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Fibrewise Multi-Compact and Locally Multi- Compact Spaces

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ABSTRACT

The aim of the research is to apply fibrewise multi-emisssions of the paramount separation axioms of normally topology namely fibrewise multi-T0. spaces, fibrewise multi-T1 spaces, fibrewise multi-R0 spaces, fibrewise multi-Hausdorff spaces, fibrewise multi-functionally Hausdorff spaces, fibrewise multi-regular spaces, fibrewise multi-completely regular spaces, fibrewise multi-normal spaces and fibrewise multi-functionally normal spaces. Also we give many score regarding it.

1.Introduction

We beginning our work with the concept of category of Fibrewise (br$\boldsymbol{ie}$fly, 𝔽.𝕎.) sets on a known set, named the base set. If the base set is stated with D then 𝔽.𝕎. set on D apply of a set E with a function X is X: E → $\boldsymbol{D}$, named the projection (briefly, project.). For every point d of D the fibre on d is the subset $\boldsymbol{E_d = X^{-1}}$(d) of E; fibres will be empty let we do not require X to be surjection, also for every subset $\boldsymbol{D}^*$ o$\boldsymbol{f}$ D we regard $\boldsymbol{E_{D^*}} = \boldsymbol{X^{-1}}(\boldsymbol{D}^*)$ as a 𝔽.𝕎. set on $\boldsymbol{D}^*$ with the project. determined by X. A multi- function [2] Ω of a set E in to F is a correspondence such that Ω ($\boldsymbol{e}$) is a nonempty subset of F for every e ∈ E. We will denote such a multi- function by $\boldsymbol{\Omega}$: E → F . For a multi- function Ω, the upper and lower inverse set of a set K of F, will be denoted by Ω^+(K) and Ω^-(K) respectively that is Ω^+(K) = {$\boldsymbol{e}$ ∈ E : Ω($\boldsymbol{e}$) ⊆ K} and Ω^-(K) = {$\boldsymbol{e}$ ∈ E: Ω($\boldsymbol{e}$) ∩ V ≠ ∅ }.

Definition 1.1. [8] Suppose that E and F are 𝔽.𝕎. sets on D, with project. X_E: E → D and Y_F: F → D, respectively, a function Ω: E → F is named to be 𝔽.𝕎. if $Y_F O\Omega = X_E$, that is to say if $\Omega(X_d) \subset$ Fd for every point d of D.

It should be noted that a 𝔽.𝕎. function Ω: E → F on D determines, by restriction, 𝔽.𝕎. function ΩD*: ED*→ FD* on D* for every D* of D.

Let {Er} be an indexed family of 𝔽.𝕎. sets on D the 𝔽.𝕎. product $\prod_D E_r$ is stated, as a 𝔽.𝕎. set on D, and comes included with the family of 𝔽.𝕎. projection $\pi_r : \prod_D E_r \to E_r$. Specifically, the 𝔽.𝕎. product is stated as the subset of the normally product $\prod E_r$ where in the fibres are the products of the relevant fibers of the strain E_r. The 𝔽.𝕎. product is recognized with the following Cartesian property: for every 𝔽.𝕎. set E on D the 𝔽.𝕎. functions Ω: E → $\prod_r E_r$ correspond exactly to the families of 𝔽.𝕎. functions {Ω_r}, with $\Omega_r = \pi_r o\Omega$: E → Er. For example if Er = E for every index r the diagonal Δ : E → $\prod_D$ E, is stated so that $\pi_r o\Delta$ = idE for every r. If {Er} is as before, the 𝔽.𝕎. coproduct $\coprod_D$Er is with stated, as 𝔽.𝕎. set on D, and comes included with the family of F.W. insertions σ: $E_r \to \coprod_D E_r$, specifically the 𝔽.𝕎. coproduct synchronize, as a set, with the normally coproduct (saparated union), the fibres being the coproducts of the relevant fipers of the summands E_r. The 𝔽.𝕎. coproduct is recognized by the following Cartesian property, for every 𝔽.𝕎. set E on D the 𝔽.𝕎. functions φ: $\coprod_D E_r$ → E correspond exactly to the families of 𝔽.𝕎. functions {φ_r}, where in $\varphi_r = \varphi o\sigma_r$: E_r → E. For example, if E_r = E for every index r the codiagonal ∇: $\coprod_D$E → E is stated so that, $\nabla o\sigma_r$ = idE for every r. The notation $E\times_D$ F is used for the 𝔽.𝕎. product in the case of the family {E, F}, of two 𝔽.𝕎. sets and similarity for finite

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families generally. As well as, we builte on some of the result in [1,6,7-18]. For other concepts or informataon that are undefined here we follow nearly I.M.James [8], R.Engelking [7] and N. Bourbaki [6].

Recall that [8] Let D be topological space, the 𝔽.𝕎. topology space (briefly, 𝔽.𝕎.T.S.) on a 𝔽.𝕎. set E on D, mean any topology on E for that the project. X is continuous.

Remark 1.1. [8]

(a) The smaller topology is the topology trace with X, where in the open sets of E are exactly the pre image of the open sets of D, this is named the 𝔽.𝕎. indiscrete topology.

(b) The 𝔽.𝕎.T.S. on D is stated to be a 𝔽.𝕎. set on D with a 𝔽.𝕎.T.S.

We regard the topology product $D \times T$, for any topological space T, as a 𝔽.𝕎.T.S. on D using the category of first projection. The equivalences in the category of 𝔽.𝕎.T.S. are named 𝔽.𝕎.T. equivalences. If E is 𝔽.𝕎.T. equivalent to $D \times T$, for some topological space T, we say that E is trivial, as a 𝔽.𝕎.T.S. on D. In 𝔽.𝕎.T. the form neighborhood (briefly, ηℙⅆ) is used in exactly in the same sense as it is in normally topology, but the forms 𝔽.𝕎. basic may ned some illustration, so let E be 𝔽.𝕎.T.S. on D, if e is a point of Ed where in $d \in D$, appear a family N(e) of ηℙⅆ of e in E as 𝔽.𝕎. basic if as every ηℙⅆ H of e we have $Ew \cap K \subset H$, for some element K of N(e) and ηℙⅆ W of d in D. As exampe, in the case of the topological product $D \times T$, where in T is a topological spaces, the family of Cartesian products $D \times N(t)$, where in N(t) runs through the ηℙⅆs of t, is 𝔽.𝕎. basic for (d, t).

Definition 1.2. [8] The F.W. functions Ω: E→ F; E and F are 𝔽.𝕎. spaces on D is named:

(a) continuous (briefly, cont.) if every $e \in Ed$; $d \in D$, the $\Omega^{-1}(e)$ is open set of e.

(b) open if for every $e \in E_d$, $d \in D$, the direct image of every open set of e is an open set of $\Omega(e)$.

Definition 1.3. [8] The 𝔽.𝕎.T.S. E on D is named 𝔽.𝕎. closed (resp., open) if the project. X is closed (resp., open) functions.

Definition 1.4. [5] Let Ω: E → F be a multi-function. Then Ω is upper cont. (briefly, U. cont.) iff $\Omega^{+}(K)$ open in E for all V open in F. That is, $\Omega^{+}(K) = \{x \in E: \Omega(x) \subseteq K\}$. $K \subseteq F$.

Definition 1.5. [5] Let Ω: E → F be a multi-function. Then Ω is lower cont. (briefly, L. cont.) iff $\Omega^{-}(K)$ open in E for all K open in F. That is, $\Omega^{-}(K) = \{e \in E: \Omega(e) \cap K \neq \emptyset \}$. $K \subseteq F$

Let Ω: E → F be a multi-function. Then Ω is multi cont. (briefly, M. cont.) iff it is U. cont. and L. cont.

2. Fibrewise Multi-Compact and Locally Multi-Compact Spaces

In this segment we study 𝔽.𝕎. multi-compact and 𝔽.𝕎. locally multi-compact spaces as a generalizations of well-known ideas multi-compact and locally multi-compact topological spaces.

Definition 2.1. The function $\Omega : E \longrightarrow F$ is named upper proper (briefly, U.$\wp$.). If it is upper continuous, closed and $\forall f \in F, \Omega^{-1}(f)$ is compact set .

Definition 2.2. The function $\Omega : E \longrightarrow F$ is named lower proper (briefly, L.$\wp$.). If it is lower continuous, closed and $\forall f \in F, \Omega^{-1}(f)$ is compact set.

The function $\Omega : E \longrightarrow F$ is named multi-proper (briefly, M.$\wp$.). If it is U.$\wp$ and L.$\wp$..

For example, Assume that $(\mathbb{R}, \tau)$ where in τ is the topology with basic whose members are of the form (a, b) and $(a, b) - \mathbb{N}, \mathbb{N} = \{1\backslash n; n \in \mathbb{Z}^{+}\}$ and $E = \mathbb{N}$. Define $\Omega : (\mathbb{R}, \tau) \to (\mathbb{R}, \tau)$ by Ω (e) = e, so Ω is M.$\wp$. function.

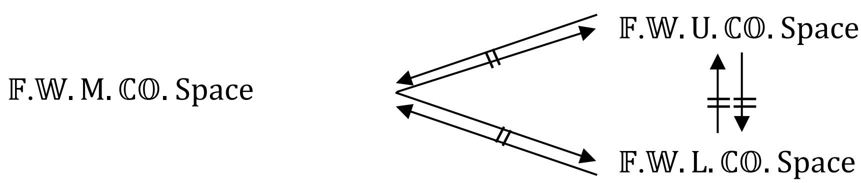
A function $\Omega : E \longrightarrow Y$ is a 𝔽.𝕎. and M.$\wp$. function, so Ω is named 𝔽.𝕎. M.$\wp$. function.

Definition 2.3. The $\mathbb{F}.\mathbb{W}.\mathrm{T.S.}$ E on D is named a $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{C}\mathbb{O}.$, when the projection function X is $\mathrm{U}.\wp.$.

Definition 2.4. The $\mathbb{F}.\mathbb{W}.\mathrm{T.S.}$ E on D is named a $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{C}\mathbb{O}.$, when the projection function X is $\mathrm{L}.\wp.$.

The $\mathbb{F}.\mathbb{W}.\mathrm{T.S.}$ E on D is named a $\mathbb{F}.\mathbb{W}.\mathrm{M}.\mathbb{C}\mathbb{O}.$ if it is $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{C}\mathbb{O}.$ and $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{C}\mathbb{O}.$.

Remark 2.1.
- (a) Every $\mathbb{F}.\mathbb{W}.\mathrm{M}.\mathbb{C}\mathbb{O}.$ space is $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{C}\mathbb{O}.$ space, but the convers is not true.
- (b) Every $\mathbb{F}.\mathbb{W}.\mathrm{M}.\mathbb{C}\mathbb{O}.$ space is $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{C}\mathbb{O}.$ space, but the convers is not true.
- (c) The $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{C}\mathbb{O}.$ space and $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{C}\mathbb{O}.$ space are independence.



Planned 2.1.

Example 2.1.

(a) Let $E = \{a,b,c\}$, $\tau_{(E)}$ = discrete topology. $D = \{1,2\}$, $\rho = \{\emptyset, D, \{1\}\}$. Define the project. $X: (E, \tau_{(E)}) \to (D,\rho)$ by $X(a) = X(b) = X(c) = \{1\}$

1. E is $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{C}\mathbb{O}$.S., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{C}\mathbb{O}$., and $\mathbb{F}.\mathbb{W}.\mathrm{M}.\mathbb{C}\mathbb{O}$.S.
2.

(b) Let $E = N$, with the cofinite topology τ_{cof} and let $D = \{a,b,c\}$ with the topology $\rho = \{\emptyset, D, \{a\}, \{a,b\}\}$. Define multi-function

3. $X : (\mathbb{R},\tau) \to (D,\rho)$ by $X(e) = \begin{cases} \{a\}; & e \le 0 \\ \{a,c\}; & e > 0 \end{cases}$

4. E is $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{C}\mathbb{O}$.S., but not $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{C}\mathbb{O}$.S. and not $\mathbb{F}.\mathbb{W}.\mathrm{M}.\mathbb{C}\mathbb{O}$.S.

5.

(c) Let $E = \{a,b,c\}$, $\tau_{(E)}$= discrete topology and $D = \{a,b\}$ with the topology $\rho = \{\emptyset, D, \{a\}\}$. Define multi-function

6. $X : (\mathbb{R},\tau) \to (D,\rho)$ by $X(e) = \begin{cases} \{a\}; & e \le 0 \\ \emptyset; & e > 0 \end{cases}$

7. E is $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{C}\mathbb{O}$.S., but not $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{C}\mathbb{O}$.S. and not $\mathbb{F}.\mathbb{W}.\mathrm{M}.\mathbb{C}\mathbb{O}$.S.
8.

(d) Let $E = \mathbb{R}$ with the usual topology τ and $D = \{a,b,c\}$ with the topology $\rho = \{\emptyset, D, \{a\}\}$. Define multi-function

9. $X : (\mathbb{R},\tau) \to (D,\rho)$ by $X(e) = \begin{cases} \{a\}; & e < 0 \\ \{a,b\}; & e = 0 \\ \{c\}; & e > 0 \end{cases}$

10. E is not $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{C}\mathbb{O}$.S., not $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{C}\mathbb{O}$.S. and not $\mathbb{F}.\mathbb{W}.\mathrm{M}.\mathbb{C}\mathbb{O}$.S.

Proposition 2.1. The $\mathbb{F}.\mathbb{W}.\mathrm{T.S.}$ E on D is $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{C}\mathbb{O}.$ (resp., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{C}\mathbb{O}.$) iff E is a $\mathbb{F}.\mathbb{W}.$closed and every fibre of E is $\mathbb{C}\mathbb{O}.$.

Proof. ($\Longrightarrow$) Let E be a $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{CO}.$ (resp., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{CO}.$) space, so the projection function $\mathrm{X}: \mathrm{E} \to \mathrm{D}$ is $\mathrm{U}.\wp.$ (resp., $\mathrm{L}.\wp.$) function(i.e., X is a closed and for every $\mathrm{d} \in \mathrm{D}$, $\mathrm{E_d}$ is $\mathbb{CO}$.., So E is an $\mathbb{F}.\mathbb{W}.$ closed and all fibre of E is $\mathbb{CO}$..

($\Longleftarrow$) Let E be $\mathbb{F}.\mathbb{W}.$ closed and all fibre d of D, Ed is $\mathbb{CO}$., therefore the projection function $\mathrm{X} : \mathrm{E} \to \mathrm{D}$ is a closed and X is U. continuous (resp., L. continuous), and for every $\mathrm{d} \in \mathrm{D}$, $\mathrm{E_d}$ is $\mathbb{CO}$.. So E is $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{CO}.$ (resp., F.W.L.CO.).

Corollary 2.1. The $\mathbb{F}.\mathbb{W}.\mathrm{T.S.}$ E on D is $\mathbb{F}.\mathbb{W}.\mathrm{M}.\mathbb{CO}.$ iff E is a $\mathbb{F}.\mathbb{W}.$ closed and every fibre of E is $\mathbb{CO}$.

Proposition 2.2. Let E be a $\mathbb{F}.\mathbb{W}.\mathrm{T.S.}$ on D. Then E is $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{CO}.$ (resp., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{CO}.$) iff for every fibre $\mathrm{E_d}$ of E and every covering Γ of $\mathrm{E_d}$ by open sets of E there exists a ηℙ𝕕 W of d such that, a finite subfamily of Γ covers $\mathrm{E}_{\mathcal{W}}$.

Proof. ($\Longrightarrow$) Let E be a $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{CO}.$ (resp$\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{CO}.$) space, thus the projection function $\mathrm{X}: \mathrm{E} \to \mathrm{D}$ is $\mathrm{U}.\wp.$ (resp., $\mathrm{L}.\wp.$) function, so that $\mathrm{E_d}$ is $\mathbb{F}.\mathbb{W}.\mathbb{CO}.$) for every $\mathrm{d} \in \mathrm{D}$. Assume that Γ is a covering of $\mathrm{E_d}$ in open sets of E for every $\mathrm{d} \in \mathrm{D}$ and let $\mathrm{E_W} = \cup\, \mathrm{E_d}$ for all $\mathrm{d} \in \mathcal{W}$. Since $\mathrm{E_d}$ is $\mathbb{F}.\mathbb{W}.\mathbb{CO}.$ for every $\mathrm{d} \in \mathcal{W} \in \mathrm{D}$ and the union of $\mathbb{F}.\mathbb{W}.\mathbb{CO}.$ sets is a $\mathbb{F}.\mathbb{W}.\mathbb{CO}.$, but $\mathrm{E_W}$ is a $\mathbb{F}.\mathbb{W}.\mathbb{CO}.$. So, there exists a ηℙ𝕕 W of d such that a finite subfamily of Γ covers $\mathrm{E_W}$.

($\Longleftarrow$) Let E be $\mathbb{F}.\mathbb{W}.\mathrm{T.S.}$ on D, thus the projection function $\mathrm{X}: \mathrm{E} \to \mathrm{D}$ exist. T.P. X is $\mathrm{U}.\wp.$ (resp., $\mathrm{L}.\wp.$) . So X is U. continuous (resp., L. continuous) and for all $\mathrm{d} \in \mathrm{D}$, $\mathrm{E_d}$ is $\mathcal{F}.\mathcal{W}.\mathbb{CO}.$)by taking $\mathrm{E_d} = \mathrm{E_W}$. By Proposition (2.1), therefore X is closed. So, X is $\mathrm{U}.\wp.$ (resp., $\mathrm{L}.\wp.$) and E is $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{CO}.$ (resp., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{CO}.$).

Corollary 2.2. Let E be a $\mathbb{F}.\mathbb{W}.\mathrm{T.S.}$ on D. Then E is $\mathbb{F}.\mathbb{W}.\mathrm{M}.\mathbb{CO}.$ iff for every fibre $\mathrm{E_d}$ of E and every covering Γ of $\mathrm{E_d}$ by open sets of E there exists a ηℙ𝕕 W of d such that, a finite subfamily of Γ covers $\mathrm{E}_{\mathcal{W}}$.

Proposition 2.3. Let $\Omega : \mathrm{E} \longrightarrow \mathrm{F}$ be $\mathrm{U}.\wp.$ (resp., $\mathrm{L}.\wp.$), closed $\mathbb{F}.\mathbb{W}.$ function, where in E and F are $\mathbb{F}.\mathbb{W}.\mathrm{T.S.}$ on D. If F is $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{CO}.$ (resp., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{CO}.$) then, so is E.

Proof. Assume that$\Omega : \mathrm{E} \longrightarrow \mathrm{F}$ is $\mathrm{U}.\wp.$ (resp., $\mathrm{L}.\wp.$), closed $\mathbb{F}.\mathbb{W}.$ function and F is F is $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{CO}.$ (resp., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{CO}.$) space i.e., the projection function $\mathrm{X_F}: \mathrm{F} \to \mathrm{D}$ is a $\mathrm{U}.\wp.$ (resp., $\mathrm{L}.\wp.$). T.P. E is $\mathcal{F}.\mathcal{W}.\mathrm{U}.\mathbb{CO}.$ (resp., $\mathcal{F}.\mathcal{W}.\mathrm{L}.\mathbb{CO}.$) space i.e., the projection function $\mathrm{X_E}: \mathrm{E} \to \mathrm{D}$ is $\mathrm{U}.\wp.$ (resp., $\mathrm{L}.\wp.$). So, it is obvious that $\mathrm{X_E}$ is U. continuous (resp., L. continuous), let H be a closed. subset of $\mathrm{X}_{\tilde{\mathrm{d}}}$, where $\mathrm{d} \in \mathrm{D}$. However, Ω is a closed, so Ω (H) is a closed subset of $\mathrm{F_d}$. By $\mathrm{X_E}$ is a closed, so $\mathrm{X_E}\big(\Omega\,(\mathrm{H})\big)$ is a closed in D. However $\mathrm{E}\big(\Omega\,(\mathrm{H})\big) = (\mathrm{X_E} \mathrm{o}\, \Omega)(\mathrm{H}) = \mathrm{X_E}(\mathrm{H})$ is closed in D ,so $\mathrm{X_E}$ is a closed. Let $\mathrm{d} \in \mathrm{D}$, since $\mathrm{X_E}$ is $\mathrm{U}.\wp.$ (resp., $\mathrm{L}.\wp.$), so $\mathrm{F_d}$ is $\mathbb{CO}$.). Now let $\{\mathrm{U}_{\imath}: \imath \in \Lambda\}$ be a family of open sets of E such that, $\mathrm{F_d} \subset \mathrm{U}_{\imath\in\Lambda} \mathrm{U}_{\imath}$. If $\mathrm{f} \in \mathrm{F_d}$, subsistent a finite subset M(f) of Λ such that $\Omega^{-1}(\mathrm{f}) \subset \cup_{\imath\in\mathcal{M}(d)} \mathrm{U_i}$. Because Ω is closed function, so by Proposition (2.2) subsistent a open set $(\mathcal{V})_{\mathrm{f}}$ of F such that $\mathrm{f} \in (\mathcal{V})_{\mathrm{f}}$ and $\Omega^{-1}((\mathcal{V})_{\mathrm{f}}) \subset \cup_{\imath\in\mathcal{M}(\mathrm{f})} \mathrm{U}_{\imath}$. By $\mathrm{F_f}$ is $\mathbb{CO}$., subsistent a finite subset K of $\mathrm{F_f}$ such that, $\mathrm{F_d} \subset \cup_{\mathrm{f}\in\mathrm{K}} (\mathcal{V})_{\mathrm{f}}$. So $\Omega^{-1}(\mathrm{F_d}) \subset \cup_{\mathrm{f}\in\mathrm{K}} \Omega^{-1}(\mathcal{V})_{\mathrm{f}} \subset \cup_{\mathrm{f}\in\mathrm{K}} \cup_{\imath\in\mathcal{M}(\mathrm{f})} \mathcal{U}_{\mathrm{i}}$. Thus if $\mathcal{M} = \cup_{\mathrm{f}\in\mathrm{K}} \mathcal{M}(\mathrm{f})$, then $\mathcal{M}$ is a finite subset of Λ and $\Omega^{-1}(\mathrm{F_d}) \subset \cup_{\imath\in\mathcal{M}} \mathrm{U_i}$. Then $\Omega^{-1}(\mathrm{F_d}) = \Omega^{-1}\big(\mathrm{X_F^{-1}}(\mathrm{d})\big) = (\mathrm{X_F} \mathrm{o} \Omega)^{-1}(\mathrm{d}) = \mathrm{X_E^{-1}}(\mathrm{d}) = \mathrm{E_d}$ and $\mathrm{E_d} \subset \cup_{\imath\in\mathcal{M}} \mathrm{U}_{\imath}$, then $\mathrm{E_d}$ is a $\mathbb{CO}$.. Therefore, $\mathrm{X_E}$ is $\mathrm{U}.\wp.$ (resp., $\mathrm{L}.\wp.$) and E is a $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{CO}.$ (resp., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{CO}.$).

Corollary 2.3. Let $\Omega : \mathrm{E} \longrightarrow \mathrm{F}$ be $\mathrm{M}.\wp.$, closed $\mathbb{F}.\mathbb{W}.$ function, where in E and F are $\mathbb{F}.\mathbb{W}.\mathrm{T.S.}$ on D. If F is $\mathbb{F}.\mathbb{W}.\mathrm{M}.\mathbb{CO}.$then, so is E.

The category of $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{CO}.$ (resp., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{CO}.$ and $\mathbb{F}.\mathbb{W}.\mathrm{M}.\mathbb{CO}.$) spaces is finitely multiplicative as mentioned in:

Proposition 2.4. Suppose that $\{\mathrm{E_r}\}$ be a family of $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{CO}.$ (resp., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{CO}.$) spaces on D. Therefore after the $\mathbb{F}.\mathbb{W}.\mathrm{T}.$ product $\mathrm{E} = \prod_{\mathrm{D}} \mathrm{E_r}$, is $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{CO}.$ (resp., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{CO}.$) .

Proof. Let $\{\mathrm{E_r}\}$ and F are $\mathbb{F}.\mathbb{W}.\mathrm{T.S.}$ on D. When E is $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{CO}.$ (resp., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{CO}.$) , so the projection function $\mathrm{X} \times \mathrm{id_F}: \mathrm{E} \times_{\mathrm{D}} \mathrm{F} \equiv \mathrm{F}$ is $\mathrm{U}.\wp.$ (resp., $\mathrm{L}.\wp.$). When F is also $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{CO}.$ (resp., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{CO}.$) , therefore is $\mathrm{E} \times_{\mathfrak{B}} \mathrm{F}$ by Proposition (2.3).

Corollary 2.4. Suppose that $\{\mathrm{E_r}\}$ be a family of $\mathbb{F}.\mathbb{W}.\mathrm{M}.\mathbb{CO}.$ spaces on D. Therefore after the $\mathbb{F}.\mathbb{W}.\mathrm{T}.$ product $\mathrm{E} = \prod_{\mathrm{D}} \mathrm{E_r}$,is $\mathbb{F}.\mathbb{W}.\mathrm{M}.\mathbb{CO}.$.

Proposition 2.5. Let E is $\mathbb{F}.\mathbb{W}.\mathrm{T.S.}$ on D. Suppose that $\mathrm{E_i}$ is $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{CO}.$ (resp., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{CO}.$) for every member Ei of a finite covering of E. Then E is $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{CO}.$ (resp., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{CO}.$).

Proof. Let E be a $\mathbb{F}.\mathbb{W}.\mathrm{T.S.}$ on D. Currently, the projection function $\mathrm{X} : \mathrm{E} \to \mathrm{D}$ subsistent. T.P. X is $\mathrm{U}.\wp.$ (resp., $\mathrm{L}.\wp.$). Currently, it is obvious that X is U. continuous (resp., L. continuous). Since $\mathrm{E}_{\imath}$ is $\mathcal{F}.\mathcal{W}.\mathrm{U}.\mathbb{CO}.$ (resp., $\mathcal{F}.\mathcal{W}.\mathrm{L}.\mathbb{CO}.$), then the projection function $\mathrm{X}_{\imath}: \mathrm{E_i} \to \mathrm{D}$ is a closed. and for all $\mathrm{d} \in \mathrm{D}$, $(\mathrm{E_i})_{\mathrm{d}}$ is $\mathbb{CO}$. for every member $\mathrm{E}_{\imath}$ of a finite covering of E Assume that H is a closed. subset of E, then $\mathrm{X}(\mathrm{H}) = \cup\, \mathrm{X_i}(\mathrm{E_i} \cap \mathrm{H})$ which is a finite union of closed sets and so X is a closed. Assume that $\mathrm{d} \in \mathrm{D}$, then $\mathrm{E_d} = \cup\, (\mathrm{E}_{\imath})_{\mathrm{d}}$ which is a finite union of $\mathbb{CO}$. sets and so $\mathrm{E_d}$ is a $\mathbb{CO}$..Thus, X is $\mathrm{U}.\wp.$ (resp., $\mathrm{L}.\wp.$) and E is $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{CO}.$ (resp., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{CO}.$).

Corollary 2.5. Let E is $\mathbb{F}.\mathbb{W}.\mathrm{T.S.}$ on D. Suppose that E_i is $\mathbb{F}.\mathbb{W}.\mathrm{M}.\mathbb{CO}.$ for every member Ei of a finite covering of E. Then E is $\mathbb{F}.\mathbb{W}.\mathrm{M}.\mathbb{CO}.$.

Proposition 2.6. Let E be $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{CO}.$ (resp., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{CO}.$) space on D. So E_{D^*} is $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{CO}.$ (resp., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{CO}.$) space on D^* for every subspace D^* of D.

Proof. Suppose that E is $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{CO}.$ (resp., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{CO}.$) i.e., the projection function $X_E : E \to D$ is $\mathrm{U}.\wp.$ (resp., $\mathrm{L}.\wp.$) To show that E_{D^*} is $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{CO}.$ (resp., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{CO}.$) space over D^* i.e., the projection function $X_{D^*} : E_{D^*} \to D^*$ is $\mathrm{U}.\wp.$ (resp., $\mathrm{L}.\wp.$) Currently, it is obvious that X_{D^*} is U. continuous(resp., L. continuous). Assume that H is a closed subset of E, then $H \cap E_{D^*}$ is a closed in a subspace E_{D^*} and $X_{D^*}(H \cap E_{D^*}) = X(H \cap D^*)$ which is closed set in D^*, so X is a closed. Let $d \in D$, therefore $(E_{D^*})_d = E_d \cap E_{D^*}$ which is a $\mathbb{CO}.$ set in E_{D^*}. So, X_{D^*} is $\mathrm{U}.\wp.$ (resp., $\mathrm{L}.\wp.$) and E_{D^*} is $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{CO}.$ (resp., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{CO}.$) over D^*.

Corollary 2.6. Let E be $\mathbb{F}.\mathbb{W}.\mathrm{M}.\mathbb{CO}.$ space on D. So E_{D^*} is $\mathbb{F}.\mathbb{W}.\mathrm{M}.\mathbb{CO}.$ space on D^* for every subspace D^* of D.

Proposition 2.7. Let E be a $\mathbb{F}.\mathbb{W}.\mathrm{T.S}$ on D Suppose that (E_{D_i}) is $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{CO}.$ (resp., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{CO}.$) on (D_i) for every member (D_i) of an open covering of D. Then E is $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{CO}.$ (resp., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{CO}.$) on D.

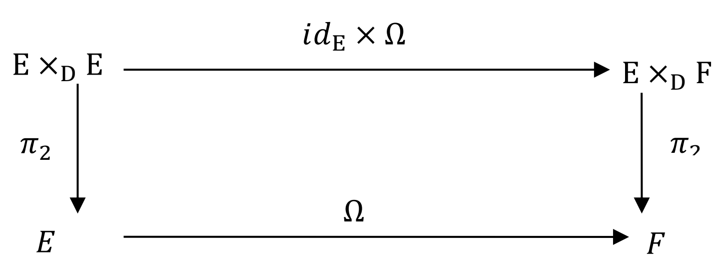
Proof. Suppose that E is $\mathbb{F}.\mathbb{W}.\mathrm{T.S.}$ on D, then the projection function $X_E : E \to D$ subsistent. T.P. X_E is $\mathrm{U}.\wp.$ (resp., $\mathrm{L}.\wp.$). Currently, it is obvious that X_E is U. continuous (resp., L. continuous). Since E_{D_i} is $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{CO}.$ (resp., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{CO}.$) on D_i, therefore the projection function $X_{D_i} : E_{D_i} \to D_i$ is $\mathrm{U}.\wp.$ (resp., $\mathrm{L}.\wp.$)for all member D_i of an open covering of D Assume that H is a closed subset of E, then we have $X_E(H) = \cup X_{D_i(E)}(E_{D_i} \cap H)$ which is a union of closed sets and so X_E is a closed Suppose that $d \in D$ then $E_d = \cup (E_{D_i})_d$ for every $d = \{d_i\} \in D_i$. Since $E_{D_{i_d}}$ is $\mathbb{CO}.$ in E_{D_i} and the union of $\mathbb{CO}.$ sets is $\mathbb{CO}.$, we have E_d is a $\mathbb{CO}.$. So, X_E is a $\mathrm{U}.\wp.$ (resp., $\mathrm{L}.\wp.$) and E is a $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{CO}.$ (resp., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{CO}.$).

Corollary 2.7. Let E be a $\mathbb{F}.\mathbb{W}.\mathrm{T.S}$ on D Suppose that (E_{D_i}) is $\mathbb{F}.\mathbb{W}.\mathrm{M}.\mathbb{CO}.$ on (D_i) for every member (D_i) of an open covering of D. Then E is $\mathbb{F}.\mathbb{W}.\mathrm{M}.\mathbb{CO}.$ on D.

Actually, the final result is also holds for locally finite closed coverings, instead of open coverings.

Proposition 2.8. A function $\Omega : E \longrightarrow F$ is a $\mathbb{F}.\mathbb{W}.$ function, where E and F are $\mathbb{F}.\mathbb{W}.\mathrm{T.S.}$ on D. If E is $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{CO}.$ (resp., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{CO}.$) and $id_E \times \Omega : E \times_D E \to E \times_D F$ is $\mathrm{U}.\wp.$ (resp., $\mathrm{L}.\wp.$) and closed, then Ω is $\mathrm{U}.\wp.$ (resp., $\mathrm{L}.\wp.$).

Proof. Regard the commutative figure shown below



Planned 2.2.

If E is $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{CO}.$ (resp., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{CO}.$), so π_2 is $\mathrm{U}.\wp.$ (resp., $\mathrm{L}.\wp.$). Condition $id_E \times \Omega$ is additionally $\mathrm{U}.\wp.$ (resp., $\mathrm{L}.\wp.$) and closed then $\pi_2 \circ (id_E \times \Omega) = \Omega \circ \pi_2$ is $\mathrm{U}.\wp.$ (resp., $\mathrm{L}.\wp.$), and so Ω itself is$\mathrm{U}.\wp.$ (resp., $\mathrm{L}.\wp.$).

Corollary 2.8. A function $\Omega : E \longrightarrow F$ is a $\mathbb{F}.\mathbb{W}.$ function, where E and F are $\mathbb{F}.\mathbb{W}.\mathrm{T.S.}$ on D. If E is $\mathbb{F}.\mathbb{W}.\mathrm{M}.\mathbb{CO}.$ and $id_E \times \Omega : E \times_D E \to E \times_D F$ is $\mathrm{M}.\wp.$ and closed, then Ω is $\mathrm{M}.\wp.$

The next new concept in this segment is given by the following:

Definition 2.5. A $\mathbb{F}.\mathbb{W}.\mathrm{T.S.}$ E on D is named $\mathbb{F}.\mathbb{W}.$locally upper compact (briefly, $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{U}.\mathbb{CO}.$) if for every point e of E_d, where in $d \in D$, subsistent a $\eta\mathbb{Pd}$ $\mathcal{W}$ of d and an open set $U \subset E_{\mathcal{W}}$ of e such that, the closure of U in $E_{\mathcal{W}}$ (i.e., $E_{\mathcal{W}} \cap Cl(U)$) is $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{CO}.$ on $\mathcal{W}$.

Definition 2.6. A $\mathbb{F}.\mathbb{W}.\mathrm{T.S.}$ E on D is named $\mathbb{F}.\mathbb{W}$.locally lower compact (briefly, $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{L}.\mathbb{CO}$.) if for every point e of E_d, where in $d \in D$, subsistent a $\eta\mathbb{P}\mathbb{d}$ $\mathcal{W}$ of d and an open set $U \subset E_{\mathcal{W}}$ of e such that, the closure of U in $E_{\mathcal{W}}$ (i.e., $E_{\mathcal{W}} \cap \mathrm{Cl}(U)$) is $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{CO}$. on $\mathcal{W}$.

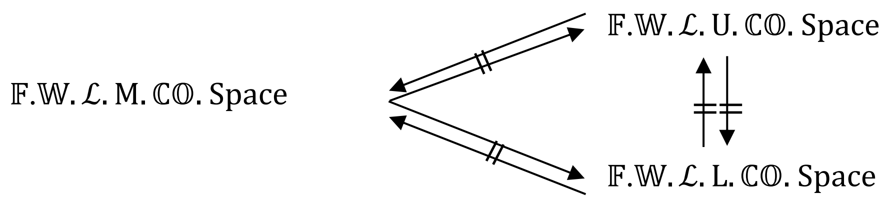
The $\mathbb{F}.\mathbb{W}.\mathrm{T.S.}$ E on D is named $\mathbb{F}.\mathbb{W}$. locally multi-compact (briefly, $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{M}.\mathbb{CO}$.) if it is $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{U}.\mathbb{CO}$. and $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{L}.\mathbb{CO}$..

Remark 2.2.

(a) Every $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{M}.\mathbb{CO}$. space is $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{U}.\mathbb{CO}$. space, but the convers is not true.

(b) Every $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{M}.\mathbb{CO}$. space is $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{L}.\mathbb{CO}$. space, but the convers is not true.

(c) The $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{U}.\mathbb{CO}$. space and $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{L}.\mathbb{CO}$. space are independence.



Planned 2.3.

Example 2.2.

(a) Let $E = \{a, b, c\}$, $\tau_{(E)}$ = discrete topology. $D = \{1,2\}$, $\rho = \{\emptyset, D, \{1\}\}$. Define the project. $X: (E, \tau_{(E)}) \to (D, \rho)$ by $X(a) = X(b) = X(c) = \{1\}$

E is $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{U}.\mathbb{CO}$.S., $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{L}.\mathbb{CO}$., and $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{M}.\mathbb{CO}$.S.

(b) Let $E = \mathbb{R}$,with the usual topology τ and let $D = \{a, b, c\}$ with the opology $\rho = \{\emptyset, D, \{a\}, \{a, b\}\}$. Define multi-function

$$X : (\mathbb{R}, \tau) \to (D, \rho) \text{ by } X(e) = \begin{cases} \{a\} ; & e \leq 0 \\ \{a, c\} ; & e > 0 \end{cases}$$

E is $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{L}.\mathbb{CO}$.S., but not $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{U}.\mathbb{CO}$.S. and not $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{M}.\mathbb{CO}$.S.

(c) Let E is infinite set with $\tau_{(E)}$= discrete topology and $D = \{a, b\}$ with the topology $\rho = \{\emptyset, D, \{a\}\}$. Define multi-function

$$X : (\mathbb{R}, \tau) \to (D, \rho) \text{ by } X(e) = \begin{cases} \{a\} ; & e \leq 0 \\ \emptyset ; & e > 0 \end{cases}$$

E is $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{U}.\mathbb{CO}$.S., but not $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{L}.\mathbb{CO}$.S. and not $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{M}.\mathbb{CO}$.S.

(d) Let $E = \mathbb{R}$ with the usual topology τ and $D = \{a, b, c\}$ with the opology$\rho = \{\emptyset, D, \{a\}\}$. Define multi-function

$$X : (\mathbb{R},\tau) \to (D,\rho) \text{ by } X(e) = \begin{cases} \{a\}\,;\ e < 0 \\ \{a,b\}\,;\ e = 0 \\ \{c\}\,;\ e > 0 \end{cases}$$

E is not $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{U}.\mathbb{CO}$.S., not $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{L}.\mathbb{CO}$.S. and not $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{M}.\mathbb{CO}$.S.

Remark 2.3. $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{CO}.$ (resp., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{CO}.$ and $\mathbb{F}.\mathbb{W}.\mathrm{M}.\mathbb{CO}.$) spaces are necessarily $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{U}.\mathbb{CO}.$(resp., $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{L}.\mathbb{CO}.$ and $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{M}.\mathbb{CO}.$) by taking $\mathcal{W} = \mathrm{D}$ and $\mathrm{E}_{\mathcal{W}} = \mathrm{E}$. , however the reverse does not need to be correct, as the following example.

Example2.3. Assume that $(\mathrm{E},\tau_{\mathrm{dis}})$ where E is infinite set and τ is discrete topology, thus $(\mathrm{E},\sigma\tau_{\mathrm{dis}})$ $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{M}.\mathbb{CO}.$ on $\mathbb{R}$, since for all e of E_{d}, where $\mathrm{d} \in \mathrm{D}$, subsistent a $\eta\mathbb{P}\mathbb{d}$ $\mathcal{W}$ of d and an open set $\mathrm{U} \subset \mathrm{E}_{\mathcal{W}}$ of e such that, the closure of U in $\mathrm{E}_{\mathcal{W}}$ (i.e., $\mathrm{E}_{\mathcal{W}} \cap \mathrm{Cl(U)}$) is $\mathcal{F}.\mathcal{W}.\mathcal{L}.\mathrm{M}.\mathbb{CO}.$ on $\mathbb{R}$. Also the product space $\mathrm{D} \times \mathrm{T}$ is $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{U}.\mathbb{CO}.$(resp., $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{L}.\mathbb{CO}.$ and $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{M}.\mathbb{CO}.$) on D,for all $\mathcal{L}.\mathrm{U}.\mathbb{CO}.$(resp., $\mathcal{L}.\mathrm{L}.\mathbb{CO}.$ and $\mathcal{L}.\mathrm{M}.\mathbb{CO}.$)space T.

Closed subspaces of $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{U}.$ (resp., $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{L}.$) spaces are $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{U}.$ (resp., $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{L}.$) spaces,. Actually we have.

Proposition 2.9. A function $\Omega : \mathrm{E} \longrightarrow \mathrm{E}^*$ is a closed $\mathbb{F}.\mathbb{W}$.embedding, where E and E* are $\mathbb{F}.\mathbb{W}.\mathrm{T.S.}$ on D. E is $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{U}.\mathbb{CO}.$(resp., $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{L}.\mathbb{CO}.$)when E* is $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{U}.\mathbb{CO}.$(resp., $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{L}.\mathbb{CO}.$).

Proof. Let e of E_{d}, where $\mathrm{d} \in \mathrm{D}$, subsistent a $\eta\mathbb{P}\mathbb{d}$ $\mathcal{W}$ of d and an open set $\mathrm{U} \subset \mathrm{E}_{\mathcal{W}}$ of e such that, the closure of U in $\mathrm{E}_{\mathcal{W}}$ (i.e., $\mathrm{E}_{\mathcal{W}} \cap \mathrm{Cl(U)}$) is $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{U}.\mathbb{CO}.$(resp., $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{L}.$) on $\mathcal{W}$. Then $\Omega^{-1}(\mathrm{U}) \subset \mathrm{E}_{\mathcal{W}}$ is an open set of e such that, the closure $\mathrm{E}_{\mathcal{W}} \cap \mathrm{Cl}\big(\Omega^{-1}(\mathrm{U})\big) = \Omega^{-1}(\mathrm{E}^*_{\mathcal{W}} \cap \mathrm{Cl(U)})$ of $\Omega^{-1}(\mathrm{U})$ in $\mathrm{E}_{\mathcal{W}}$ is $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{U}.\mathbb{CO}.$(resp., $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{L}.$) on $\mathcal{W}$. Therefore, E is $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{U}.\mathbb{CO}.$(resp., $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{L}.$).

Corollary 2.9. A function $\Omega : \mathrm{E} \longrightarrow \mathrm{E}^*$ is a closed $\mathbb{F}.\mathbb{W}.$ embedding, where E and E* are $\mathbb{F}.\mathbb{W}.\mathrm{T.S.}$ on D. E is $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{M}.\mathbb{CO}.$ when E* is $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{M}.\mathbb{CO}.$

The category of $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{U}.\mathbb{CO}.$(resp., $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{L}.$). spaces is finitely multiplicative as mentioned in . .

Proposition 2.10. Let $\{\mathrm{E_i}\}$ be finite family of $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{U}.\mathbb{CO}.$(resp., $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{L}.$) spaces onD. Then the $\mathbb{F}.\mathbb{W}.\mathrm{T}.$ product $\mathrm{E} = \prod_{\mathrm{D}}(\mathrm{E_i})$ is $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{U}.\mathbb{CO}.$(resp., $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{L}.$).

11. Proof. In the same way of proof of Proposition (4).

Corollary 2.10. Let $\{\mathrm{E_i}\}$ be finite family of $\mathcal{F}.\mathcal{W}.\mathcal{L}.\mathrm{M}.\mathbb{CO}.$ spaces onD. Then the $\mathcal{F}.\mathcal{W}.\mathrm{T}.$ product $\mathrm{E} = \prod_{\mathrm{D}}(\mathrm{E_i})$ is $\mathcal{F}.\mathcal{W}.\mathcal{L}.\mathrm{M}.\mathbb{CO}.$

3. Fibrewise Multi-Compact (resp., Locally Multi-Compact) Spaces and Some Fibrewise Multi-Separation Axioms

Now we give a series of results in which give relationships between F.W.multi-compactness (F.W. locally multi-compactness in some cases) and some F.W. multi-separation axioms which are discussed in [8,9].

Definition 3.1.[9] The $\mathbb{F}.\mathbb{W}$.T.S. E on D is amed $\mathbb{F}.\mathbb{W}$. upper Hausdorff (briefly, $\mathbb{F}.\mathbb{W}$.U. Hausd.) if whenever e1,e2 $\in \mathrm{E}^+_{\mathrm{d}}$, where in $\mathrm{d} \in \mathrm{D}$ and e1 $\neq$ e2, there exist separated open sets U1, U2 of e1, e2 in E.

Definition 3.2.[9] The $\mathbb{F}.\mathbb{W}$.T.S. E on D is amed $\mathbb{F}.\mathbb{W}$. lower Hausdorff (briefly, $\mathbb{F}.\mathbb{W}$.L. Hausd.) if whenever e1,e2 $\in \mathrm{E}^-_{\mathrm{d}}$, where in $\mathrm{d} \in \mathrm{D}$ and e1 $\neq$ e2, there exist separated open sets U1, U2 of e1, e2 in E.

The $\mathbb{F}.\mathbb{W}$.T.S. E on D is amed $\mathbb{F}.\mathbb{W}$. multi-Hausdorff (briefly, $\mathbb{F}.\mathbb{W}$.M. Hausd.) if E is $\mathbb{F}.\mathbb{W}$.U. Hausd. and $\mathbb{F}.\mathbb{W}$.L. Hausd..

Definition 3.3.[9] The $\mathbb{F}.\mathbb{W}$.T.S. E on D is amed $\mathbb{F}.\mathbb{W}$. upper regular (briefly, $\mathbb{F}.\mathbb{W}$.U. re.) if for every point e ∈ E_d^+, where in d ∈ D, and for every open set V of e in E, there exists a η$\mathbb{P}\mathbb{d}$ W of d in D and an open set U of e in E_w^+ such that closure of U in is E_w^+ancluding in V (i.e. $E_w^+ \cap Cl(U) \subset V$).

Definition 3.4.[9] The $\mathbb{F}.\mathbb{W}$.T.S. E on D is amed $\mathbb{F}.\mathbb{W}$. lower regular (briefly, $\mathbb{F}.\mathbb{W}$.L. re.) if for every point e ∈ E_d^-, where in d ∈ D, and for every open set V of e in E, there exists a η$\mathbb{P}\mathbb{d}$ W of d in D and an open set U of e in E_w^- such that closure of U in is E_w^-ancluding in V (i.e. $E_w^- \cap Cl(U) \subset V$).

The $\mathbb{F}.\mathbb{W}$.T.S. on D is amed $\mathbb{F}.\mathbb{W}$. multi- regular (briefly, $\mathbb{F}.\mathbb{W}$.M. re.), if E is $\mathbb{F}.\mathbb{W}$.U. re and $\mathbb{F}.\mathbb{W}$ L. re.

Definition 3.5.[9] The $\mathbb{F}.\mathbb{W}$.T.S. E on D is amed $\mathbb{F}.\mathbb{W}$. upper normal (briefly, F.W.U. no.) if for every point d of D and every pair H, K of separated closed sets of E, there exist a η$\mathbb{P}\mathbb{d}$ W of d and a pair of separated open sets U, V of $E_w^+ \cap H$, $E_w^+ \cap K$ in E_w^+.

Definition 3.6.[9] The $\mathbb{F}.\mathbb{W}$.T.S. E on D is amed $\mathbb{F}.\mathbb{W}$. lower normal (briefly, F.W.L. no.) if for every point d of D and every pair H, K of separated closed sets of E, there exist a η$\mathbb{P}\mathbb{d}$ W of d and a pair of separated open sets U, V of $E_w^- \cap H$, $E_w^- \cap K$ in E_w^-.

The $\mathbb{F}.\mathbb{W}$.T.S. on D is amed $\mathbb{F}.\mathbb{W}$. multi-normal (briefly, F.W.M. no.), if E is $\mathbb{F}.\mathbb{W}$.U. no. and F.W.L. no.

Proposition 3.1. Suppose that E be $\mathbb{F}.\mathbb{W}.\mathcal{L}.U.\mathbb{C}\mathbb{O}.$(resp., $\mathbb{F}.\mathbb{W}.\mathcal{L}.L.\mathbb{C}\mathbb{O}.$) and $\mathbb{F}.\mathbb{W}$.U. re. (resp., $\mathbb{F}.\mathbb{W}$.L. re.) on D. Then for every point e of Ed, where in d ∈ D, and every open set V of e in E, there exists an open set U of e in E_W^+(resp., E_W^-) such that the closure $E_W^+ \cap Cl(U)$ (resp., $E_W^- \cap Cl(U)$) of U in E_W^+(resp., E_W^-) is $\mathbb{F}.\mathbb{W}.U.\mathbb{C}\mathbb{O}.$(resp., $\mathbb{F}.\mathbb{W}.L.\mathbb{C}\mathbb{O}.$) on W and contained in V.

Proof. Let E be $\mathbb{F}.\mathbb{W}.\mathcal{L}.U.\mathbb{C}\mathbb{O}.$(resp., $\mathbb{F}.\mathbb{W}.\mathcal{L}.L.\mathbb{C}\mathbb{O}.$) there exists a η$\mathbb{P}\mathbb{d}$ W* of d in D and an open set U* of e in E_{W^*} such that the closure $E_{W^*}^+ \cap Cl(U^*)$ of U^* in $E_{W^*}^+$(resp., $E_{W^*}^- \cap Cl(U^*)$ of U^* in $E_{W^*}^-$) is $\mathbb{F}.\mathbb{W}.U.\mathbb{C}\mathbb{O}.$(resp., $\mathbb{F}.\mathbb{W}.L.\mathbb{C}\mathbb{O}.$) on W*. Let E be $\mathbb{F}.\mathbb{W}$.U. re. (resp., $\mathbb{F}.\mathbb{W}$.L. re.) there exists a η$\mathbb{P}\mathbb{d}$ $W \subset W^*$ of d and an open set U of e in E_W^+ (resp., E_W^-) such that the closure $E_W^+ \cap Cl(U)$ of U in E_W^+(resp., $E_W^- \cap Cl(U)$ of U in E_W^-) is contained in $E_W^+ \cap U^* \cap V$(resp., $E_W^- \cap U^* \cap V$). Now $E_W^+ \cap Cl(U^*)$ (resp., $E_W^- \cap Cl(U^*)$) is $\mathbb{F}.\mathbb{W}.U.\mathbb{C}\mathbb{O}.$(resp., $\mathbb{F}.\mathbb{W}.L.\mathbb{C}\mathbb{O}.$) on W, since $E_{W^*}^+ \cap Cl(U^*)$(resp., $E_{W^*}^- \cap Cl(U^*)$) is $\mathbb{F}.\mathbb{W}.U.\mathbb{C}\mathbb{O}.$(resp., $\mathbb{F}.\mathbb{W}.L.\mathbb{C}\mathbb{O}.$) on W^*, and $E_W^+ \cap Cl(U)$ (resp., $E_W^- \cap Cl(U)$) is closed in $E_W^+ \cap Cl(U^*)$(resp., $E_W^- \cap Cl(U^*)$). Hence $E_W^+ \cap Cl(U)$ (resp., $E_W^- \cap Cl(U)$) is $\mathbb{F}.\mathbb{W}.U.\mathbb{C}\mathbb{O}.$(resp., $\mathbb{F}.\mathbb{W}.L.\mathbb{C}\mathbb{O}.$) on W and contained in V.

Corollary 3.1. Suppose that E be $\mathbb{F}.\mathbb{W}.\mathcal{L}.M.\mathbb{C}\mathbb{O}.$ and $\mathbb{F}.\mathbb{W}$.M. re. on D. Then for every point e of Ed, where in d ∈ D, and every open set V of e in E, there exists an open set U of e in EW such that the closure EW ∩ Cl(U) of U in E_W is $\mathbb{F}.\mathbb{W}.M.\mathbb{C}\mathbb{O}.$ on W and contained in V.

Proposition 3.2. Let Ω : E → F be an open, U. continuous(resp., L. continuous), $\mathbb{F}.\mathbb{W}$. surjection, where in E and F are $\mathbb{F}.\mathbb{W}.T.S.$ on D. If E is $\mathbb{F}.\mathbb{W}.\mathcal{L}.U.\mathbb{C}\mathbb{O}.$(resp., $\mathbb{F}.\mathbb{W}.\mathcal{L}.L.\mathbb{C}\mathbb{O}.$) and $\mathbb{F}.\mathbb{W}$.U. re. (resp., $\mathbb{F}.\mathbb{W}$.L. re.) then, so is F.

Proof. Let f be a point of Fd, where in d ∈ D, and let V be an open set of f in F. Pick any point e of $\Omega^{-1}(f)$. Then $\Omega^{-1}(V)$ is an open set of e in E. Let E be $\mathbb{F}.\mathbb{W}.\mathcal{L}.U.\mathbb{C}\mathbb{O}.$(resp., $\mathbb{F}.\mathbb{W}.\mathcal{L}.L.\mathbb{C}\mathbb{O}.$) there exists a η$\mathbb{P}\mathbb{d}$ W of d in D and an open set U of e in EW such that the closure $E_W^+ \cap Cl(U)$ (resp., $E_W^- \cap Cl(U)$) of U in E_W is $\mathbb{F}.\mathbb{W}.U.\mathbb{C}\mathbb{O}.$(resp., $\mathbb{F}.\mathbb{W}.L.\mathbb{C}\mathbb{O}.$) on W and contained in $\Omega^{-1}(V)$. Then Ω(U) is an open set of f in FW, sine **Ω** is open, and closure $F_W^+ \cap Cl(\Omega(U))$ of Ω(U) in F_W^+(resp., $F_W^- \cap Cl(\Omega(U))$ of Ω(U) in F_W^-) is $\mathbb{F}.\mathbb{W}.U.\mathbb{C}\mathbb{O}.$(resp., $\mathbb{F}.\mathbb{W}.L.\mathbb{C}\mathbb{O}.$) on W and contained in V.

Corollary 3.2. Let Ω : E → F be an open, M. continuous, $\mathbb{F}.\mathbb{W}$. surjection, where in E and F are $\mathbb{F}.\mathbb{W}.T.S.$ on D. If E is $\mathbb{F}.\mathbb{W}.\mathcal{L}.M.\mathbb{C}\mathbb{O}.$ and $\mathbb{F}.\mathbb{W}$.M. re. then, so is F.

Proposition 3.3. Suppose that E be $\mathbb{F}.\mathbb{W}.\mathcal{L}.U.\mathbb{C}\mathbb{O}.$(resp., $\mathbb{F}.\mathbb{W}.\mathcal{L}.L.\mathbb{C}\mathbb{O}.$) and $\mathbb{F}.\mathbb{W}$.U. re. (resp., $\mathbb{F}.\mathbb{W}$.L. re.) on D. Let C be $\mathbb{C}\mathbb{O}.$ subset of E_d, where in d ∈ D, and since V is an open set of C in E. Then there exists a η$\mathbb{P}\mathbb{d}$ W of d in D and an open set U of C in EW such that the closure $E_W^+ \cap Cl(U)$ of U in E_W^+(resp., $E_W^- \cap Cl(U)$ of U in E_W^-) is $\mathbb{F}.\mathbb{W}.U.\mathbb{C}\mathbb{O}.$(resp., $\mathbb{F}.\mathbb{W}.L.\mathbb{C}\mathbb{O}.$) on W and contained in V.

Proof. Let E be $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{U}.\mathbb{C}\mathbb{O}.$(resp., $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{L}.\mathbb{C}\mathbb{O}.$) there exists for every point e of C a $\eta\mathbb{P}\mathbb{d}$ We of d in D and an open set Ue of e in EWe such that the closure $E^{+}_{We} \cap Cl(U_e)$ of U_e in E^{+}_{We}(resp., $E^{-}_{We} \cap Cl(U_e)$ of U_e in E^{-}_{We}) is $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{C}\mathbb{O}.$(resp., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{C}\mathbb{O}.$) on We and contained in V. The family {Ue ; e ∈ C} constitutes a covering of the $\mathbb{C}\mathbb{O}.$ C with open sets of E. Extract a finite sub covering indexed with $e_1, \ldots, e_n$ say. Take W to be the intersection $W_{e_1} \cap \ldots \cap W_{e_n}$, and take U to be the restriction to E_W of the union $U_{e_1} \cup \ldots \cup U_{e_n}$. Then W is a $\eta\mathbb{P}\mathbb{d}$ of d in D and U is an open set of C in E_W such that the closure $E^{+}_{W} \cap Cl(U)$ of U in E^{+}_{W}(resp., $E^{-}_{W} \cap Cl(U)$ of U in E^{-}_{W}) is $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{C}\mathbb{O}.$(resp., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{C}\mathbb{O}.$) on W Then W E be $\mathbb{F}$.E$\mathbb{W}$.U. re. (resp., $\mathbb{F}$.$\mathbb{W}$.L. re.) there exists and contained in V.

Corollary 3.3. Suppose that E be $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{M}.\mathbb{C}\mathbb{O}.$ and $\mathbb{F}$.$\mathbb{W}$.M. re. on D. Let C be $\mathbb{C}\mathbb{O}.$ subset of E_d, where in d ∈ D, and since V is an open set of C in E. Then there exists a $\eta\mathbb{P}\mathbb{d}$ W of d in D and an open set U of C in EW such that the closure $E^{+}_{W} \cap Cl(U)$ of U in E^{+}_{W}(resp., $E^{-}_{W} \cap Cl(U)$ of U in E^{-}_{W}) is $\mathbb{F}.\mathbb{W}.\mathrm{M}.\mathbb{C}\mathbb{O}.$ on W and contained in V.

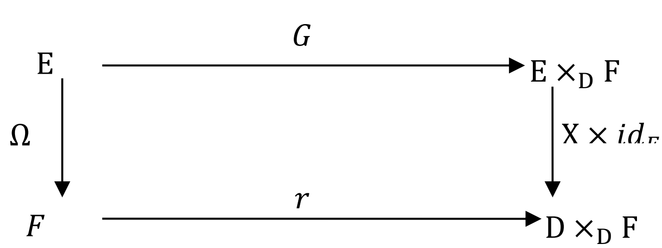
Proposition 3.4. Let $\Omega : E \to F$ be U.$\wp$.(resp., L.$\wp$.), $\mathbb{F}.\mathbb{W}.$ surjection, where in E and F are $\mathbb{F}.\mathbb{W}.\mathrm{T}.\mathrm{S}.$ on D. If E is $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{U}.\mathbb{C}\mathbb{O}.$(resp., $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{L}.\mathbb{C}\mathbb{O}.$) and $\mathbb{F}$.$\mathbb{W}$.U. re. (resp., $\mathbb{F}$.$\mathbb{W}$.L. re.) then, so is F.

Proof. Let f ∈ Fd, where in d ∈ D, and let V be an open set of f in F. Then $\Omega^{+}(V)$(resp., $\Omega^{+}(V)$)is an open set of $\Omega^{-1}(f)$ in E. Let E be $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{U}.\mathbb{C}\mathbb{O}.$(resp., $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{L}.\mathbb{C}\mathbb{O}.$). Since $\Omega^{-1}(f)$ $\mathbb{C}\mathbb{O}.$, by Proposition (3.3) there exists a $\eta\mathbb{P}\mathbb{d}$ W of d in D and an open set U of $\Omega^{-1}(f)$ in E^{+}_{W}(resp., E^{-}_{W}) such that the closure $E^{+}_{W} \cap Cl(U)$ (resp., $E^{-}_{W} \cap Cl(U)$) of U in E^{+}_{W}(resp., E^{-}_{W}) is $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{C}\mathbb{O}.$(resp., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{C}\mathbb{O}.$) on W and contained in $\Omega^{+}(V)$(resp., $\Omega^{+}(V)$). Since Ω is closed there exists an open set U^{*} of f in F^{+}_{W}(resp., F^{-}_{W}) such that $\Omega^{+}(U^{*}) \subset U$(resp., $\Omega^{-}(U^{*}) \subset U$). Then the closure $F^{+}_{W} \cap Cl(U^{*})$ of U^{*} in F^{+}_{W}(resp., $F^{-}_{W} \cap Cl(U^{*})$ of U^{*} in F^{-}_{W}) is $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{C}\mathbb{O}.$(resp., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{C}\mathbb{O}.$) is contained in $\Omega(E^{+}_{W} \cap Cl(U))$(resp., $\Omega(E^{-}_{W} \cap Cl(U))$) and so is $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{C}\mathbb{O}.$(resp., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{C}\mathbb{O}.$) on W. Since $F^{+}_{W} \cap Cl(U^{*})$ (resp., $F^{-}_{W} \cap Cl(U^{*})$) is contained in V this shows that F is $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{U}.\mathbb{C}\mathbb{O}.$(resp., $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{L}.\mathbb{C}\mathbb{O}.$).

Corollary 3.4. Let $\Omega : E \to F$ be U.$\wp$.(resp., L.$\wp$.), $\mathbb{F}.\mathbb{W}.$ surjection, where in E and F are $\mathbb{F}.\mathbb{W}.\mathrm{T}.\mathrm{S}.$ on D. If E is $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{M}.\mathbb{C}\mathbb{O}.$ and $\mathbb{F}$.$\mathbb{W}$.M. re. then, so is F.

Proposition 3.5. Let $\Omega : E \to F$ be U. continuous(resp., L. continuous) $\mathbb{F}.\mathbb{W}.$ function, where in E is $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{C}\mathbb{O}.$(resp., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{C}\mathbb{O}.$) space and F is $\mathbb{F}$.$\mathbb{W}$.U. Hausd. (resp., $\mathbb{F}$.$\mathbb{W}$.L. Hausd.) space on D. Then Ω is U.$\wp$. (resp., L.$\wp$.).

Proof. Consider the figure shown below, where in r is the standard $\mathbb{F}.\mathbb{W}.\mathrm{T}.$ equivalence and G is the $\mathbb{F}.\mathbb{W}.$ graph of Ω



Planned 3.1.

Now G closed embedding, with Proposition(2.10) in [8], let F be $\mathbb{F}$.$\mathbb{W}$.U. Hausd. (resp., $\mathbb{F}$.$\mathbb{W}$.L. Hausd.). Thus G is U.$\wp$.(resp., L.$\wp$.). Also X is U.$\wp$.(resp., L.$\wp$.) and so $X \times id_F$ is U.$\wp$.(resp., L.$\wp$.). Hence $(X \times id_F)\theta\, G = r\ \theta\ \Omega$ is U.$\wp$.(resp., L.$\wp$.) and so Ω is U.$\wp$. (resp., L.$\wp$.), since r is a $\mathbb{F}$.$\mathbb{W}$.T. equivalenc.

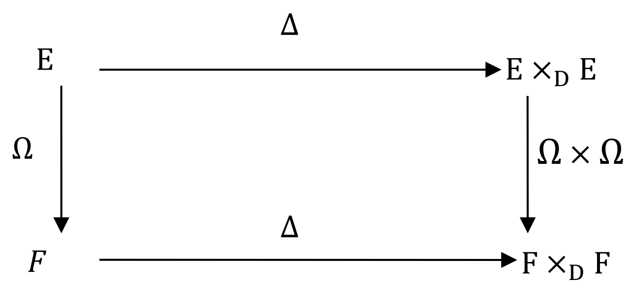
Corollary 3.5. Let $\Omega : E \to F$ be M. continuous $\mathbb{F}.\mathbb{W}.$ function, where in E is $\mathbb{F}.\mathbb{W}.\mathrm{M}.\mathbb{C}\mathbb{O}.$ space and F is $\mathbb{F}$.$\mathbb{W}$.M. Hausd. space on D. Then Ω is U.$\wp$. (resp., L.$\wp$.).

Corollary 3.6. Let $\Omega : E \rightarrow F$ be M. continuous $\mathbb{F}.\mathbb{W}.$ injection, where in is $\mathbb{F}.\mathbb{W}.\mathrm{M}.\mathbb{C}\mathbb{O}.$ space and F is $\mathbb{F}.\mathbb{W}.$M. Hausd. on D. Then Ω is closed embedding

The corollary is often used in the case when Ω is surjective to show that Ω is a a $\mathbb{F}.\mathbb{W}.\mathrm{T}.$ equivalenc.

Proposition 3.6. Let $\Omega : E \rightarrow F$ be $\mathrm{U}.\wp.(\mathrm{resp.}, \mathrm{L}.\wp.)$, $\mathbb{F}.\mathbb{W}.$ surjection where in E and F are $\mathbb{F}.\mathbb{W}.\mathrm{T}.\mathrm{S}.$ on D. If E is $\mathbb{F}.\mathbb{W}.$U. Hausd. (resp., $\mathbb{F}.\mathbb{W}.$L. Hausd.) then so is F.

Proof. Since Ω is $\mathrm{U}.\wp.(\mathrm{resp.}, \mathrm{L}.\wp.)$ surjection so is $\Omega \times \Omega$, in the following figure



Planned 3.2.

The diagonal $\Delta(E)$ closed, since E is $\mathbb{F}.\mathbb{W}.$U. Hausd. (resp., $\mathbb{F}.\mathbb{W}.$L. Hausd.), hence $((\Omega \times \Omega) O \Delta)(E) = (\Delta O \Omega)(E)$ is closed. But $(\Delta O \Omega)(E) = \Delta(F)$, since Ω is surjective, and so F is $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{C}\mathbb{O}.$(resp., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{C}\mathbb{O}.$), as asserted.

Corollary 3.7. Let $\Omega : E \rightarrow F$ be $\mathrm{M}.\wp.$, $\mathbb{F}.\mathbb{W}.$ surjection where in E and F are $\mathbb{F}.\mathbb{W}.\mathrm{T}.\mathrm{S}.$ on D. If E is $\mathbb{F}.\mathbb{W}.$M. Hausd. then so is F.

Proposition 3.7. Let E be $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{C}\mathbb{O}.$(resp., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{C}\mathbb{O}.$) and $\mathbb{F}.\mathbb{W}.$U. Hausd. (resp., $\mathbb{F}.\mathbb{W}.$L. Hausd.) space on D. Then E is $\mathbb{F}.\mathbb{W}.$U. re. (resp., $\mathbb{F}.\mathbb{W}.$L. re.).

Proof. Let $e \in E_d$, where in $d \in D$, and let U be an open set of e in E. Since E is $\mathbb{F}.\mathbb{W}.$U. Hausd. (resp., $\mathbb{F}.\mathbb{W}.$L. Hausd.) there exists for each point $e^* \in E_d$ such that $e^* \notin U$ an open set V_{e^*} of e and an open set ${V^*}_{e^*}$ of e* which do not intersect. Now the family of open sets ${V^*}_{e^*}$, for $e^* \in (E-U)_d^+(\mathrm{resp.}, (E-U)_d^-)$, forms a covering of $(E-U)_d^+(\mathrm{resp.}, (E-U)_d^-)$. Since $E-U$ is closed in E and therefore $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{C}\mathbb{O}.$(resp., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{C}\mathbb{O}.$) there exists, by Proposition(2.2), a $\eta\mathbb{P}\mathbb{d}$ W of d in D such that $E_W^+ - (E_W^+ \cap U)(\mathrm{resp.}, E_W^- - (E_W^- \cap U))$ is covered with a finite subfamily, indexed with$e_1^*, \ldots, e_n^*$, say. Now the intersection $V = V_{e1}^* \cap \ldots \cap V_{en}^*$, is an open set of e which does not meet the open set $V^* = {V_{e1}^*}^* \cup \ldots \cup {V_{en}^*}^*$ of $E_W^+ - (E_W^+ \cap U)(\mathrm{resp.}, E_W^- - (E_W^- \cap U))$. Therefore the closure $E_W^+ \cap \mathrm{Cl}(V)$ of $E_W^+ \cap V$ in $E_W^+(\mathrm{resp.}, E_W^- \cap \mathrm{Cl}(V)$ of $E_W^- \cap V$ in $E_W^-)$ is contained in U, as asserted.

Corollary 3.8. Let E be $\mathbb{F}.\mathbb{W}.\mathrm{M}.\mathbb{C}\mathbb{O}.$ and $\mathbb{F}.\mathbb{W}.$M. Hausd. space on D. Then E is $\mathbb{F}.\mathbb{W}.$M. re..

We extend this last result to.

Proposition 3.8. Let E be $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{U}.\mathbb{C}\mathbb{O}.$(resp., $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{L}.\mathbb{C}\mathbb{O}.$) and $\mathbb{F}.\mathbb{W}.$U. Hausd. (resp., $\mathbb{F}.\mathbb{W}.$L. Hausd.) space on D. Then E is $\mathbb{F}.\mathbb{W}.$U. re. (resp., $\mathbb{F}.\mathbb{W}.$L. re.).

Proof. Let $e \in E_d$, where in $d \in D$, and let V be an open set of e in E. Since W is a $\eta\mathbb{P}\mathbb{d}$ W of $d \in D$ and let U be an open set of $e \in E_W$ such that the closure $E_W^+ \cap \mathrm{Cl}(U)$ of U in $E_W^+(\mathrm{resp.}, E_W^- \cap \mathrm{Cl}(U)$ of U in $E_W^-)$ is $\mathbb{F}.\mathbb{W}.\mathrm{U}.\mathbb{C}\mathbb{O}.$(resp., $\mathbb{F}.\mathbb{W}.\mathrm{L}.\mathbb{C}\mathbb{O}.$) on D. Then $E_W^+ \cap \mathrm{Cl}(U)$ (resp., $E_W^- \cap \mathrm{Cl}(U)$) is $\mathbb{F}.\mathbb{W}.$U. re. (resp., $\mathbb{F}.\mathbb{W}.$L. re. on W, by Proposition(3.7), since $E_W^+ \cap \mathrm{Cl}(U)(\mathrm{resp.}, E_W^- \cap \mathrm{Cl}(U))$ is $\mathbb{F}.\mathbb{W}.$U. Hausd. (resp., $\mathbb{F}.\mathbb{W}.$L. Hausd.) on W. So there exists a $\eta\mathbb{P}\mathbb{d}$ $W^* \subset W$ of $d \in D$ and an open set U* of $e \in E_{W^*}$ such that the closure $E_{W^*}^+ \cap \mathrm{Cl}(U^*)$ of $U^*(\mathrm{resp.}, E_{W^*}^- \cap \mathrm{Cl}(U^*)$ of $U^*)$ in E_{W^*} is contained in $U \cap V \subset V$, as required.

Corollary 3.9. Let E be $\mathbb{F}.\mathbb{W}.\mathcal{L}.\mathrm{M}.\mathbb{C}\mathbb{O}.$ and $\mathbb{F}.\mathbb{W}.$M. Hausd. space on D. Then E is $\mathbb{F}.\mathbb{W}.$M. re..

Proposition 3.9. Let E be 𝔽.𝕎.U. re. (resp., 𝔽.𝕎.L. re.) space on D and let K be 𝔽.𝕎.U.ℂ𝕆.(resp., 𝔽.𝕎.L.ℂ𝕆.) subset of E. Let d be a point of D and let V be an open set of Kd in E. Then there exists a η℘𝕕 W of d in D and an open set U of KW in EW such that closure $E^+_W \cap Cl(U)$ of U in E^+_W(resp., $E^+_W \cap Cl(U)$ of U in E^+_W) is contained in V.
Proof. We may let Kd is non-empty since otherwise we can take U = E^+_W(resp., E^-_W), where in W = D − X(E − V). Since V is an open set of each point e of Kd there exists, with 𝔽.𝕎.U. re. (resp., 𝔽.𝕎.L. re.), a η℘𝕕 W of d in D and an open set Ue ⊂ E^+_{We}(resp., E^-_{We}) of e such that the closure $E^+_{We} \cap$ Cl(Ue) of Ue in E^+_{We}($E^-_{We} \cap$ Cl(Ue) of Ue in E^-_{We})is contained in V. The family of open sets { E^+_{We} ∩ Ue(resp., E^-_{We} ∩ Ue); e ∈ Kd} covers Kd and so there exists a η℘𝕕 W* of d and a finite subfamily indexed with e_1, …,e_n say, which covers KW. Then the conditions are satisfied with

$$W = W^* \cap W_{e_1} \cap \ldots \cap W_{e_n}, U = U_{e_1} \cup \ldots \cup U_{e_n}.$$

Corollary 3.10. Let E be 𝔽.𝕎.M. re. space on D and let K be 𝔽.𝕎.M.ℂ𝕆. subset of E. Let d be a point of D and let V be an open set of Kd in E. Then there exists a η℘𝕕 W of d in D and an open set U of KW in EW such that closure $E_W \cap Cl(U)$ of U in E_W is contained in V.

Corollary 3.11. Let E be .𝕎.M.ℂ𝕆. and 𝔽.𝕎.M. re. on D. Then E is 𝔽.𝕎. M. no..

Proposition 3.10. Let E be 𝔽.𝕎.U. re. (resp., 𝔽.𝕎.L. re.) on D and let K be 𝔽.𝕎.U.ℂ𝕆.(resp., 𝔽.𝕎.L.ℂ𝕆.) subset of E. Let {Vi; i = 1, …, n} be a covering of Kd, where in d ∈ D with open sets of E. Then there exists a η℘𝕕 W of d and a covering {Ui; i = 1, …, n} of KW with open sets of E^+_W(resp., E^-_W) such that the closure $E^+_W \cap Cl(U_i)$ of U_i(resp., $E^-_W \cap Cl(U_i)$ of U_i) in E^+_W(resp., E^-_W) is contained in Vi.

Proof. Write V= $V_2 \cup \ldots \cup V_n$, so that E−V is closed in E. Hence K ∩ (E − V) is closed in K and so 𝔽.𝕎.U.ℂ𝕆.(resp., 𝔽.𝕎.L.ℂ𝕆.). Applying the previous result to the open V1 of Kd ∩ $(E-V)^+_d$(resp., $(E-V)^-_d$) we obtain a η℘𝕕 W of d and an open set U of KW ∩ $(E - V)_W$ such that $E^+_W \cap Cl(U) \subset V_1$(resp., $E^-_W \cap Cl(U) \subset V_1$). Now K ∩ V and K ∩ (E − V) cover K, hence V and U cover KW. Thus $U = U_1$ is the first step in the shrinking process. We continue with repeating the argument for {U1, V2, …, Vn}, so as to shrink V2, and so on. Hence the result is obtained.

Corollary 3.12. Let E be 𝔽.𝕎.M. re. on D and let K be 𝔽.𝕎.M.ℂ𝕆. subset of E. Let {Vi; i = 1, …, n} be a covering of Kd, where in d ∈ D with open sets of E. Then there exists a η℘𝕕 W of d and a covering {Ui; i = 1, …, n} of KW with open sets of EW such that the closure $E^+_W \cap Cl(U_i)$ of U_i(resp., $E^-_W \cap Cl(U_i)$ of U_i) in E_W is contained in Vi.

Proposition 3.11. Let Ω : E → F be U.𝓅.(resp., L.𝓅.), U. open(resp., L. open) 𝔽.𝕎. surjection, where in E and F are 𝔽.𝕎.T.S. on D. If E is 𝔽.𝕎.U. re. (resp., 𝔽.𝕎.L. re.) then so is F.

Proof. Let E be 𝔽.𝕎.U. re. (resp., 𝔽.𝕎.L. re.). Let f be appoint of Fd, where in d ∈ D, and let V be an open set of f in F. Then $\Omega^+(V)$(resp., $\Omega^-(V)$) is an open set of the ℂ𝕆.Ω^{-1}(f) in E. with Proposition(3.9), therefore, there exists a η℘𝕕 W of d in D and an open set U of Ω^{-1}(f) in EW such that the cloure $E^+_W \cap Cl(U)$ of U(resp., $E^-_W \cap Cl(U)$ of U) in E^+_W(resp., −) is contained in $\Omega^+(V)$(resp., $\Omega^-(V)$).

Now since Ω_W is closed there exists an open set V* of f in F^+_W(resp., F^-_W) such that $\Omega^+(V^*) \subset$ U(resp., $\Omega^-(V^*) \subset$ U), and then the closure $E^+_W \cap Cl(V^*)$ of V^*(resp., $E^-_W \cap Cl(V^*)$ ofV^*) in E^+_W(resp., E^-_W) is contained in V since,

$$Cl(V^*) = Cl(\Omega(\Omega^+(V^*))(resp., Cl(\Omega(\Omega^-(V^*))) = \Omega(Cl\Omega^+(V^*)) \subset \Omega(Cl(U)) \subset \Omega(\Omega^+(V)) \subset V(resp., \Omega(Cl\Omega^-(V^*)) \subset \Omega(Cl(U)) \subset \Omega(\Omega^-(V)) \subset V).$$

Thus, F is 𝔽.𝕎.U. re. (resp., 𝔽.𝕎.L. re.), as asserted.

Corollary 3.13. Let Ω: E → F be U.𝓅. (resp., L.𝓅.), M. open 𝔽.𝕎. surjection, where in E and F are 𝔽.𝕎.T.S. on D. If E is 𝔽.𝕎.M. re. then so is F.

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