Nearly Maximal Submodules

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Abstract

In this paper we introduce the concept of nearly maximal submodules as a generalization of the class of maximal submodules, where a proper submodule N of M is called nearly maximal, if whenever a submodule W of M containing N properly implies that W+J(M)=M. Various properties of nearly maximal submodules are considered. Also we define an NM-module, which is a module in which every proper nonzero submodule is nearly maximal; we study some properties of this class of modules.

Key words: Maximal submodules, Almost maximal submodules, Nearly maximal submodules, Nearly Jacobson radical of modules, Semimaximal submodules.

1. Introduction

Let R be a commutative ring with identity and let M be a unitary left R-module. It is well known that a proper submodule N of M is called maximal, if whenever W is a submodule of M with $N \subset W \subseteq M$ implies that W = M, equivalently, there is no proper submodule of M containing N properly [(1)]. Inaam and Riyadh in [(2)]introduced the concept of almost maximal submodules as a generalization of the class of maximal submodules, where a submodule N of M is called almost maximal, if whenever W is an essential submodule of M with $N \subset W$ implies that W = M, where a submodule K of M is said to be an essential if for every submodule L of M with $K \cap L=(0)$ implies that L=(0) [(1)]. Hatem in[(3)] introduced another generalization which is called semimaximal submodules, where a submodule N of an R-module M is called semimaximal, if $\frac{M}{N}$ is a semisimple R-module. Shwkaea in [(4)] gave a class of weak maximal submodules as another generalization of maximal submodules, which are submodules such that $\frac{M}{N}$ is F-regular modules, where an R-module M is called F-regular, if every submodule of M is a pure [(5)].

In this paper, we introduce the concept of nearly maximal submodules which is another generalization of the class of maximal submodules, where a proper submodule N of M is called nearly maximal, if whenever a submodule W of M containing N properly implies that W+J(M)=M, where J(M) is the Jacobson radical of M.

In section 2, we investigate the main properties of the class of nearly maximal submodules. Also we study the relationship between this concept and maximal submodules. In section 3, we define the class of NM-module which is a module in which every proper nonzero submodule is a nearly maximal. We study the hereditary property between NM-modules over a ring R and the ring R itself, and we study the direct sum of two NM-modules.

2. Nearly Maximal Submodules

In this section we introduce a class of nearly maximal submodule which is a generalization of maximal submodules. We study the main properties of this type of submodules. Firstly we begin by the following definition.

Definition (2.1): A proper submodule N of an R-module M is called nearly maximal, if whenever a submodule W of M containing N properly implies that W+J(M)=M, where J(M) is the Jacobson radical of M. A proper ideal I of a ring R is called nearly maximal, if I is a nearly maximal submodule of the R-module R.

Remarks and Examples (2.2):

1. It is clear that every maximal submodule is nearly maximal, but the converse is not true in general. In fact the submodule of integer number Z in the Z-module Q is nearly maximal but not maximal, where Q is the ring of rational numbers.
2. If M is a simple module then (0) is a nearly maximal submodule of M.
3. Local module has only one nearly maximal submodule, which is the unique maximal submodule.
4. If M has no maximal submodule, then J(M)=M and hence every submodule of M is nearly maximal, such as Z_{P^∞} as Z-module and Q as Z-module.
5. If J(M)=(0) and N is a nearly maximal submodule, then N is a maximal. So if M is Z-regular (F-regular) then every nearly maximal submodule is maximal, where an R-module M is called Z-regular if every cyclic submodule of M is a projective and a direct summand [(6)].
6. Let N and W be a proper submodules of an R-module M such that $N \subseteq W$. If N is a nearly maximal submodule of M, then W is a nearly maximal submodule of M.

   Proof: Let K be a submodule of M such that $W \subset K$. Since $N \subseteq W$, so $N \subset K$. But N is a nearly maximal submodule, thus K+J(M)=M.
7. If I and J are two proper submodules of an R-module M such that $I \cap J$ is a nearly maximal submodule, then both of Iand J are nearly maximal submodules.

   The converse of (7) is not true in general, for example in the Z-module Z_{12}, both of $I = \{\bar{0}, \bar{3}, \bar{6}, \bar{9}\}$ and $J = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\}$ are nearly maximale submodule but $I \cap J = \{\bar{0}, \bar{6}\}$ is not nearly maximal submodule. In fact the proper submodule containing $I \cap J$ are only I& J, and $I + J(Z_{12}) \neq Z_{12}$, also $J + J(Z_{12}) \neq Z_{12}$.
8. Nearly maximal submodules need not be an almost maximal submodules for example, in the Z-module Q, the submodule Z is a nearly maximal (2.2)(1), but it is not almost maximal since there exists a submodule $\frac{1}{2}Z$ of Q such that $Z \subseteq \frac{1}{2}Z$ and $\frac{1}{2}Z$ is an essential submodule of Q. I think the two concepts are independent, but I can't find an example to complete this claim.
9. Semimaximal submodules and nearly maximal submodules are independent, for examples the submodule (6) in Z is semimaximal since $\frac{Z}{(6)} \cong Z_6$ is a semisimple module[(3)], but it is not nearly maximal submodule since there exists a submodule (2) of Z such that $(2) + J(Z) \neq (0)$. On the other hand, in the Z-module Q, the submodule Z is a nearly maximal (2.2)(1), but it is not semimaximal since $\frac{Q}{Z}$ is not semisimple module.

Proposition (2.3): Let M_1 and M_2 be R-modules, and let $f: M_1 \to M_2$ be an epimorphism. If N is a nearly maximal submodule of M_1 such that ker f $\subseteq$ N, then f(N) is a nearly maximal submodule of M_2.

Proof: Let W be a submodule of M_2 such that f(N)$\subset$W, we must show that W+J(M_2)= M_2. Since N is a nearly maximal submodule, and N is a proper submodule of M_1, also since kerf $\subseteq$ N, then we get f(N) is a proper submodule of M_2. On the other hand f(N) $\subset$ W, then N$\subset$ $f^{-1}(W)$, hence $f^{-1}(W)+J(M_1)$ $=M_1$. This implies that $f(f^{-1}(W)+J(M_1))$ $=f(M_1)=M_2$. Thus $f(f^{-1}(W)) + f(J(M_1)) = M_1$, but f is an epimorphism so W + J(M_2). Thus f(N) is a nearly maximal submodule of M_2.

Corollary (2.4): If a submodule N is a nearly maximal submodule of an R-module M, then $\frac{N}{W}$ is a nearly maximal submodule of $\frac{M}{W}$ for every submodule W of M such that W contained in N.

Remark (2.5): If N is a nearly maximal submodule of an R-module M, then it is not necessarily that (N:M) is a nearly maximal ideal of R. For example the submoduleZ in the Z-module Q is a nearly maximal, but the ideal $(Z:_Z Q)$ = (0) is not nearly maximal ideal in the ring Z.

Recall that an R-module M is called multiplication, if for each submodule N of M there exists an ideal I of R such that N=IM [7]. Now we have the following.

Proposition (2.6): Let M be a finitely generated faithful multiplication R-module. A submodule N of M is nearly maximal if and only if there exists a nearly maximal ideal I of R such that N=IM.

Proof:⇒) Suppose that N is a nearly maximal submodule. Since M is a multiplication module, then N=IM for some ideal I of R. If I is not nearly maximal ideal, so there exists an ideal K of R containing I properly such that I$\subset$ K+J(R) $\subset$ R, this implies that IM $\subset$ KM+J(R)M $\subset$ RM. But J(R)M$\subset$ J(M), thus N $\subset$ KM+J(M) $\subset$ M, that is N is not nearly maximal submodule which a contradict with our assumption.

⇐) Assume that N is not nearly maximal submodule, then there exists a proper submodule L of M with N $\subset$ L and N $\subset$ L+J(M) $\subset$ M. Since M is a multiplication module, so L=KM for some ideal K of R, and by assumption there exists a nearly ideal I of R such that N=IM. Now IM $\subset$ KM + J(R)M$\subset$ RM. Since M is a finitely generated faithful and multiplication, so by [8],(Theorem.(3.1)), we get I$\subset$ K+J(R) $\subset$ R, that is I is not nearly maximal which is a contradiction.

Corollary (2.7): Let M be a finitely generated faithful and multiplication module. A submodule N of M is nearly maximal if and only if [N:M] is nearly maximal ideal of R.

Proof: Since M is a multiplication, then for some ideal I of R. Note that N=[N:M]M [9], and by Prop (2.6) we get the result.

Recall that a proper submodule N of an R-module M is called weakly prime, if whenever 0≠rx∈N for r∈R and x∈M implies that either x∈N or r∈[N:M] [10], and an ideal I of a ring R is called weakly prime if whenever a,b∈R where 0≠ ab∈I, implies that either a∈I or b∈I[11]. It is clear that a zero ideal of any ring R is weakly prime ideal, also every prime ideal is weakly prime [12].

Remark (2.8): It is well known that every maximal submodule is a prime submodule, we see now that there is no direct implication between nearly maximal submodule and weakly prime submodule. In fact it is clear that (0) is weakly prime submodule of the Z-module Z, but it is clear that (0) is not nearly maximal submodule of Z. On the other hand, in the Z-module Q, Z is a nearly maximal of, but it is not weakly prime submodule since$2.\frac{1}{2} = 1\in$ Z, but neither $\frac{1}{2}\in$Z nor 2∈[Z:Q]=(0).However, we prove the following in a certain class of rings.

Theorem (2.9): Let I be a nontrivial proper ideal of a principal ideal domain R (briefly P.I.D). Then the following statements are equivalent.

1. Iis an almost maximal ideal.
2. Iis a nearly maximal ideal.
3. Iis a maximal ideal.
4. Iis a prime ideal.
5. Iis a weakly prime ideal.

Proof:(1)⇔ (3) ⇔ (4)⇔ (5): See [2]. So it is enough to show that **(1)⇔ (2).**

(1) ⇒ (2): LetI be an almost maximal ideal of R, and let J be an ideal of R such that I⊂ J. Since R is a P.I.D, so it is clear that J is an essential ideal of in R. But I is an almost maximal ideal of R, thus J=R. Hence J + J(R) = R, that is I is a nearly maximal ideal.

(2) ⇒ (1): Let J be an essential ideal of R such that I⊂ J. Since R is a P.I.D, so it is clear that J(R) = (0). But I is a nearly maximal ideal of R, therefore J + J(R) =R and hence J = R, that is I is an almost maximal ideal of R.

In the following proposition we show under the class of almost maximal submodule, every nearly maximal submodule of $\frac{M}{N}$ is a maximal submodule.

Proposition (2.10): Let M be an R-module and let N be a submodule of M. If N is an almost maximal submodule of M, then every nearly maximal submodule of $\frac{M}{N}$ is a maximal submodule.

Proof: Let$\frac{L}{N}$ be a nearly maximal submodule of $\frac{M}{N}$. Suppose there exists a proper submodule $\frac{W}{N}$ of $\frac{M}{N}$ containing $\frac{L}{N}$ properly and $\frac{L}{N}\subset\frac{W}{N}\subseteq\frac{M}{N}$. This implies that $\frac{L}{N}\subset\frac{W}{N}\subseteq\frac{W}{N}$ + J($\frac{M}{N}$). Since $\frac{L}{N}$ is a nearly maximal submodule of $\frac{M}{N}$, then $\frac{W}{N}$ + J($\frac{M}{N}$) = $\frac{M}{N}$, but N is an almost maximal submodule of M, therefore $\frac{M}{N}$ is a semisimplemodule [(2) Theorem (1.10)], that is J($\frac{M}{N}$) = (0), and hence $\frac{W}{N}$ = $\frac{M}{N}$. Thus $\frac{W}{N}$ is a maximal submodule of $\frac{M}{N}$.

Recall that an R-module M is called cosemisimple, if J($\frac{M}{N}$) =(0) for each submodule N of M [11]. Then we have the following.

Proposition (2.11): Let M be an R-module, then:

1. If M is a Semisimple module, then every nearly maximal submodule of M is maximal.
2. If M is a cosemisimple module, then every nearly maximal submodule of $\frac{M}{N}$ is a maximal submodule, for each submodule N of M.
3. If N is a semimaximal submodule of M, then every nearly maximal submodule of $\frac{M}{N}$ is a maximal submodule, for each submodule N of M.

Proof: All type of modules M in (1),(2),(3) has the property J(M) =(0), and by using (2.2)(5) we get the result.

Recall that a submodule N of an R-module M is called weak maximal if $\frac{M}{N}$ia an F-regular module [4].So we have the following proposition.

Proposition (2.12): Let M be an R-module, and let N be a weak maximal submodule of M. Then every nearly maximal submodule of $\frac{M}{N}$ is a maximal.

Proof: Since N is a weak maximal submodule of M, then$\frac{M}{N}$ is an F-regular module. Hence J(M) = (0), and by (2.2)(5) we get the result.

We end this section by the following two examples about the direct sum of two nearly maximal submodules. The first one show that for R-modules M_1 and M_2, if N_1 is nearly maximal submodule of M_1 and N_2 is nearly maximal submodule of M_2 then it is not necessarily that $N_1 \oplus N_2$ is a nearly maximal submodule of $M_1 \oplus M_1$, as follows.

Example (2.13): Consider the Z-module Z, and assume that M= Z⊕Z as Z-module. The submodules 2Z and 3Z are nearly maximal submodule, but 2Z⊕3Z is not nearly maximal submodule of Z⊕Z since $2Z\oplus 3Z \subset Z\oplus 3Z + J(Z) \subset Z\oplus Z$. Moreover, Z⊕3Z is a proper submodule of Z⊕Z. Thus 2Z⊕3Z is not nearly maximal submodule of Z⊕Z.

The other example shows that if both of N_1 and N_2 are nearly maximal submodules of an R-module M, then $N_1 \oplus N_2$ is not nearly maximal submodule of M.

Example (2.14): Consider the Z-module Z_6, each of the submodules $(\bar{2})$ and $(\bar{3})$ are nearly maximal submodules of Z_6, but $(\bar{2}) \oplus (\bar{3}) = Z_6$ is not nearly maximal submodule of Z_6 since any nearly submodule must be proper and $Z_6 \not\subset Z_6$.

3. NM-Modules:

In this section we introduce the concept of NM-modules, beginning with the following definition.

Definition (3.1): An R-module M is called NM-module, if every proper nonzero submodule of M is nearly maximal. And a ring R is called NM-ring if every proper nonzero ideal of R is a nearly maximal.

Examples (3.2):

1. Z_6 as Z-module is NM-module.
2. Z as Z-module is not NM-module, since the submodule (6) of Z is not nearly maximal.
3. If M has no maximal submodule, then M is an NM-module, such as Q as Z-module.

Proposition (3.3): A direct summand of NM-module is an NM-submodule.

Proof: Let M be an NM-module and suppose that $M = M = M_1 \oplus M_2$, where both of M_1 and M_2 are submodules of M. Let N_1 be a proper nonzero submodule of M_1 and let W_1 be a submodule of M_1 such that $N_1 \subset W_1$ with $N_1 \subset W_1 + J(M_1) \subset M_1$. Now $N_1 \oplus M_2 \subset W_1 + J(M_1) \oplus M_2$, but $N_1 \oplus M_2$ is a nearly maximal submodule of M, thus $(W_1 + J(M_1)) \oplus M_2 + J(M) = M$. But $J(M) = J(M_1) \oplus J(M_2)$, so that $W_1 + J(M_1) + M_2 + (J(M_1) \oplus J(M_2)) \subseteq W_1 + J(M_1) \oplus M_2$. Therefore $W_1 + J(M_1) \oplus M_2 + J(M_2) = M$. This implies that $W_1 + J(M_1) = M_1$, hence N_1 is a nearly maximal submodule of M_1. That is M_1 is an NM-submodule.

Proposition (3.4): An epimomorphic image of an NM-module is an NM-module.

Proof: Follows from Prop (2.3).

The following theorem gives the hereditary property between an NM-module over a ring R and R itself.

Theorem (3.5): Let M be a finitely generated faithful multiplication module. Then M is an NM- module if and only if R is an NM-module.

Proof:⇒) Assume that M is an NM-module, and let I be a proper nonzero ideal of R. Put N = IM. By assumption N is a nearly maximal submodule of M, hence by Prop (2.6) the ideal I is a nearly maximal.

⇐) Suppose that R is an NM-module and let N be a proper nonzero submodule of M. Since M is a multiplication, so there exists an ideal I of R such that N = IM. By assumption I is a nearly maximal ideal of R, therefore by Prop (2.6) N is a nearly maximal submodule, that is M is an NM-module.

We end this work by the following theorem which deals with the direct sum of two NM-modules.

Theorem (3.6): Let M be an R-module such that $M=M_1\oplus M_2$ where M_1 and M_2 be NM-module. If $ann_R M_1 + ann_R M_2 = R$, then M is an NM-module.

Proof: Let N be a proper nonzero submodule of M and let K be a submodule of M containing N. Since $ann_R M_1 + ann_R M_2 = R$, then $N = N_1\oplus N_2$ for some submodules N_1 of M_1 and N_2 of M_2, and $K = K_1\oplus K_2$ for some submodules K_1 of M_1 and K_2 of M_2[(13) Theorem (4.2)), Now; $N_1\oplus N_2 \subset K_1\oplus K_2 + J(M_1\oplus M_2) \subseteq M_1\oplus M_2$. But $J(M_1\oplus M_2) = J(M_1)\oplus J(M_2)$, so we get $N_1\subset K_1 + J(M_1) \subseteq M_1$ and $N_2\subset K_2 + J(M_2) \subseteq M_2$. Since $N_1\subseteq M_1$ and M_1 is an NM-module, then $K_1 + J(M_1) = M_1$. Similarly we have $K_2 + J(M_2) = M_2$, and this implies that $K_1\oplus K_2 + J(M_1\oplus M_2) = M_1\oplus M_2$, hence $K + J(M) = M$, so $N_1\oplus N_2 = N$ is a nearly maximal submodule of M. That is M is an NM-module.

Open problem: Before giving the problem, we need to introduce the following definition.

Definition: Let M be an R-module. A nearly Jacobson radical of M is denoted by NJ(M), and we define it as follows:

If M has nearly maximal submodules, then:

$NJ(M) = \cap$ {N| where N is a nearly maximal submodule of M}

and if M hasn't nearly maximal submodules, then we say that NJ(M) = M.

Now, we know that $J(M) = \sum_i S_i$, S_i is a small submodule of M, where a submodule N of an R-module M is called small if for every submodule L of M, if N +L =M then L= M [10]. The question is what about NJ(M)? .i.e does there is a relation between NJ(M) and the small submodules of M, or maybe we need to define analogous concept of small submodules?.

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