكن استخدام صيغة الاتجاهين في الحقيقي لاثبات ان الغاية غير موجودة في العقدي بمعنى ناخذ اي اتجاهين و نثبت لهم غايات مختلفة بالتالي الغاية غير موجودة.

Example(3): Show that $\lim_{z\to 0}\frac{\bar{z}}{2z}$ does not exist

Solution:

If $z = x$ then $\lim_{z\to 0}\frac{\bar{z}}{2z} = \lim_{x\to 0}\frac{x}{2x} = \frac{1}{2}$ … … … … … … … … … . . . (1)

If $z = iy$ then $\lim_{z\to 0}\frac{\bar{z}}{2z} = \lim_{iy\to 0}\frac{-i}{-2iy} = -\frac{1}{2}$ … … … … … … . . (2)

from $(1)\&(2)$ implies $\lim_{z\to 0}\frac{\bar{z}}{2z}$ does not exist.

14- Theorem on Limits

We can expedite our statement of limits by establishing a connection between limits of function of a complex variable and limits of real-valued functions of two variables. Since limits of the later type are studied in calculus, we use their definition and properties freely.

Theorem(1): If $f(z) = u(x,y) + iv(x,y), z = x + iy \& z_\circ = x_\circ + iy_\circ$,

$w_\circ = u_\circ + iv_\circ$, then $\lim_{z\to z_\circ} f(z) = w_\circ$ … (1) if and only if

$$\lim_{(x,y)\to(x_\circ,y_\circ)} u(x,y) = u_\circ \& \lim_{(x,y)\to(x_\circ,y_\circ)} v(x,y) = v_\circ \ldots (2)$$

Proof: We first assume that **limits(2)** hold and obtain **limits(1)**

from (2) we get $\forall\epsilon > 0\ \exists\delta_1,\delta_2$ such that

$|u - u_\circ| < \frac{\epsilon}{2}$, whenever $0 < \sqrt{(x - x_\circ)^2 + (y - y_\circ)^2} < \delta_1$ … … … (3)

and $|u - u_\circ| < \frac{\epsilon}{2}$, whenever $0 < \sqrt{(x - x_\circ)^2 + (y - y_\circ)^2} < \delta_2$ … … … (4)

Let $\delta = min\{\delta_1,\delta_2\}$ then $|f(z) - w_\circ| = |(u + iv) - (u_\circ + iv_\circ)| =$

$$|(u - u_\circ) + i(v - v_\circ)| \le |u - u_\circ| + |v - v_\circ| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

whenever

$$0 < |z - z_\circ| = |(x + iy) - (x_\circ + iy_\circ)| = |(x - x_\circ) + i(y - y_\circ)|$$
$$= \sqrt{(x - x_\circ)^2 + (y - y_\circ)^2} < \delta$$

That is **limit**$(\mathbf{1})$ hold. Now, let start with (1) that limit hold, this means $\forall \epsilon > 0 \; \exists \; \delta > 0$ such that

$|f(z) - w_\circ| = |(u + iv) - (u_\circ + iv_\circ)| < \epsilon \ldots\ldots\ldots (5)$ whenever

$0 < |z - z_\circ| = |(x + iy) - (x_\circ + iy_\circ)| < \delta \ldots\ldots\ldots (6)$

But $|u - u_\circ| \leq |(u - u_\circ) + i(v - v_\circ)| = |(u + iv) - (u_\circ + iv_\circ)| < \epsilon$

$$|v - v_\circ| \leq |(u - u_\circ) + i(v - v_\circ)| = |(u + iv) - (u_\circ + iv_\circ)| < \epsilon$$

and $|(x + iy) - (x_\circ + iy_\circ)| = |(x - x_\circ) + i(y - y_\circ)|$

Hence, it follows from inequities (5) and (6) that

$|u - u_\circ| < \epsilon$ and $|v - v_\circ| < \epsilon$ whenever $0 < \sqrt{(x - x_\circ)^2 + (y - y_\circ)^2} < \delta$ this establishes **limits**$(\mathbf{2})$ and the proof of the theorem is complete.■

Example$(\mathbf{1})$**:** Find by above theorem $\lim_{z \to 0}(z^2 + iz)$

Solution:

$f(z) = z^2 + iz = (x^2 - y^2) + i2(xy + 1), u(x, y) = x^2 - y^2$

$$v(x, y) = 2(xy + 1)$$

$$z \to 0, (x, y) \longrightarrow (0,0) \Rightarrow x \longrightarrow 0 \& y \longrightarrow 0$$

$$u_\circ = \lim_{(x,y)\to(0,0)} u(x, y) = \lim_{(x,y)\to(0,0)} (x^2 - y^2) = 0$$

$$v_\circ = \lim_{(x,y)\to(0,0)} v(x, y) = \lim_{(x,y)\to(0,0)} 2(xy + 1) = 2$$

$$\therefore \lim_{z \to 0}(z^2 + iz) = i2$$

Theorem(2): Assume that $\lim_{z\to z_\circ} f(z) = L$ & $\lim_{z\to z_\circ} g(z) = M$, then

a. $\lim_{z\to z_\circ}[f(z) \pm g(z)] = L \pm M$
b. $\lim_{z\to z_\circ}\left[\frac{f(z)}{g(z)}\right] = \frac{L}{M}, M \neq 0$
c. $\lim_{z\to z_\circ}[f(z).\, g(z)] = L.\, M$

Proof: Use the corresponding theorem for function of two real variables

Corollary(1):

a. If $p(z)$ is a polynomial then $\lim_{z\to z_\circ} p(z) = p(z_\circ)$
b. If $p(z)$ & $Q(z)$ are polynomial , $Q(z) \neq 0$ then $\lim_{z\to z_\circ}\left[\frac{f(z)}{Q(z)}\right] = \frac{f(z_\circ)}{Q(z_\circ)}$

Example(2): Find by above theorem $\lim_{z\to 2+i}(2z^3 + iz + 1)$

Solution:

$$\lim_{z\to 2+i}(2z^3 + iz + 1) = 2(2+i)^3 + i(2+i) + 1 = 2(2 + i11) + i2 - 1 + 1$$

$$= 2(2 + i11) + i2 = 4 + i22 + i2 = 4 + i24$$

Proposition(1): If $\lim_{z\to z_\circ} f(z) = L$ then $\lim_{z\to z_\circ}|f(z)| = |L|$

Proof: Assume $\lim_{z\to z_\circ} f(z) = L$, let $\epsilon > 0$ be given and let $\delta = \delta_1$, $\exists\delta_1 > 0$ such that $0 < |z - z_\circ| < \delta_1 \Rightarrow |f(z) - L| < \epsilon$

Assume $0 < |z - z_\circ| < \delta = \delta_1$ and so $\big||f(z)| - |L|\big| \leq |f(z) - L| < \epsilon$. ■

15- Theorem involving the point at infinity

It is often useful to include the point at infinity, denoted by ∞, with the complex plane.

Definition$(\boldsymbol{Extended\ complex\ plane})$**:** The extended complex plane is the extended complex plane together with the point at infinity(∞).

ملاحظة: المستوي العقدي لا يحتوي على(∞) لان (∞)ليست عدد عقدي و لكن عندما نضيف ال(∞) يسمى $(Extended\ complex\ plane)$.

Note: To visualize the point at infinity, one can use stereographic projection. We construct a (one-one) corresponding between the extended complex plane and the unit sphere**.**

$$S^2 = \{(x_1, x_2, x_3): x_1^2 + x_2^2 + x_3^2 = 1\}, \varphi: S^2 \to \mathbb{C} \cup \{\infty\},$$

$$\varphi(x_1, x_2, x_3) = \frac{x_1 + ix_2}{1 - x_3}, \varphi(0,0,1) = \infty(\text{غير معرفة})$$

Remark: The sphere S^2 is known as the Riemann sphere.

Definition$(\boldsymbol{\epsilon - Neighborhood\ of}\ (\infty))$**:** The point closed to (∞) in the extended complex plane correspond to points close to the North pole (N). That is

$\left\{z \in \mathbb{C}: |z| > \frac{1}{\epsilon}\right\}$ Corresponds to $\{(x_1, x_2, x_3) \in S^2: \delta < x_3 < 1\}$.

ملاحظات:

1. نحن نعلم ممكن تحديد قيمة z من خلال القيمة المطلقة $|z|$ و هذه القيمة اما ان تكون كبيرة او صغيرة بالتالي عندما نقول $z \to \infty$ هذا يعني ان $|z|$ كبيرة وهذه النقطة تسمى (∞) ولا يوجد $(-\infty)$ في الاعداد العقدية لان اصغر قيمة للعدد العقدي الصفر.
2. بما ان (ϵ) عدد صغير حتى يكبر نقسم عليه و بما ان $|z|$ كبيرة جدا هذا يعني $|z| > \frac{1}{\epsilon}$ ومثال ذلك:

$$\text{If } \epsilon = 0.001 = \frac{1}{1000} \Rightarrow |z| > \frac{1}{\frac{1}{1000}} \Rightarrow |z| > 1000$$

Definition$(\boldsymbol{infinite\ limits})$**:**

a. We say that $lim_{z\to z_\circ} f(z)=\infty$, if for each $\epsilon>0$ there exist $\delta>0$ such that $0<|z-z_\circ|<\delta \Rightarrow |f(z)|>\frac{1}{\epsilon}$.

b. We say that $lim_{z\to\infty} f(z)=L$, if for each for each $\epsilon>0$ there exist $\delta>0$ such that $|z|>\frac{1}{\delta}\Rightarrow |f(z)-L|<\epsilon$.

c. We say that $lim_{z\to\infty} f(z)=\infty$, if for each for each $\epsilon>0$ there exist $\delta>0$ such that $|z|>\frac{1}{\delta}\Rightarrow |f(z)|>\frac{1}{\epsilon}$.

Theorem(1): If $z_\circ$ and L are points in the z −plane and w −plane respectively then

a. $lim_{z\to z_\circ} f(z)=\infty \leftrightarrow lim_{z\to z_\circ}\frac{1}{f(z)}=0$

b. $lim_{z\to\infty} f(z)=L \leftrightarrow lim_{z\to 0} f\left(\frac{1}{z}\right)=L$

c. $lim_{z\to\infty} f(z)=\infty \leftrightarrow lim_{z\to 0}\frac{1}{f\left(\frac{1}{z}\right)}=0$

ملاحظات:

1. اذا كانت $f(z)$ تذهب الى ∞ هذا يعني ان مقلوبها يذهب الى الصفر.

2. اذا كانت z تذهب الى ∞ هذا يعني ان $\frac{1}{z}$ تذهب الى الصفر.

Proof:

a. We start the proof by the first of limits (a) means that

$\forall\epsilon>0\ \exists\ \delta>0$ such that $|f(z)|>\frac{1}{\epsilon}$ whenever $0<|z-z_\circ|<\delta$ … … … … … . (1)

Let $f(z)=w\Rightarrow w$ lies in ϵ −neighborhood $|w|>\frac{1}{\epsilon}$ of ∞, whenever z lies in $0<|z-z_\circ|<\delta$, statement(1) can be written $\left|\frac{1}{f(z)}-0\right|<\epsilon$ whenever $0<|z-z_\circ|<\delta$.

b. The first of limits (b) mean

$\forall\epsilon>0\ \exists\ \delta>0$ such that $|f(z)-L|<\epsilon$ whenever $|z-z_\circ|<\frac{1}{\delta}$ … … … … … . (2)

Replace z by $\frac{1}{z}$ in (5) we get $\left|f\left(\frac{1}{z}\right)-L\right|<\epsilon$ whenever $\left|\frac{1}{z}\right|>\frac{1}{\delta}=0<|z-0|<\delta$.

c. The first of limits (c) is to be interpreted as saying that

$\forall \epsilon > 0 \; \exists \; \delta > 0$ such that $|f(z)| > \frac{1}{\epsilon}$ whenever $|z| > \frac{1}{\delta} \dots \dots \dots \dots \dots \dots \dots \dots \dots .. (3)$

Replace z by $\frac{1}{z}$ in (3) we get $\left|\frac{1}{f\left(\frac{1}{z}\right)} - 0\right| < \epsilon$ whenever $0 < |z - 0| < \delta$. And this gives the second of limits (c).

Examples:

1- Show that $lim_{z\to i} \frac{z^2+2}{z-i} = \infty$

Solution: we have to show $lim_{z\to i} \frac{1}{f(z)} = 0 \Rightarrow lim_{z\to i} \frac{z-i}{z^2+2} = \frac{0}{1} = 0$

2- Show that $lim_{z\to \infty} \frac{z+i}{z+1} = 1$

Solution: we have to show $lim_{z\to 0} f\left(\frac{1}{z}\right) = 1 \Rightarrow lim_{z\to 0} \frac{\frac{1}{z}+i}{\frac{1}{z}+1} = lim_{z\to 0} \frac{1+iz}{1+z} = 1$

3- Show that $lim_{z\to \infty} \frac{i2z^3-2z}{z^2+3-i} = \infty$

Solution: we have to show

$$lim_{z\to 0} \frac{1}{f\left(\frac{1}{z}\right)} = lim_{z\to 0} \frac{\left(\frac{1}{z}\right)^2+3-i}{\frac{i2}{z^3}-\frac{2}{z}} = lim_{z\to 0} \frac{z(1+(3-i)z^2)}{(i2-2z^2)} = 0.$$