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NEW SEVEN-PARAMETER MITTAG-LEFFLER FUNCTION WITH CERTAIN ANALYTIC PROPERTIES

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Abstract. In this paper, a new seven-parameter Mittag-Leffler function of a single complex variable is proposed as a generalization of the standard Mittag-Leffler function, certain generalizations of Mittag-Leffler function, hypergeometric function and confluent hypergeometric function. Certain essential analytic properties are mainly discussed, such as radius of convergence, order, type, differentiation, Mellin-Barnes integral representation and Euler transform in the complex plane. Its relation to Fox-Wright function and H-function is also developed.

1. Introduction

The higher transcendental function, Mittag-Leffler function, was introduced in 1903 by the Swedish mathematician Gosta Mittag-Leffler for one complex variable concurring one parameter as, [15]:

$$E_\tau(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\tau k + 1)}, \tag{1.1}$$

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where z $\in \mathbb{C}$ and $Re(\tau) > 0$. It is define an entire function of an order $\rho = \frac{1}{Re(\tau)}$ and type $\sigma = 1$ and considered as a slight generalization of the exponential function, preserving certain properties of it. Eminently, this function has attracted the numerous attention of researchers due to its role in solving common problems in analytic function theory, treating problems with fractional order integral and differential equations, and motivating the description of numerous problems involving the problems of computer science, food science, physics, and engineering; see [2, 5, 9, 11, 14]. Decades ago until the present, many generalizations and extensions of Mittag-Leffler function have been studied; among the considerable generalizations, we mention those are crucial in our work.

In 1905, Wiman presented the first generalization of Mittag-Leffler function, known as Wiman's function or two-parameter Mittag-Leffler function defined as [23]:

$$E_{\tau,\lambda}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\tau k + \lambda)}, \tag{1.2}$$

where $z \in \mathbb{C}, Re(\tau) > 0$ and $Re(\lambda) > 0$.

In 1960, Dzherbashian submitted a four-parameter Mittag-Leffler function as follows [4]:

$$E_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\tau_1 k + \lambda_1)\ \Gamma(\tau_2 k + \lambda_2)}, \tag{1.3}$$

where $z, \lambda_1, \lambda_2 \in \mathbb{C}$ and τ_1, τ_2 are positive real numbers.

In 1971, Prabhakar gave an innovative generalization of Mittag-Leffler function as a function of three parameters using the Pochhammer symbol, defined as [18]:

$$E^{\gamma}_{\tau,\lambda}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k\ z^k}{\Gamma(\tau k + \lambda)\ k!}, \tag{1.4}$$

where $z \in \mathbb{C}, Re(\tau) > 0, Re(\lambda) > 0$ and $\mathrm{Re}(\gamma) > 0$.

In 1994, Luchko and Yakubovich generalized Mittag-Leffler function to multi-index (2m-parameter) function defined as [13]:

$$E((\tau, \lambda)_m; z) = \sum_{k=0}^{\infty} \frac{z^k}{\prod_{i=1}^{m} \Gamma(\tau_i k + \lambda_i)}, \tag{1.5}$$

where $z, \lambda_i \in \mathbb{C}\ \ (i = 1, ..., m), \tau_i \in \mathbb{R}$ and $(\tau_1^2 + ... + \tau_n^2) \neq 0$.

In 2007, Shukla and Prajapati defined a different generalization of the Mittag-Leffler function by using the generalized Pochhammer symbol, defined as [21]:

$$E_{\tau,\lambda}^{\gamma,q}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{qk} \ z^k}{\Gamma(\tau k + \lambda) \ k!}, \tag{1.6}$$

where $z, \tau, \lambda, \gamma \in \mathbb{C}, Re(\tau) > 0, Re(\lambda) > 0, Re(\gamma) > 0$ and $q \in (0, 1)$.

In 2009, Salim presented another four-parameter Mittag-Leffler function that is not a special case of multi index (2m-parameter) function (1.5), defined as [20]:

$$E_{\tau,\lambda}^{\gamma,\delta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k \ z^k}{\Gamma(\tau k + \lambda) \ (\delta)_k \ k!}, \tag{1.7}$$

where $z \in \mathbb{C}, Re(\tau) > 0, Re(\lambda) > 0, Re(\gamma) > 0$ and $Re(\delta) > 0$.

In 2011, Paneva-Konovska proposed a further multi-index generalization of Mittag-Leffler function known as multi index (3m-parameter) Mittag-Leffler function defined as [17]:

$$E_{(\tau_i),(\lambda_i)}^{(\gamma_i),m}(z) = \sum_{k=0}^{\infty} \frac{(\gamma_1)_k \ ... \ (\gamma_m)_k}{\Gamma(\tau_1 k + \lambda_1) \ ... \ \Gamma(\tau_m k + \lambda_m)} \frac{z^k}{(k!)^m}, \tag{1.8}$$

where $z, \tau_i, \lambda_i, \gamma_i \in \mathbb{C} \ \ (i = 1, ..., m)$ and $Re(\tau_i) > 0$.

In 2021, Özeslan and Fernandez considered, in their study, a five-parameter Mittag-Leffler function as a special case of the multi index Mittag-Leffler function (1.8), [16]:

$$E_{\tau_1,\lambda_1,\tau_2,\lambda_2}^{\gamma}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\tau_1 k + \lambda_1) \ \Gamma(\tau_2 k + \lambda_2)} \frac{z^k}{k!}, \tag{1.9}$$

where $z, \tau_1, \tau_2, \lambda_1, \lambda_2, \gamma \in \mathbb{C}$ and $Re(\tau_1 + \tau_2) > 0$.

Since then, Mittag-Leffler functions acquired wide attention for studying different many properties because of their relation to the fractional calculus and its application in varied sciences; see [3, 7, 12].

This paper devoted to propose a new function with seven complex parameters and single complex variable generalizes the standard Mittag-Leffler function, several generalizations of Mittag-Leffler function, hypergeometric function, and Confluent hypergeometric function, then review its special cases. Additionally, we elaborate on certain beneficial analytic properties such as radius of convergence, order, type, differentiation, Mellin-Barnes integral representation, and a recurrence relation, further seek its connection to Fox-Wright function and H-function.

2. Preliminaries

Throughout our work, we need the following well-known formulas, functions, and facts:

(1) *Stirling's formulas* for gamma function [1]:

$$\Gamma(z)=\sqrt{2\pi}(z)^{z-\frac{1}{2}}e^{-z}\left[1+O\left(\frac{1}{z}\right)\right],\ z\to\infty,\ |\arg(z)|<\pi, \tag{2.1a}$$

$$\Gamma(z)\sim\sqrt{2\pi}\,(z)^{z-\frac{1}{2}}\ e^{-z},\quad |\arg(z)|<\pi, \tag{2.1b}$$

$$\Gamma(az+b)\sim\sqrt{2\pi}\,(az)^{az+b-\frac{1}{2}}\ e^{-az},\quad a>0,\ |\arg(z)|<\pi. \tag{2.1c}$$

(2) *Asymptotic formulas* [1, 6]:

$$\frac{\Gamma(z+a)}{\Gamma(z+b)}=z^{a-b}\left[1+\frac{(a-b)(a-b-1)}{2z}+O\left(\frac{1}{z^2}\right)\right],\ z\to\infty,\ |\arg(z)|<\pi, \tag{2.2a}$$

$$\frac{\Gamma(z+a)}{\Gamma(z+b)}\sim z^{a-b}\ ,\ z\to\infty,\ |\arg(z)|<\pi. \tag{2.2b}$$

(3) *Beta function* is defined for $Re(z)>0$ and $Re(w)>0$ as [19]:

$$\beta(z,w)=\int_0^1 t^{z-1}(1-t)^{w-1}dt \tag{2.3a}$$

or in terms of gamma function for $z,w\in\mathbb{C}\backslash\mathbb{Z}_0^-$ as:

$$\beta(z,w)=\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}. \tag{2.3b}$$

(4) *Hypergeometric function* is defined for $z,a,b\in\mathbb{C}$ and $c\in\mathbb{C}\backslash\mathbb{Z}_0^-$ as [19]:

$${}_2F_1(a,b;c;z)=\sum_{k=0}^{\infty}\frac{(a)_k\ (b)_k}{(c)_k}\frac{z^k}{k!}, \tag{2.4}$$

where $|z|<1$, and the notation $(.)_k$ is the Pochhammer symbol which is defined for $z\in\mathbb{C}$ as:

$$(z)_k=\frac{\Gamma(z+k)}{\Gamma(k)}=\begin{cases} z(z+1)...(z+k-1), & k\in\mathbb{N},\\ (z)_0=1. \end{cases} \tag{2.5a}$$

Notice that,

$$(z)_{m+n}=(z)_m(z+m)_n. \tag{2.5b}$$

(5) *Confluent hypergeometric function* is defined for $z,a\in\mathbb{C}$ and $b\in\mathbb{C}\backslash\mathbb{Z}_0^-$ as [19]:

$${}_1F_1(a;b;z)=\sum_{k=0}^{\infty}\frac{(a)_k}{(b)_k}\frac{z^k}{k!}. \tag{2.6}$$

(6) *Mellin-Barnes integral* is an integral that generally has the form [12]:

$$\frac{1}{2\pi i}\int_C \phi(s) z^s ds, \tag{2.7}$$

where $z \in \mathbb{C}$; C is a contour in the complex plane initiate at $p - i\infty$ and terminate at $p + i\infty$ with $Re(s) = p$, and the integral kernel $\phi(s)$ assumed to has the form:

$$\phi(s) = \frac{g_1(s)\ g_2(s)}{g_3(s)\ g_4(s)}, \tag{2.8}$$

where $g_1(s), g_2(s), g_3(s)$ and $g_4(s)$ are product of gamma function.

(7) *H-function* is defined via Mellin-Barnes integral as [10]:

$$\begin{aligned} H^{m,n}_{p,q}(z) &\equiv H^{m,n}_{p,q}\left[z \middle| \begin{array}{c} (a_1,\tau_1),...,(a_p,\tau_p) \\ (b_1,\lambda_1),...,(b_q,\lambda_q) \end{array} \right] \\ &= \frac{1}{2\pi i}\int_C \mathcal{H}^{m,n}_{p,q}(s)\ \ z^{-s} ds \end{aligned} \tag{2.9}$$

with

$$\mathcal{H}^{m,n}_{p,q}(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j+\lambda_j s)\prod_{i=1}^{n}\Gamma(1-a_i-\tau_i s)}{\prod_{i=n+1}^{p}\Gamma(a_i+\tau_i s)\prod_{j=m+1}^{q}\Gamma(1-b_j-\lambda_j s)}, \tag{2.10}$$

where m, n, p, q are integers such that $0 \leq m \leq q$ and $0 \leq n \leq p$, for $a_i, b_j \in \mathbb{C}$ and $\tau_i, \lambda_j \in \mathbb{R}^+$ $(i = 1, ..., p; j = 1, ..., q)$.

(8) *Fox-Wright function* [12]

$${}_p\Psi_q\left[\begin{array}{c} (a_i,\tau_i)_{1,p} \\ (b_j,\lambda_j)_{1,q} \end{array} \middle| z \right] = \sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\tau_i k)}{\prod_{j=1}^{q}\Gamma(b_j+\lambda_j k)}\frac{z^k}{k!}, \tag{2.11}$$

where $z, a_i, b_j \in \mathbb{C}$ and $\tau_i, \lambda_i \in \mathbb{R}$ $(i = 1, ..., p$ and $j = 1, ..., q)$.

(9) *Radius of convergence for an infinite series* of the form $\sum_{k=0}^{\infty} a_k z^k$ can be found by any of the following two formulas [22]

$$R = \liminf_{k\to\infty}\frac{1}{\sqrt[k]{|a_k|}}, \tag{2.12a}$$

$$R = \lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right|, \tag{2.12b}$$

if the limit exist.

(10) *Order and type* of an entire function f(z) that represented as infinite series of the form $\sum_{k=0}^{\infty} a_k z^k$ can be found by the following formulas respectively, [8, 22]

$$\frac{1}{\rho} = \liminf_{k\to\infty} \frac{\log \frac{1}{|a_k|}}{k \log k}, \tag{2.13}$$

$$(\sigma e \rho)^{\frac{1}{\rho}} = \limsup_{k\to\infty} \left(k^{\frac{1}{\rho}} \sqrt[k]{|a_k|} \right). \tag{2.14}$$

3. Main results

This section, defines a new seven-parameter function considering one complex variable as a generalization of the standard Mittag-leffler function and some generalized Mittag-Leffler functions. Besides, it generalizes hypergeometric function and confluent hypergeometric function along with all their special cases.

Let $z, \tau_1, \tau_2 \in \mathbb{C}$ and $\min\{Re(a), Re(b), Re(c), Re(\lambda_1), Re(\lambda_2)\} > 0$. Then

$$E^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k \; k!} \frac{z^k}{\Gamma(\tau_1 k + \lambda_1)\Gamma(\tau_2 k + \lambda_2)}. \tag{3.1}$$

The following special cases can directly obtained:

(1) $E^{1,b,b}_{\tau_1,1,0,1}(z)$ gives the standard Mittag-Leffler function defined in (1.1).

(2) $E^{1,b,b}_{\tau_1,\lambda_1,0,1}(z)$ gives Wiman's function defined in (1.2).

(3) $E^{1,b,b}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z)$ gives Dzherbashian four-parameter Mittag-Leffler function defined in (1.3).

(4) $E^{a,b,b}_{\tau_1,\lambda_1,0,1}(z)$ gives the three-parameter Mittag-Leffler function defined in (1.4).

(5) $E^{a,1,c}_{\tau_1,\lambda_1,1,1}(z)$ gives Salim four-parameter Mittag-Leffler function defined in (1.7).

(6) $E^{a,b,b}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z)$ gives the five-parameter Mittag-Leffler function defined in (1.9).

(7) $E^{a,b,c}_{0,1,0,1}(z)$ gives the hypergeometric function defined in (2.4).

(8) $E^{a,1,c}_{1,1,0,1}(z)$ gives the confluent hypergeometric function defined in (2.6).

In addition, all the consequence special cases of the above functions. One may asks about the region of the complex plane in which the function (3.1) converges? The following theorem determine its radius of convergence there.

Theorem 3.1. (Radius of Convergence) *For any $z \in \mathbb{C}$, the series* (3.1) *converges in the whole complex plane.*

Proof. Write the series (3.1) in the form

$$\sum_{k=0}^{\infty} \ell_k z^k,$$

where

$$\ell_k = \frac{(a)_k \, (b)_k}{(c)_k \;\, k!\,\Gamma(\tau_1 k + \lambda_1)\,\Gamma(\tau_2 k + \lambda_2)}.$$

In order to obtain the radius of convergence of this series; we use formula (2.12b). Beforehand, we use expression (2.5a) to write the coefficient ℓ_k in terms of gamma function

$$\ell_k = \frac{\Gamma(c)\Gamma(a+k)\Gamma(b+k)}{\Gamma(a)\Gamma(b)\Gamma(c+k)\Gamma(k+1)\Gamma(\tau_1 k+\lambda_1)\Gamma(\tau_2 k+\lambda_2)}.$$

Accordingly

$$\left|\frac{\ell_k}{\ell_{k+1}}\right| = \left|\frac{(c+k)(k+1)}{(a+k)(b+k)}\right| \left|\frac{\Gamma(\tau_1 k+\tau_1+\lambda_1)}{\Gamma(\tau_1 k+\lambda_1)}\right| \left|\frac{\Gamma(\tau_2 k+\tau_2+\lambda_2)}{\Gamma(\tau_2 k+\lambda_2)}\right|.$$

Applying formula (2.12b) we have

$$R = \lim_{k\to\infty} \left(\left|\frac{(c+k)(k+1)}{(a+k)(b+k)}\right| \left|\frac{\Gamma(\tau_1 k+\tau_1+\lambda_1)}{\Gamma(\tau_1 k+\lambda_1)}\right| \left|\frac{\Gamma(\tau_2 k+\tau_2+\lambda_2)}{\Gamma(\tau_2 k+\lambda_2)}\right| \right).$$

It is obvious that the value of the first term of above expression equal to 1. For the other two terms, we use formula (2.2a) to get

$$\frac{\Gamma(\tau_1 k+\tau_1+\lambda_1)}{\Gamma(\tau_1 k+\lambda_1)} = k^{\tau_1}\left[1+\frac{\tau_1(\tau_1-1)}{2k}+O\left(\frac{1}{k^2}\right)\right], \quad k\to\infty$$

and

$$\frac{\Gamma(\tau_2 k+\tau_2+\lambda_2)}{\Gamma(\tau_2 k+\lambda_2)} = k^{\tau_2}\left[1+\frac{\tau_2(\tau_2-1)}{2k}+O\left(\frac{1}{k^2}\right)\right], \quad k\to\infty.$$

We can easily find that the limit as k goes to infinity for the above expressions gives infinity. That yields $R=\infty$. □

In the next theorem, we confirm the analyticity of the function (3.1) in the whole complex plane and estimate its order and type, respectively.

Theorem 3.2. (Order and Type) *The function* (3.1) *is an entire function of order* $\rho = \frac{1}{Re(\tau_1+\tau_2)}$ *and type* $\sigma = \left(\frac{1}{\rho|\tau_1|}\right)^{\rho Re(\tau_1)} \left(\frac{1}{\rho|\tau_2|}\right)^{\rho Re(\tau_2)}$.

Proof. From Theorem 3.1 and according to well-known fact in the complex analysis, we observe that the series defined in (3.1) is an entire function, thus we can infer its order by using formula (2.13). Note that

$$\frac{1}{|\ell_k|} = \left| \frac{\Gamma(a)\Gamma(b)\Gamma(c+k)\Gamma(k+1)\Gamma(\tau_1 k+\lambda_1)\Gamma(\tau_2 k+\lambda_2)}{\Gamma(c)\Gamma(a+k)\Gamma(b+k)} \right|.$$

Applying Stirling's formulas given in (2.1b) and (2.1c) on each gamma function in the above expression, we imply

$$\frac{1}{|\ell_k|} \sim \left| \frac{2\pi\Gamma(a)\Gamma(b)}{\Gamma(c)} \right| |k|^{c-a-b+1} |\tau_1 k|^{\tau_1 k+\lambda_1-\frac{1}{2}} |\tau_2 k|^{\tau_2 k+\lambda_2-\frac{1}{2}} e^{-k(\tau_1+\tau_2)}.$$

It follows that

$$\begin{aligned}\frac{\log\frac{1}{|\ell_k|}}{k\log k} \sim\ & \frac{\log\left|\frac{2\pi\Gamma(a)\Gamma(b)}{\Gamma(c)}\right| + \left(\tau_1 k+\lambda_1-\frac{1}{2}\right)\log|\tau_1| + \left(\tau_2 k+\lambda_2-\frac{1}{2}\right)\log|\tau_2|}{k\log k} \\ & + \frac{\lambda_1+\lambda_2+c-a-b}{k} - \frac{(\tau_1+\tau_2)}{\log k} + \tau_1+\tau_2.\end{aligned}$$

Consequently, and due to formula (2.13), we get

$$\frac{1}{\rho} = \liminf_{k\to\infty} \frac{\log\frac{1}{|\ell_k|}}{k\log k} = Re(\tau_1+\tau_2).$$

Immediately, we find that the function (3.1) has the order

$$\rho = \frac{1}{Re(\tau_1+\tau_2)}. \tag{3.2}$$

Respectively, we will estimate the type of the function (3.1), to do this we use Stirling's formula (2.1b) and (2.1c) on the coefficient ℓ_k, so

$$|\ell_k|^{\frac{1}{k}} \sim \left|\frac{\Gamma(c)}{2\pi\Gamma(a)\Gamma(b)}\right|^{\frac{1}{k}} |k|^{\frac{a+b-c-\tau_1-\tau_2-\lambda_1-\lambda_2}{k}} |\tau_1|^{-\tau_1-\left(\frac{\lambda_1+\frac{1}{2}}{k}\right)} |\tau_2|^{-\tau_2-\left(\frac{\lambda_2+\frac{1}{2}}{k}\right)} e^{\tau_1+\tau_2}.$$

Now apply formula (10) then use the result (3.2), we see that

$$\limsup_{k\to\infty}\left(k^{\frac{1}{\rho}}|\ell_k|^{\frac{1}{k}}\right) \sim |\tau_1|^{-Re(\tau_1)} |\tau_2|^{-Re(\tau_2)} e^{Re(\tau_1+\tau_2)}.$$

Alternatively from formula (2.14), we have

$$(\sigma e\rho)^{\frac{1}{\rho}} = \left(\frac{\sigma e}{\tau_1+\tau_2}\right)^{Re(\tau_1+\tau_2)}.$$

Equating the above two expressions then simplify the result, we obtain

$$\sigma^{Re(\tau_1+\tau_2)} = \left(\frac{\tau_1+\tau_2}{|\tau_1|}\right)^{Re(\tau_1)}\left(\frac{\tau_1+\tau_2}{|\tau_2|}\right)^{Re(\tau_2)}.$$

Therefore, the function (3.1) is of the type

$$\sigma = \left(\frac{1}{\rho|\tau_1|}\right)^{\rho Re(\tau_1)}\left(\frac{1}{\rho|\tau_2|}\right)^{\rho Re(\tau_2)}.$$

□

The following theorem, afford a major differentiation formula involving the *mth* derivative of the function (3.1).

Theorem 3.3. (*mth* Derivative) *For* $m \in \mathbb{N}$*, the function* (3.1) *satisfy the following relation:*

$$\left(\frac{d}{dz}\right)^m E^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z) = \frac{(a)_m(b)_m}{(c)_m}\, E^{a+m,b+m,c+m}_{\tau_1,\tau_1 m+\lambda_1,\tau_2,\tau_2 m+\lambda_2}(z). \tag{3.3}$$

Proof. For the left-hand side, we have

$$\begin{aligned}\left(\frac{d}{dz}\right)^m E^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z) &= \sum_{k=0}^{\infty}\frac{(a)_k\,(b)_k}{(c)_k\,k!\Gamma(\tau_1 k+\lambda_1)\Gamma(\tau_2 k+\lambda_2)}\left(\frac{d}{dz}\right)^m z^k\\ &= \sum_{k=0}^{\infty}\frac{(a)_k\,(b)_k}{(c)_k\,\Gamma(\tau_1 k+\lambda_1)\Gamma(\tau_2 k+\lambda_2)}\frac{z^{k-m}}{(k-m)!}\\ &= \sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_{m+n}}{(c)_{m+n}\,n!}\frac{z^n}{\Gamma(\tau_1(n+m)+\lambda_1)\Gamma(\tau_2(n+m)+\lambda_2)}.\end{aligned}$$

In order to identify the above expression with function (3.1), we use the Pochhammer property (2.5b), we conclude

$$\begin{aligned}\left(\frac{d}{dz}\right)^m E^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z) &= \sum_{n=0}^{\infty}\frac{(a)_m(a+m)_n(b)_m(b+m)_n}{(c)_m(c+m)_n\,n!}\\ &\quad\times\frac{z^n}{\Gamma(\tau_1 n+\tau_1 m+\lambda_1)\Gamma(\tau_2 n+\tau_2 m+\lambda_2)}\\ &= \frac{(a)_m(b)_m}{(c)_m}\, E^{a+m,b+m,c+m}_{\tau_1,\tau_1 m+\lambda_1,\tau_2,\tau_2 m+\lambda_2}(z).\end{aligned}$$

□

Many important features reveal from the integral representations of the Mittag-Leffler function, thus as a main type of this representations, we focus to obtain the Mellin-Barnes integral representation for the function (3.1) in the following theorem.

Theorem 3.4. (Mellin-Barnes Integral) *For each $z \in \mathbb{C}$ with $|\arg(z)| < \pi$, the function* (3.1) *has the following Mellin-Barnes integral representation:*

$$E^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z) = \frac{\Gamma(c)}{2\pi i \Gamma(a)\Gamma(b)} \int_{\mathcal{C}} \frac{\Gamma(s)\Gamma(a-s)\Gamma(b-s)}{\Gamma(c-s)\Gamma(\lambda_1-\tau_1 s)\Gamma(\lambda_2-\tau_2 s)}(-z)^{-s} ds, \tag{3.4}$$

where $\mathcal{C}$ is the integration contour beginning at $\lambda - i\infty$ going to $\lambda + i\infty$ with $0 < \lambda < \min\{Re(a), Re(b)\}$ splitting all the poles at $s = -k$, $(k \in \mathbb{N}_0)$ to the left and the poles at both $s = a + m$ and $s = b + n$, $(m, n \in \mathbb{N}_0)$ to the right.

Proof. To evaluate the integral (3.4) within the complex plane, we close the contour $\mathcal{C}$ such that only the poles at $s = -k$, $(k \in \mathbb{N}_0)$ contribute. Thus, consider

$$\Omega = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{\Gamma(s)\Gamma(a-s)\Gamma(b-s)}{\Gamma(c-s)\Gamma(\lambda_1-\tau_1 s)\Gamma(\lambda_2-\tau_2 s)}(-z)^{-s} ds. \tag{3.5}$$

From using the residue theorem, we obtain

$$\begin{aligned}
\Omega &= \sum_{k=0}^{\infty} \lim_{s\to -k} \left[(s+k)\frac{\Gamma(s)\Gamma(a-s)\Gamma(b-s)}{\Gamma(c-s)\Gamma(\lambda_1-\tau_1 s)\Gamma(\lambda_2-\tau_2 s)}(-z)^{-s}\right] \\
&= \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)(-z)^k}{\Gamma(c+k)\Gamma(\lambda_1+\tau_1 k)\Gamma(\lambda_2+\tau_2 k)} . \lim_{s\to -k}\left[\frac{\Gamma(s+k-1)}{(s+k-1)\ \dots\ s}\right] \\
&= \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)(-z)^k}{\Gamma(c+k)\Gamma(\lambda_1+\tau_1 k)\Gamma(\lambda_2+\tau_2 k)} . \frac{(-1)^k}{k!}.
\end{aligned}$$

By simplifying the above expression and using the Pochhammer definition (2.5a), we get

$$\Omega = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} E^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z).$$

Return with this result to expression (3.5), we find that

$$\frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\Gamma(s)\Gamma(a-s)\Gamma(b-s)}{\Gamma(c-s)\Gamma(\lambda_1-\tau_1 s)\Gamma(\lambda_2-\tau_2 s)}(-z)^{-s} ds = \frac{\Gamma(a)\Gamma(b)}{\Gamma(c)} E^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z),$$

which immediately gives the intended result. □

The following theorem discuss the Euler transform for the function (3.1), which gives a connection between this function and the Fox-wright function.

Theorem 3.5. (Euler Transform) *Let* $u, v \in \mathbb{C}$ *and* $\mu > 0$, *then the function* (3.1) *satisfy the following relation:*

$$\begin{aligned}&\int_0^1 z^{u-1}(1-z)^{v-1}E^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(xz^{\mu})\ dz\\&\quad=\frac{\Gamma(c)\Gamma(v)}{\Gamma(a)\Gamma(b)}\ {}_3\Psi_4\left[\begin{array}{c}(a,1),(b,1),(u,\mu)\\(c,1),(\tau_1,\lambda_1),(\tau_2,\lambda_2),(u+v,\mu)\end{array}\Big|\ x\right].\end{aligned}\tag{3.6}$$

Proof. Take the left-hand side of the above expression then by means of the function (3.1) and the beta function (2.3a), (2.3b) respectively, we have

$$\begin{aligned}&\int_0^1 z^{u-1}(1-z)^{v-1}E^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(xz^{\mu})\ dz\\&\quad=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k\ k!}\frac{x^k}{\Gamma(\tau_1k+\lambda_1)\Gamma(\tau_2k+\lambda_2)}\ \beta(\mu k+u,v)\\&\quad=\frac{\Gamma(c)\Gamma(v)}{\Gamma(a)\Gamma(b)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(b+k)\Gamma(\mu k+u)}{\Gamma(c+k)\Gamma(\tau_1k+\lambda_1)\Gamma(\tau_2k+\lambda_2)\Gamma(\mu k+u+v)}\frac{x^k}{k!}\end{aligned}$$

comparing the above expression with the Fox-Wright function (2.11), we obtain the desired result. □

As one an important functional relations, the next theorem establish a recurrence relation for the function (3.1).

Theorem 3.6. (Recurrence Relation)

$$E^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z)=\lambda_1E^{a,b,c}_{\tau_1,\lambda_1+1,\tau_2,\lambda_2}(z)+\tau_1z\frac{d}{dz}E^{a,b,c}_{\tau_1,\lambda_1+1,\tau_2,\lambda_2}(z).\tag{3.7}$$

Proof. By virtue of the function (3.1) and easy simplification the right-hand side becomes as

$$\begin{aligned}&\lambda_1\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k\ k!}\frac{z^k}{\Gamma(\tau_1k+\lambda_1)\Gamma(\tau_2k+\lambda_2)}+\tau_1k\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k\ k!}\frac{z^k}{\Gamma(\tau_1k+\lambda_1)\Gamma(\tau_2k+\lambda_2)}\\&\quad=(\tau_1+\lambda_1k)\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k\ k!}\frac{z^k}{\Gamma(\tau_1k+\lambda_1)\Gamma(\tau_2k+\lambda_2)}.\end{aligned}$$

By using the recurrence relation of the gamma function, that is,

$$\Gamma(z+1)=z\Gamma(z),$$

we acquire the left-hand side of the required relation. □

4. Explicit formulas

As all the Mittag-Leffler functions, the function (3.1) has a relation with certain elementary and special functions; some are special cases and others has connection as functional relations as we mentioned previously in Section 3. This segment, investigates the relation between the function (3.1) and both the Fox-Wright function and the H-function.

If we use the Pochhammer definition (2.5a) to rewrite the function (3.1) then compare the resulting formula with the definition of Fox-wright function (2.11), we find

$$E^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\ {}_2\Psi_3\left[\begin{array}{c} (a,1),(b,1) \\ (c,1),(\tau_1,\lambda_1),(\tau_2,\lambda_2) \end{array}\middle|\ z\right]. \tag{4.1}$$

Moreover, in agreement with expression (3.4) and definitions (2.9), (2.10) it is not difficult to observe that the function (3.1) appears as a special case of the H-function,

$$E^{a,b,c}_{\tau_1,\lambda_1,\tau_2,\lambda_2}(z) = H^{1,2}_{2,4}\left[z\ \middle|\ \begin{array}{c} (1-a,1),(1-b,1) \\ (0,1),(1-c,1),(1-\lambda_1,\tau_1),(1-\lambda_2,\tau_2) \end{array}\right]. \tag{4.2}$$

5. Conclusion and discussion

The central idea of this work is to define a new function of one complex variable and seven complex parameters as an exclusive generalization of the standard Mittag-Leffler function. It is noteworthy to declare that this function generalizes some another special functions for instance, hypergeometric function, confluent hypergeometric function and several generalizations of the standard Mittag-Leffler function with all their consequence elementary special cases.

Additionally, a detailed discussion was investigated for many essential properties, specifically the properties whose assumed as a basic in the theory of entire functions such as radius of convergence, order and type. Subsequently, we study further important properties as derivation, Mellin-Barnes integral representation, Euler transform and recurrence relation.

As another tendency of this work, we managed to establish certain explicit formulas for our new function in order to describe it in terms of the Fox-wright function and the H-function that are significant in many areas of application one can motivate for future work.

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