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FILTER BASES AND ω-PERFECT FUNCTIONS

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المرشحات الأساسية والدوال التامة من النمط- ω

الخلاصة

في هذا البحث قدمنا بعض التعميمات لبعض التعريفات وهي، التقارب المغلق لنقطة، الاتجاه المباشر المغلق لمجموعة، التقارب لمجموعة من النمط- ω على الأغلب، نقطة تكاثف من النمط- ω على الأغلب، المجموعة المغلقة من النمط- ωH ذات العلاقة، الدوال المستمرة من النمط- ω، الدوال المستمرة الضعيفة من النمط- ω، الدوال المتراصة من النمط- ω، المجموعة الصلبة من النمط- ω، الدوال المغلقة من النمط- ω على الأغلب، والدوال التامة من النمط- ω مع العديد من النتائج المتعلقة بها.

Abstract

In this paper we introduce some generalizations of some definitions which are, closure converge to a point, closure directed toward a set, almost ω-converges to a set, almost condensation point, a set ωH-closed relative, ω-continuous functions, weakly ω-continuous functions, ω-compact functions, ω-rigid a set, almost ω-closed functions and ω-perfect functions with several results concerning them.

1. Introduction and Preliminaries:

The notion "filter" first appeared in Riesz (1) and the setting of convergence in terms of filters was sketched by Cartan in (2) and (3) and was developed by Bourbaki in (4). Whyburn in (5) introduced the notion directed toward a set and the generalization of this notion is studied in section 3, Dickman and Porter in (6) introduced the notion almost convergence, Porter and Thomas in (7) introduced the notion quasi-H-closed and the analogues of this notions are studied in section 4, Levine in (8) introduced the notion θ-continuous functions, Andrew and Whittlesy in (9) introduced the notion weakly θ-continuous functions, Dickman in (6) introduced the notions θ-compact functions, θ-rigid a set, almost closed functions and the analogues of these notions are studied in section 5, Whyburn in (5) introduced the notion θ-perfect functions and the analogue of this notion is studied in section 6.

In this work, the neighborhood denoted by nbd, for a subset A of a topological space X, the closure of A denoted by cl(A) and ω denotes the cardinal number of integers.

2. Basic Definitions:

Definition(2.1), (4): A filter $\Im$ on a set X is a nonempty collection of nonempty subsets of X with the properties:

(a) If $F_1, F_2 \in \Im$, then $F_1 \cap F_2 \in \Im$,

(b) If $F \in \Im$ and $F \subseteq F^* \subseteq X$, then $F^* \in \Im$.

Definition(2.2), (4): A filter base $\Im$ on a set X is a nonempty collection of nonempty subsets of X such that if $F_1, F_2 \in \Im$ then $F_3 \subseteq F_1 \cap F_2$ for some $F_3 \in \Im$.

The filter generated by a filter base $\Im$ consists of all supersets of elements of $\Im$. An open filter base on a space X is a filter base with open members. The set $\aleph_x$ of all nbds of $x \in X$ is a filter on X, and any nbd base at x is a filter base for $\aleph_x$. This filter called the nbd filter at x, (10).

Definition(2.3), (4): A filter base $\Im$ on a space X is said to converge to $x \in X$ (written as $\Im \to x$) iff every open set U about x contains some element $F \in \Im$. We say $\Im$ has x as a cluster point (or $\Im$ cluster at x) iff every open set U about x meets each element $F \in \Im$.

Clear that if $\Im \to x$, then $\Im$ cluster at x, (10).

Definition(2.4), (4): If $\Im$ and G are filter bases on X, we say that G is finer than $\Im$ (written as $\Im < G$) if for each $F \in \Im$, there is $G \in G$ such that

$G \subseteq F$ and that $\Im$ meets G if $F \cap G \neq \phi$ for every $F \in \Im$ and $G \in G$. If also $\Im \neq G$ then G is said to be strictly finer than $\Im$.

Notice, $\Im \to x$ iff $\aleph_x < \Im$ (means, $\Im$ finer than $\aleph_x$). (10).

Definition(2.5), (5): A filter base $\Im$ on a space X is said to be directed toward a set $A \subseteq X$, provided every filter base finer than $\Im$ has a cluster point in A. (Note: No filter base can be directed toward the empty set).

Definition(2.6), (4): A filter $\Im$ is said to be an ultrafilter if there is no strictly finer filter G than $\Im$ (in other words, the ultrafilter is the maximal element in the ordered set of all filters on X).

Definition(2.7), (4): A point x of a space X is called a condensation point of the set $A \subseteq X$ if every nbd of the point x contains an uncountable subset of this set.

Clearly the set of condensation points of a set A is closed.

Definition(2.8), (11): A subset of a space X is called ω-closed if it contains all its condensation points. Also $cl^{\omega}A$ will denote the intersection of all ω-closed sets which contains A. i.e., $cl^{\omega}A = \cap\{F: F$ is ω-closed and $A \subseteq F\}$, then A is ω-closed iff $A = cl^{\omega}A$.

3. Filter Bases and Closure Directed Toward a Set:

Lemma(3.1), (5): Let $f: X \to Y$ be an injective function.

(a) If $\Im = \{F : F \subseteq X\}$ is a filter base in X, then $f(\Im) = \{f(F) : F \in \Im\}$ is a filter base in Y.

(b) If $G = \{G : G \subseteq f(X)\}$ is a filter base in f(X), $\Im = \{f^{-1}(G) : G \in G\}$ is a filter base in X. For any nonempty set A in X and any filter base G in f(A), then $\{A \cap f^{-1}(G) : G \in G\}$ is a filter base in A.

(c) If $\Im = \{F : F \subseteq X\}$ is a filter base in X, $G = \{f(F) : F \in \Im\}$, G^* is finer than G, and $\Im^* = \{f^{-1}(G^*) : G^* \in G^*\}$, then the collection of sets $\Im^{**} = \{F \cap F^*$ for all $F \in \Im$ and $F^* \in \Im^*\}$ is finer than both of $\Im$ and $\Im^*$.

Now, we will give generalizations of definitions (2.3) and (2.5) as follows.

Definition(3.2): A filter base $\Im$ on a space X is said to closure converges to $x \in X$ (written as $\Im \rightsquigarrow x$) iff every open set U about x, the cl(U) contains some element $F \in \Im$. We say $\Im$ has x as a closure cluster point (or $\Im$ closure cluster at x) iff every open set U about x the clU meets each element $F \in \Im$.

Clear that if $\Im \rightsquigarrow x$, then $\Im$ closure cluster at x. $cl(\aleph_x)$ is used to denote the filter base $\{clU : U \in \aleph_x\}$. Notice, $\Im \rightsquigarrow x$ iff $cl(\aleph_x) < \Im$.

Definition(3.3): A filter base $\Im$ on a space X is said to be closure directed toward a set $A \subseteq X$, provided every filter base finer than $\Im$ has a closure cluster point in A.

Whyburn prove in (5) theorem(2,d), theorem(2,e) and theorem(3). We show that these theorems remains true if we replace "converge" and "directed toward" by the more general concept of "closure converges" and "closure directed toward" and obtain the same conclusion which are in theorem(3.4), theorem(3.5) and theorem(3.6).

Theorem(3.4): A filter base $\Im$ in a space X closure converges to a point x iff $\Im$ is closure directed toward x.

Proof: ($\Rightarrow$) If $\Im \rightsquigarrow x$, every open set U about x, the cl(U) contains a member of $\Im$ and thus contains a member of any filter base $\Im^*$ finer than $\Im$, so that $\Im^*$ actually closure converges to x.

($\Leftarrow$) If $\Im$ is closure directed toward x, it must closure converge to x. For if not, there exists an open set U in X about x such that cl(U) contains no element of $\Im$. Denote by $\Im^*$ the family of sets $F^*=F\cap(X-cl(U))$ for $F \in \Im$, then the sets F^* are nonempty (if not, then $F^*=\phi$ and $F\cap(X-cl(U))=\phi$, so $F\subseteq X-(X-cl(U))$ and $F\subseteq cl(U)$ which is contradiction with the cl(U) contains no element of $\Im$). Also $\Im^*$ is a filter base and indeed it is finer than $\Im$, because given $F_1^*=F_1\cap(X-cl(U))$ and $F_2^*=F_2\cap(X-cl(U))$, there is an $F_3\subseteq F_1\cap F_2$ and this gives $F_3^*=F_3\cap(X-U)\subseteq F_1\cap F_2\cap(X-U) = F_1\cap(X-U)\cap F_2\cap(X-U)$. By construction x is not a closure cluster point of $\Im^*$. This is a contradiction, and thus $\Im \rightsquigarrow x$.

Theorem(3.5): Let $f : X \to Y$ be an injective function and given $B \subset Y$. If for each filter base G in f(X) closure directed toward a point $y \in B$, the inverse filter $M = \{f^{-1}(G) : G \in G\}$ is closure directed toward $f^{-1}(y)$, then for any filter base $\Im$ of sets in f(X) closure directed toward a set B, $E = \{f^{-1}(F) : F \in \Im\}$ is closure directed toward $A = f^{-1}(B)$.

Proof: By hypothesis "for each filter base G in f(X) closure directed toward a point $y \in B$, the inverse filter $M = \{f^{-1}(G) : G \in G\}$ is closure directed toward $f^{-1}(y)$" any $y \in B$ which is a closure cluster point of a filter base finer than $\Im$ must be in f(X). Thus not only is $B \cap f(X) \neq \phi$, but , since any filter base $\Im$ of sets in f(X) closure directed toward a set B, also $\Im$ is closure directed toward $B \cap f(X)$. Thus we may assume $B \subseteq f(X)$. Let M be a filter base finer than E. Then $G = \{f(M) : M \in M\}$ finer than $\Im$ by lemma (3.1, a). Thus G has a closure cluster point z in B and a filter base G* finer than G closure converges to z and thus is closure directed toward z. By hypothesis $M^* = \{f^{-1}(G^*) : G^* \in G^*\}$ is closure directed toward $f^{-1}(z)$. Also by lemma (3.1, c), M and M* have a common filter base M** finer than of them. Thus M** has a closure cluster point x in $f^{-1}(z)$. Since x is then a closure cluster point of M and $x \in f^{-1}(z) \subset A$, our conclusion follows.

Theorem(3.6): A function $f : X \rightarrow Y$ is closed and has compact point inverses iff for each filter base $\Im$ in f(X) closure directed toward a set B in Y, the inverse family $E = \{f^{-1}(F) : F \in \Im\}$ is closure directed toward $f^{-1}(B)$.

Proof: ($\Rightarrow$) Suppose f is closed and has compact point inverses. Then by Theorem: (3.4) and (3.5) it suffices to show that if G is a filter base in f(X) closure converging to a point y in B, then $M = \{f^{-1}(G) : G \in G\}$ is closure directed toward $f^{-1}(y)$. Suppose that to the contrary, that for some filter base M* finer than M, no point of $f^{-1}(y)$ is a closure cluster point of M*. We show, however, that this leads to the contradiction that the filter base $G^* = \{f(M^*) : M^* \in M^*\}$ finer than G cannot closure converge to y. For each $x \in f^{-1}(y)$, by supposition there is an open set U_x about x and $M_x^* \in M^*$ with $M_x^* \cap U_x = \phi$. Since $f^{-1}(y)$ is compact, it is contained in a finite union $U = \cup U_{x_i}$ of the sets U_x. Let M* is an element of M* which is contained in the intersection $\cap M_{x_i}{}^*$ and let V be the open set $Y-f(X-U)$. Then $f(M^*)\cap V=\phi$ because $M^*\subset X-clU$. Thus since $f(M^*)\in G^*$, G^* cannot have y as a closure cluster point.

($\Leftarrow$) Suppose our condition is satisfied but f is not closed. Let A be a closed set in X such that some $y\in Y-f(A)$ is a closure cluster point of f(A). Let G be a filter base of sets $f(A)\cap V$ for all open sets V in Y containing y, then G is a filter base in f(X) closure converging to y. Let $M=\{f^{-1}(G) : G\in G\}$ and $M^*=\{A\cap M : M\in M\}$. It readily follows that M^* is finer than of M. But since X–A is open and contains $f^{-1}(y)$, M^* has no closure cluster point in $f^{-1}(y)$. This is a contradiction, and thus f must be closed. Finally, to show each $f^{-1}(y)$ is compact, we have only to show that every filter base of subsets of $f^{-1}(y)$ has a closure cluster point in $f^{-1}(y)$. This is trivial for $y\in Y-f(X)$. Also for $y\in f(X)$, $\{y\}$ is a filter base in f(X) closure directed toward y. By hypothesis, $\{f^{-1}(y)\}$ must be closure directed toward $f^{-1}(y)$. This means that every filter base of sets in $f^{-1}(y)$ has a closure cluster point in $f^{-1}(y)$, so that $f^{-1}(y)$ is compact.

Corollary(3.7): A function $f : X\rightarrow Y$ is closed and has compact point inverses iff each filter base in f(X) closure converging to $y\in Y$ has inverse filter base closure directed toward $f^{-1}(y)$.

Proof: The proof is easy, so it is omitted.

Corollary(3.8): If $f : X\rightarrow Y$ is closed and has compact point inverses, then the inverse of any compact set in Y is compact.

Proof: For if K is any compact set in Y and $\Im$ is a filter base in $f^{-1}(K)$, $G=\{f(F) : F\in\Im\}$, is a filter base in K and in f(X) and is closure directed toward K. Thus $\{f^{-1}(G) : G\in G\}$ is closure directed toward $f^{-1}(K)$ so that it is finer than $\Im$ and has a closure cluster point in $f^{-1}(K)$.

4. Filter Bases and Almost ω-Convergence:

By analogue of the definition of almost convergence in (6) we can give the following definition.

Definition(4.1): Let $\Im$ be a filter base on a space X. We say $\Im$ almost ω-converges to a subset $A\subseteq X$ (written as $\Im\omega\rightsquigarrow A$) if for every cover A of A by subsets open in X, there is a finite subfamily $B\subseteq A$ and $F\in\Im$

such that $F\subseteq\cup\{cl^{\omega}B : B\in B\}$. We say $\Im$ almost ω-converges to a point $x\in X$ (written as $\Im\omega \rightsquigarrow x$) if $\Im\omega\rightsquigarrow\{x\}$.

Now, $cl(\aleph_x)\rightsquigarrow x$, where as, $cl^{\omega}(\aleph_x)\omega\rightsquigarrow x$.

Also, we introduce the following definitions:

Definition(4.2): A point $x\in X$ is called an almost condensation point of a filter base $\Im$ (written as $x\in alc_X\Im$) if $\Im$ meets $cl^{\omega}(\aleph_x)$.

For a set $A\subset X$, the almost ω-closure of A, denoted as $al_{\omega}A$ is $alc_X\{A\}$ if $A\neq\phi$ i.e., $\{x\in X$: every ω-closed nbd of x meets A$\}$ and is ϕ if $A=\phi$; A is almost ω-closed if $A=alc_{\omega}A$. Correspondingly, the almost ω-interior of A, denoted as $int_{\omega}A$, is $\{x\in X ; cl^{\omega}U\subseteq A$ for some open set U containing x$\}$; A is almost ω-interior if $A=int_{\omega}A$.

Theorem(4.3): Let $\Im$ and G be filter bases on a space X, $A\subseteq X$ and $x\in X$.

(a) If $\Im\omega\rightsquigarrow A$, then $cl^{\omega}(\aleph_A) < \Im$.

(b) If $\Im\omega\rightsquigarrow x$, iff $cl^{\omega}(\aleph_x) < \Im$.

(c) If $\Im < G$, then $alc_XG \subseteq alc_X\Im$.

(d) If $\Im < G$ and $\Im\omega\rightsquigarrow A$, then $G\omega\rightsquigarrow A$.

(e) $alc_X\Im=\cap\{al_{\omega}F : F\in\Im\}$.

(f) If $\Im\omega\rightsquigarrow x$ and $x\in A$, then $\Im\omega\rightsquigarrow A$.

(g) If $\Im\omega\rightsquigarrow A$ iff $\Im\omega\rightsquigarrow A\cap alc_X\Im$.

(h) If $\Im\omega\rightsquigarrow A$, then $A\cap alc_X\Im\neq\phi$.

(i) If $U\subseteq X$ is open, then $al_{\omega}U=clU$.

(j) If $\Im$ is a open filter base, then $al_{\omega}\Im=alc_X\Im$.

(k) If U is an open ultrafilter on X, then $U\rightsquigarrow x$ iff $U\omega\rightsquigarrow x$.

Proof: The proof is easy, so it is omitted.

By analogue of the definition of quasi-H-closed relative in (7) we can give the following definition.

Definition(4.4): The subset A of a space X is said to be quasi-ωH-closed relative to X if every cover A of A by open subsets of X contains a finite subfamily $B \subseteq A$ such that $A\subseteq\cup\{cl^{\omega}B : B\in B\}$. If X is Hausdorff, we say that A is ωH-closed relative to X. If X is quasi-ωH-closed relative to itself, then X is said to be quasi-ωH-closed (resp., ωH-closed).

Theorem(4.5): The following are equivalent for a subset $A \subseteq X$:

(a) A is quasi-ωH-closed relative to X.

(b) For every filter base $\Im$ on A, $\Im\omega \rightsquigarrow A$.

(c) For every filter base $\Im$ on A, $alc_X\Im \cap A \neq \phi$.

Proof: Clearly (a)⇒(b), and by theorem (4.3, h), (b)⇒(c). To show (c)⇒(a), let A be a cover of A by open subsets of X such that the ω-closed of the union of any finite subfamily of A is not cover A. Then $\Im = \{A \setminus cl^{\omega}{}_X(\cup_S U_s) : S$ is finite subfamily of A$\}$ is a filter base on A and $alc_X\Im \cap A = \phi$. This contradiction yields that A is quasi-ωH-closed relative to X.

The concepts of closure directed toward a set and almost ω-convergence are characterized and related in the next result.

Theorem(4.6): Let $\Im$ be a filter base on a space X and $A \subseteq X$.

(a) Then $\Im$ is closure directed towards A iff for every cover A of A by open subsets of X, there is a finite subfamily $B \subseteq A$ and an $F \in \Im$ such that $F \subseteq \cup\{cl^{\omega}B : B \in B\}$.

(b) Then for every filter base G, $\Im < G$ implies $alc_X G \cap A \neq \phi$ iff $\Im\omega \rightsquigarrow A$.

Proof: The proof of the two facts are similar; so, we will only prove the fact (b): (⇒) Suppose for every filter base G, $\Im < G$ implies $alc_X G \cap A \neq \phi$. If $\Im\omega \rightsquigarrow x$ for some $x \in A$, then by theorem (4.3, f), $\Im\omega \rightsquigarrow A$. So, suppose that for every $x \in A$, $\Im$ does not $\omega \rightsquigarrow x$. Let A be a cover of A by subsets open in X. For each $x \in A$, there is an open set U_x containing x and $V_x \in A$ such that $U_x \subseteq V_x$ and $F \setminus cl^{\omega}{}_X U_x \neq \phi$ for every $F \in \Im$. Thus, $G_x = \{F \setminus cl^{\omega}{}_X U_x : F \in \Im\}$ is a filter base on X and $\Im < G_x$. Now, $x \notin alc_X G_x$. Assume that $\cup\{G_x : x \in A\}$ forms a filter subbase with G denoting the generated filter. Then $\Im < G$ and $alc_X G \cap A = \phi$. This contradiction implies there is a finite subset $B \subseteq A$ and $F_x \in \Im$ for $x \in B$ such that. $\phi = \cap\{F_x \setminus cl^{\omega}{}_X U_x : x \in B\}$. There is $F \in \Im$ such that $F \subset \cap\{F_x :$

$x\in B\}$. It easily follows that $\phi=\cap\{F\setminus cl^{\omega}{}_X U_x : x\in B\}$ and $F\subseteq\cup\{cl^{\omega}{}_X V_x : x\in B\}$. Thus $\Im\omega\rightsquigarrow A$.

($\Leftarrow$) Suppose $\Im\omega\rightsquigarrow A$ and G is a filter base such that $\Im < G$. By theorem (4.3, d), $G\omega\rightsquigarrow A$, and theorem (4.3, h), $al_X G\cap A\neq\phi$.

5. Filter Bases and ω-Rigidity:

By analogues of definitions θ-continuous functions in (7) and weakly θ-continuous functions in (8) one can define.

Definition(5.1): A function $f : X\to Y$ is ω-continuous (resp., weakly ω-continuous) if for every $x\in X$ and every nbd V of f(x), there exists a nbd U of x in X such that $f(cl^{\omega}(U))\subseteq cl^{\omega}V$ (resp., $f(U)\subseteq cl^{\omega}V$). Clearly, every continuous function is ω-continuous.

The notions of almost ω-convergence and almost condensation can be used to characterize ω-continuous.

Theorem(5.2): Let $f : X\to Y$ be a function. The following are equivalent:

(a) f is ω-continuous.

(b) For every filter base $\Im$ on X, $\Im\omega\rightsquigarrow x$ implies $f(\Im)\to f(x)$.

(c) For every filter base $\Im$ on X, $f(alc\Im)\subseteq alc\, f(\Im)$.

(d) For every open $U\subseteq Y$, $f^{-1}(U)\subseteq int_{\omega} f^{-1}(al_{\omega}U)$.

Proof: The proof of the equivalence of (a), (b),and (d) is straightforward.

(a)⇒(c) Suppose $\Im$ is a filter base on X, $x\in alc\Im$, $F\in\Im$ and V is a nbd of f(x). There is a nbd U of x such that $f(cl^{\omega}U)\subseteq cl^{\omega}V$. Since $cl^{\omega}U\cap F\neq\phi$, then $cl^{\omega}V\cap f(F)\neq\phi$. So, $f(x)\in alc\, f(\Im)$. This shows that $f(alc\Im)\subseteq alc\, f(\Im)$.

(c)⇒(a) Let U be an ultrafilter containing $f(cl^{\omega}\aleph_x)$. Now, $f^{-1}(U)$ is a filter base since $f(X)\in U$ and $f^{-1}(U)$ meets $cl^{\omega}\aleph_x$. So, $f^{-1}(U)\cup cl^{\omega}\aleph_x$ is contained in some ultrafilter V. Now $f\,f^{-1}(U)$ is an ultrafilter base that generates U. Since $f\,f^{-1}(U)<f(V)$, then f(V) also generates U; hence $alcf(V)=alcU$. Since $x\in alc(V)$, then $f(x)\in f(alcV)\subseteq alcf(V)=alcU$. So, U meets $cl^{\omega}(\aleph_{f(x)})$ and $cl^{\omega}(\aleph_{f(x)})\subseteq\cap\{U : U \text{ ultrafilter}, U\supseteq f(cl^{\omega}\aleph_x)\}$,

(denote this intersection by G). But G is the filter generated by ($cl^{\omega}\aleph_x$) (see (4) proposition I.6.6); so $cl^{\omega}(\aleph_{f(x)}) < f(cl^{\omega}\aleph_x)$. Hence f is ω-continuous.

Corollary(5.3): If f: X→Y is ω-continuous and A⊆X, then $f(al_{\omega}A) \subseteq al_{\omega}f(A)$.

Here are some similarly proven facts about weakly ω-continuous functions.

Theorem(5.4): Let f : X→Y be a function. The following are equivalent:

(a) f is weakly ω-continuous.

(b) For every filter base $\Im$ on X, $\Im \to x$ implies $f(\Im)\omega \rightsquigarrow f(x)$.

(c) For every filter base $\Im$ on X, $f(alc\Im) \subseteq alc\ f(\Im)$.

(d) For every open U⊆Y, $f^{-1}(U) \subseteq int\ f^{-1}(cl^{\omega}U)$.

Theorem(5.5): If f : X→Y is weakly ω-continuous, then

(a) For each A⊆X, $f(cl^{\omega}A) \subseteq al_{\omega}f(A)$.

(b) For each B⊆Y, $f(cl^{\omega}(int(cl^{\omega} f^{-1}(B)))) \subseteq cl^{\omega} B$.

(c) For every open U⊆Y, $f(cl^{\omega}U) \subseteq cl^{\omega}f(U)$.

By analogues of the definitions of θ-compact functions, θ-rigid a set and almost closed in (6) we can give the following definitions.

Definition(5.6): A function f : X→Y is said to be ω-compact if for every subset K quasi-ωH-closed relative to Y, $f^{-1}(K)$ is quasi-ωH-closed relative to X.

Definition(5.7): A subset A of a space X is said to be ω-rigid provided whenever $\Im$ is a filter base on X and $A \cap alc_X\Im = \phi$, there is an open U containing A and $F \in \Im$ such that $cl^{\omega}U \cap F = \phi$.

Definition(5.8): A function f : X→Y is said to be almost ω-closed if for any set A⊆X, $f(al_{\omega}A) = al_{\omega}f(A)$.

Definition(5.9): A space X is said to be ω-Urysohn if every pair of distinct points are contained in disjoint ω-closed nbds.

Before characterizing ω-rigidity, we show that a ω-continuous, ω-compact function into a ω-Urysohn space with a certain property (the "ω-closure" and "quasi-ωH-closed relative" analogue of property α in (5)) is almost ω-closed.

Theorem(5.10): Suppose $f : X \rightarrow Y$ is a ω-continuous and ω-compact and Y is ω-Urysohn with this property: For each $B \subseteq Y$ and $y \in al_{\omega}B$, there is a subset K quasi-ωH-closed relative to Y such that $y \in al_{\omega}(K \cap B)$. Then f is almost ω-closed.

Proof: Let $A \subseteq X$. By corollary (5.3), $f(al_{\omega}A) \subseteq al_{\omega}f(A)$. Suppose $y \in al_{\omega}f(A)$. There is a subset K quasi-ωH-closed relative to Y such that $y \in al_{\omega}(K \cap f(A))$. Then $\Im = \{cl^{\omega}U \cap K \cap f(A) : U \in \aleph_y\}$, is a filter base on Y such that $\Im \omega \rightsquigarrow y$. Now, $G = \{A \cap f^{-1}(F) : F \in \Im\}$ is a filter base on $A \cap f^{-1}(K)$. Since $f^{-1}(K)$ is quasi-ωH-closed relative to X, then there is $x \in alc_X G \cap f^{-1}(K)$. By theorem (5.2), $f(x) \in alc_Y f(G) \subseteq alc_Y \Im$. Since $\Im \omega \rightsquigarrow y$ and Y is ω-Urysohn, $alc_Y \Im = \{y\}$. Thus, $y \in f(al_{\omega}A)$.

Theorem(5.11): Let A be a subset of a space X. The following are equivalent:

(a) A is ω-rigid in X.

(b) For any filter base $\Im$ on X, if $A \cap alc_X \Im = \phi$, then for some $F \in \Im$, $A \cap al_{\omega}F = \phi$.

(c) For each cover A of A by open subsets of X, there is a finite subfamily $B \subseteq A$ such that $A \subseteq int\ cl^{\omega}(\cup B)$.

Proof: The proof that (a)⇒(b) is straightforward. (b)⇒(c) Let A be a cover of A by open subsets of X and $\Im = \{\cap_{U \in B}(X \setminus cl^{\omega}U) : B$ is a finite subset of A$\}$. If $\Im$ is not a filter base, then for some finite subfamily $B \subseteq A$, $X \subseteq \cup\{cl^{\omega}U : U \in B\}$; thus, $A \subseteq X \subseteq int\ cl^{\omega}(\cup B)$ which completes the proof in the case that $\Im$ is not a filter base. So, suppose $\Im$ is a filter base. Then $A \cap alc\Im = \phi$ and there is an $F \in \Im$ such that $A \cap al_{\omega}F = \phi$. For each $x \in A$, there is open V_x of x such that $cl^{\omega}V_x \cap F = \phi$. Let $V = \cup\{V_x : x \in A\}$. Now, $V \cap F = \phi$. Since $F \in \Im$, then for some finite subfamily $B \subseteq A$, $F = \cap\{X \setminus cl^{\omega}U : U \in B\}$. It follows that $V \subseteq cl^{\omega}(\cup B)$ and hence, $A \subseteq int\ cl^{\omega}(\cup B)$.

(c)⇒(a). Let $\Im$ be a filter base on X such that $A \cap alc\Im = \phi$. For each $x \in A$ there is open V_x of x and $F_x \in \Im$ such that $cl^{\omega}V_x \cap F_x = \phi$. Now $\{V_x :$

$x\in A\}$ is a cover of A by open subsets of X; so, there is finite subset $B\subseteq A$ such that $A\subseteq \text{int } cl^{\omega}(\cup\{V_x: x\in B\})$. Let $U=\text{int } cl^{\omega}(\cup\{V_x : x\in B\})$. There is $F\in \Im$ such that $F\subseteq\cap\{F_x : x\in B\}$. Since $cl^{\omega}U=\cup\{cl^{\omega}V_x: x\in B\}$, then $cl^{\omega}U\cap F=\phi$. Thus A is ω-rigid in X.

6. Filter Bases and ω-Perfect Functions:

From corollary (3.7), corollary (3.8) and in view theorem (4.6), we say that a function $f : X\to Y$ is ω-perfect if for every filter base $\Im$ on $f(X)$, $\Im\,\omega\!\rightsquigarrow y\in Y$ implies $f^{-1}(\Im)\,\omega\!\rightsquigarrow f^{-1}(y)$.

Theorem(6.1): Let $f : X\to Y$ be a function. The following are equivalent:

(a) f is ω-perfect.

(b) For every filter base $\Im$ on X, alc $f(\Im)\subseteq f(\text{alc}\Im)$.

(c) For every filter base $\Im$ on $f(X)$, $\Im\,\omega\!\rightsquigarrow B\subseteq Y$, implies $f^{-1}(\Im)$ $\omega\!\rightsquigarrow f^{-1}(B)$.

Proof: (a)⇒(b) Suppose $\Im$ is a filter base on X and $y\in$ alc $f(\Im)$. Assume, by way of contradiction, that $f^{-1}(y)\cap \text{alc}\Im=\phi$. For each $x\in f^{-1}(y)$, there is open U_x of x and $F_x\in\Im$ such that $cl^{\omega}U_x\cap F_x=\phi$. Since $f^{-1}(cl^{\omega}\aleph_y)\,\omega\!\rightsquigarrow f^{-1}(y)$ and $\{U_x : x\in f^{-1}(y)\}$ is an open cover of $f^{-1}(y)$, there is a $V\in\aleph_y$ and a finite subset $B\subseteq f^{-1}(y)$ such that $f^{-1}(cl^{\omega}V)\subseteq\cup\{cl^{\omega}U_x : x\in B\}$.There is an $F\in\Im$ such that $F\subseteq\cap\{F_x : x\in B\}$. Thus, $F\cap f^{-1}(cl^{\omega}V)=\phi$ implying $cl^{\omega}V\cap f(F)=\phi$, a contradiction as $y\in$ alc $f(\Im)$. This shows that $y\in f(\text{alc }\Im)$.

(b)⇒(c) Suppose $\Im$ is a filter base on $f(X)$ and $\Im\,\omega\!\rightsquigarrow B\subseteq Y$. Let G be a filter base on X such that $f^{-1}(\Im) < G$. Then $\Im < f(G)$ and alc $f(G)\cap B\neq\phi$. Hence $f(\text{alc } G)\cap B\neq\phi$ and alc $G\cap f^{-1}(B)\neq\phi$. By theorem (4.6, b), $f^{-1}(\Im)$ $\omega\!\rightsquigarrow f^{-1}(B)$.

(c)⇒(a) Clearly.

Corollary(6.2): If $f : X\to Y$ is ω-perfect, then f is ω-compact.

Proof: Let K be quasi-ωH-closed relative to Y, and G be a filter base on $f^{-1}(K)$, then $f(G)$ is a filter base on K. By theorem (4.5), alcf(G)∩K≠ϕ and by theorem (6.1, b), alcG∩f^{-1}(K) ≠ϕ. By theorem (4.5), f^{-1}(K) is quasi-H-closed relative to X.

Theorem(6.3): A ω-continuous function f : X→Y is ω-perfect iff

(a) f is almost ω-closed, and

(b) point-inverses are ω-rigid.

Proof: (⇒) If f is ω-continuous and ω-perfect, then by corollaries (6.2) and (5.3), f is almost ω-closed. To show f^{-1}(y), for y∈Y, is ω-rigid, Let ℑ be a filter base on X such that f^{-1}(y)∩alcℑ=ϕ. So, y∉f(alcℑ) and by Theorem: (6.1, b), y∉alc f(ℑ). There is open U of y and F∈ℑ such that cl^{ω}U∩f(F)=ϕ. Therefore, f^{-1}(cl^{ω}U)∩F=ϕ. Since f is ω-continuous, then for each x∈f^{-1}(y), there is open V of x such that $cl^{\omega}V \subseteq f^{-1}(cl^{\omega}U)$. So, f^{-1}(y)∩cl_{ω}F=ϕ.

(⇐) Suppose a ω-continuous function f satisfies (a) and (b). Let ℑ be a filter base on f(X) such that ℑω⇝y. Let G be a filter base on X such that f^{-1}(ℑ) < G. So, ℑ < f(G) implying that y∈alc f(G). So, for every G∈G, y∈al_{ω}f(G)⊆f(al_{ω}G). Hence, f^{-1}(y)∩al_{ω}G≠ϕ for every G∈G. By (b), f^{-1}(y) ∩alcG≠ϕ. By theorem (6.1), f is ω-perfect.

Corollary(6.4): Let f : X→Y. If (a) for each A⊆X, al_{ω}f(A)⊆f(al_{ω}A) and (b) point-inverses are ω-rigid, then f is ω-perfect.

Corollary(6.5): Let f : X→Y. (a) f is almost ω-closed, and (b) point-inverses are ω-rigid, then f^{-1} preserves ω-rigidity.

Proof: Let K⊆Y be ω-rigid and ℑ be a filter base on X such that alc_Xℑ∩ f^{-1}(K)=ϕ. By corollary (6.4) and theorem (6.1), alcf(ℑ)∩K=ϕ. So, there is F∈ℑ such that al_{ω}f(F)∩K=ϕ. But al_{ω}f(F)=f(al_{ω}F). So, al_{ω}F∩f^{-1}(K)=ϕ. So, by theorem (5.11), f^{-1}(K) is ω-rigid.

Theorem(6.6): Suppose f : X→Y has ω-rigid point-inverses. Then:

(a) f is ω-continuous iff for each y∈Y and open set V containing y, there is an open set U containing f^{-1}(y) such that f(cl^{ω}U)⊆cl^{ω}V.

(b) If for each y∈Y and open set U containing f^{-1}(y), there is an open set V of y such that f^{-1} (cl^{ω}V)⊆cl^{ω}U, then for each A⊆X, al_{ω}(f(A)⊆ f(al_{ω}A).

Proof: (a) (⇒) Obvious.

(⇐) Is straightforward using theorem (5.11, c)

(b) Let $\phi \neq A \subseteq X$ and $y \notin f(al_\omega A)$. Then $f^{-1}(y) \cap al_\omega A = \phi$. Now, $\Im = \{A\}$ is a filter base and $alc\Im \cap f^{-1}(y) = \phi$. So, there is open set U continuing $f^{-1}(y)$ such that $cl^\omega U \cap A = \phi$. There is open V of y such that $f^{-1}(cl^\omega V) \subseteq cl^\omega U$. So, $cl^\omega V \cap f(A) = \phi$. Hence $y \notin al_\omega f(A)$.

The next result is closely related to theorem (6.7, b); the proof is straightforward.

Theorem(6.7): Let $f : X \rightarrow Y$. The following are equivalent:

(a) For every ω-closed $A \subseteq X$, f(A) is ω-closed.

(b) For every $B \subseteq Y$ and ω-open U containing $f^{-1}(B)$, there is ω-open V containing B such that $f^{-1}(V) \subseteq U$.

Theorem(6.8): If $f : X \rightarrow Y$ is ω-continuous and Y is ω-Urysohn, then f is ω-perfect iff for every filter base $\Im$ on X, if $f(\Im)\omega \rightsquigarrow y \in Y$, then $alc_X \Im \neq \phi$.

Proof: (⇒) Suppose f is ω-perfect and $f(\Im)\omega \rightsquigarrow y$. So, $f^{-1}f(\Im)\omega \rightsquigarrow f^{-1}(y)$. Since $f^{-1}f(\Im) < \Im$, then by theorem (4.3, d), $\Im\omega \rightsquigarrow f^{-1}(y)$, by theorem (4.3, h), alc $\Im \neq \phi$.

(⇐) Suppose for every filter base $\Im$ on X, if $f(\Im)\omega \rightsquigarrow y \in Y$, then $alc_X \Im \neq \phi$. Suppose G is a filter base on f(X) such that $G\omega \rightsquigarrow y \in Y$, and suppose H is a filter base on X such that $f^{-1}(G) < H$. Then $G = ff^{-1}(G) < f(H)$. So, $f(H)\omega \rightsquigarrow y$. Hence, $alc_X H \neq \phi$. Let $z \in Y \backslash \{y\}$. Since Y is ω-Urysohn, there are open sets U_z of z and U_y of y such that $cl^\omega U_z \cap cl^\omega U_y = \phi$. There is $H \in H$ such that $f(H) \subseteq cl^\omega U_y$. For each $x \in f^{-1}(z)$, there is open V_x of x such that $f(cl^\omega V_x) \subseteq cl^\omega U_z$. So, $cl^\omega V_x \cap H = \phi$. It follows that $f^{-1}(z) \cap alc_X H = \phi$ for each $z \in Y \backslash \{y\}$. So, $alc_X H \cap f^{-1}(y) \neq \phi$ and f is ω-perfect.

Corollary(6.9): If $f : X \rightarrow Y$ is ω-continuous, X is quasi-ωH-closed, and Y is ω-Urysohn, then f is ω-perfect.

Proof: Since X is quasi-ωH-closed, then every filter base on X has nonvoid almost condensation; now, the corollary follows directly from theorem (6.3).

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