International Journal of Applied Mathematics
& Statistical Sciences (IJAMSS)
ISSN(P): 2319-3972; ISSN(E): 2319-3980
Vol. 5, Issue 2, Feb - Mar 2016; 51-72

International Academy of Science, Engineering and Technology
Connecting Researchers; Nurturing Innovations

TOPOLOGICAL STRUCTURES USING MIXED DEGREE SYSTEMS IN GRAPH THEORY

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ABSTRACT

This paper is concerned with introducing and studying the *M*-space by using the mixed degree systems which are the core concept in this paper. The necessary and sufficient condition for the equivalence of two reflexive *M*-spaces is super imposed. In addition, the *m*-derived graphs, *m*-open graphs, *m*-closed graphs, *m*-interior operators, *m*-closure operators and *M*-subspace are introduced. From an *M*-space, a unique supratopological space is introduced. Furthermore, the *m*-continuous (*m*-open and *m*-closed) functions are defined and the fundamental theorem of the *m*-continuity is provided. Finally, the *m*-homeomorphism is defined and some of its properties are investigated.

KEYWORDS: Digraph, In-Degree System, Mixed Degree System, *M*-Space, Out-Degree System

2000 Mathematics Subject Classification: 04A05, 54A05, 05C20.

1. INTRODUCTION

For a long time, many individuals believed that abstract topological structures have limited application in the generalization of real line and complex plane or some connections to Algebra and other branches of mathematics. And it seems that there is a big gap between these structures and real life applications. We noticed that in some situations, the concept of relation is used to get topologies that are used in important applications such as computing topologies [12], recombination spaces [5, 6, and 13] and information granulation which are used in biological sciences and some other fields of applications.

Topological graph theory [1, 2, 4, 9, and 10] is a branch of mathematics, whose concepts exists not only in almost all branches of mathematics, but also in many real life applications. We believe that topological graph structure will be an important base for narrow the gap between topology and its applications.

A directed graph or digraph [11] is pair $G = (V(G), E(G))$ where $V(G)$ is a non-empty set (called vertex set) and $E(G)$ of ordered pairs of elements of $V(G)$ (called edge set). An edge of the from (v, v) is called a loop. If $v\in V(G)$, the out-degree of v is $|\{u\in V(G) : (v, u)\in E(G)\}|$ and in-degree of v is $|\{u\in V(G) : (u, v)\in E(G)\}|$. A digraph is reflexive if $(v, v)\in E(G)$ for each $v\in V(G)$, symmetric if $(v, u)\in E(G)$ implies $(u, v) \in E(G)$, transitive if $(v, u)\in E(G)$ and $(u, w)\in E(G)$ implies $(v, w)\in E(G)$, tolerance if it is reflexive and symmetric, dominance if it is reflexive and transitive, equivalence if it is reflexive and symmetric and transitive, serial if for all $v\in V(G)$ there exists $u\in V(G)$ such that $(v, u)\in E(G)$.A sub graph of a graph G is a graph each of whose vertices belong to $V(G)$ and each of whose edges belong to $E(G)$. An empty graph [3] if the vertices set and edge set is empty. A subfamily μ of X is said to supratopology [8] on X if (i) $X, \phi\in\mu$ (ii) if $A_i\in\forall i\in j$ then $\bigcup A_i\in\mu$. (X,μ) is called supratopology space. Let $G = (V (G), E (G))$ be a digraph, the digraph inverseG^{-1} [7] is specified by the same set of vertices $V (G)$ and a set of edge $E (G)^{-1} = \{(u, v): (v, u) \in E (G)\}$.

2. MIXED DEGREE SYSTEMS AND *M*-SPACES

In this section, we introduce and investigate the notions of mixed degree systems, *M*-spaces and *m*-derived graphs which are essential for our present study.

Definition 2.1.

Let $G=(V(G), E(G))$ be digraph and a vertex $v\in V(G)$.

(a) The out-degree set of v is denoted by vD and defined by: $vD=\{u\in V(G): (v, u)\in E(G)\}$ and

(b) The in-degree set of v is denoted by Dv and defined by: $Dv=\{u\in V(G): (u, v)\in E(G)\}$.

Definition 2.2.

Let $G=(V(G),E(G))$be a digraph, then the out-degree system(resp. in-degree system)of a vertex $v\in V(G)$ is denoted by $ODS(v)$ (resp.$IDS(v)$)and defined by:

$ODS(v) = \{vD\}$ (resp. $IDS(v) = \{Dv\}$).

Example 2.3.

Let $G=(V(G),E(G))$ be a digraph such that $V(G)=\{v_1,v_2,v_3,v_4,v_5\}$,$E(G)=\{(v_1,v_1),(v_1,v_2),(v_2,v_3),(v_2,v_5),(v_4,v_3),(v_4,v_4),(v_5,v_2),(v_5,v_4),(v_5,v_5)\}$.

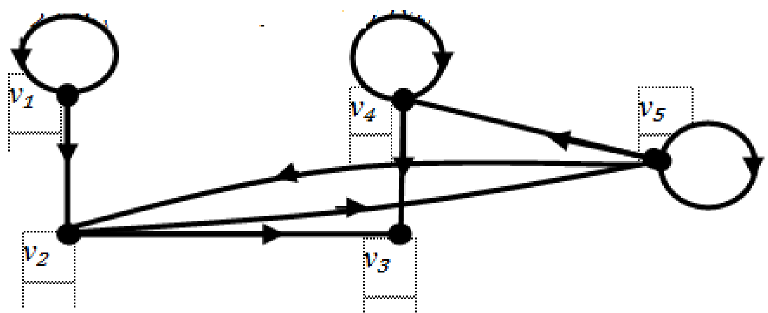

Figure 2.1: Graph *G* given in Example 2.3

Then we have $OD(v_1)=\{v_1, v_2\}$,$OD(v_2)=\{v_3, v_5\}$,$OD(v_3)=\phi$,$OD(v_4)=\{v_3, v_4\}$,$OD(v_5)=\{v_2, v_4, v_5\}$.$ODS(v_1)=\{\{v_1, v_2\}\}$,$ODS(v_2)=\{\{v_3, v_5\}\}$,$ODS(v_3)=\{\phi\}$,$ODS(v_4)=\{\{v_3, v_4\}\}$ and $ODS(v_5)=\{\{v_2, v_4, v_5\}\}$.

Also, we have $ID(v_1)=\{v_1\}$,$ID(v_2)=\{v_1,v_5\}$,$ID(v_3)=\{v_2,v_4\}$,$ID(v_4)=\{v_4,v_5\}$,$ID(v_5)=\{v_2,v_5\}$.$IDS(v_1)=\{\{v_1\}\}$,$IDS(v_2)=\{\{v_1,v_5\}\}$,$IDS(v_3)=\{\{v_2,v_4\}\}$,$IDS(v_4)=\{\{v_4,v_5\}\}$ and $IDS(v_5)=\{\{v_2, v_5\}\}$.

Definition 2.4.

Let $G=(V(G),E(G))$ be a digraph. The mixed degree system of a vertex $v\in V(G)$ is denoted by $MDS(v)$ and defined by $MDS(v)=\{ODS(v),IDS(v)\}$.

Definition 2.5.

Let $G=(V(G), E(G))$ be a digraph the mixed degree of a vertex $v\in V(G)$ is denoted by $MD(v)$ such that $MD(v)\in MDS(v)$.

Example2.6.

According to Example (2.3), the mixed degree systems are given by

$MDS(v_1)=\{\{v_1, v_2\},\{v_1\}\}$,$MDS(v_2)=\{\{v_3, v_5\},\{v_1, v_5\}\}$,$MDS(v_3)=\{\phi,\{v_2, v_4\}\}$,$MDS(v_4)=\{\{v_3, v_4\}, \{v_4, v_5\}\}$ and $MDS(v_5)=\{\{v_2, v_4, v_5\},\{v_2, v_5\}\}$.

Definition 2.7.

Let $G=(V(G),E(G))$ be a digraph and suppose that $\xi_m: V(G)\rightarrow P(P(V(G)))$ is a mapping which assigns for each v in $V(G)$ its mixed degree system in $P(P(V(G)))$.The pair (G,ξ_m) is called an M-space.

Example 2.8.

Let $G=(V(G),E(G))$ be a digraph such that $V(G)=\{v_1,v_2,v_3,v_4,v_5\}$,$E(G)=\{(v_1,v_2),(v_1,v_4),(v_2,v_2),(v_4,v_5),(v_4,v_5),(v_4,v_3),(v_5,v_5)\}$.

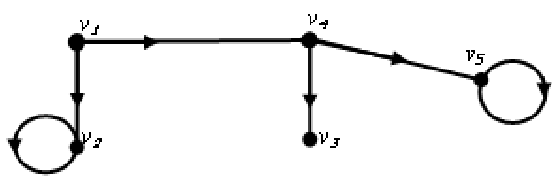

Figure 2.2: Graph *G* Given in Example 2.8

Thus we get

$\xi_m(v_1)=\{\{v_2, v_4\},\phi\}$,$\xi_m(v_2)=\{\{v_2\},\{v_1, v_2\}\}$,$\xi_m(v_3)=\{\phi,\{v_4\}\}$,$\xi_m(v_4)=\{\{v_3, v_5\},\{v_1\}\}$ and$\xi_m(v_5)=\{\{v_5\},\{v_4, v_5\}\}$.There for (G,ξ_m) is an M-space.

An M-space is defined by the mapping ξ_m and a given graph G for which there are defined two different mappings $\xi_{\square_1}$and $\xi_{\square_2}$given two different corresponding M-spaces.

It might see that the concept of M-spaces without additional assumptions on graph G is two general to embrace many properties. It will be seen however that, with suitable definitions, a whole concept of M-spaces can be developed and certain of its results find an application in generalized rough set theory.

Definition 2.9:

Let (G, ξ_m) be an M-space. A vertex v in $V(G)$ is called a limit vertex of a graph $H\subseteq G$ if every mixed degree of v contains at least one vertex of H different from v. The set of all limit vertices of a graph $H\subseteq G$ is called the m-derived graph of H and is denoted by $[V(H)]^{`}_m$, that is,

$[V(H)]^{`}_m=\{v\in V(G);\forall MD(v),MD(v)\cap(V(H)-\{v\})\neq\phi\}$.

Example 2.10.

In Example (2.8),if $H\subseteq G$,$H=(V(H),E(H))$:$V(H)=\{v_1, v_2, v_3\}$,$E(H)=\{(v_1, v_2),(v_2, v_2)\}$

Figure 2.3: Sub graph *H* of graph *G* Given in Example 2.10

Then $[V(H)]'_{m} = \{v_4\}$.

Suppose that $\Psi_m : P(V(G)) \to P(V(G))$ is am aping which assigns for every graph $H \subseteq G$ a set $\Psi_m(V(H)) \subseteq V(G)$ such that $\Psi_m(V(H)) = [V(H)]'_m$. Obviously, by Definition (2.9), the mapping Ψ_m satisfies the following properties:

(a) $\Psi_m(\phi) = \phi$,

(b) If $H \subseteq K$, then $\Psi_m(V(H)) \subseteq \Psi_m(V(K))$ for all $H, K \subseteq G$ and

(c) If $v \in \Psi_m(V(H))$, then $v \in (V(H) - \{v\})$.

Definition 2.11.

Two M-spaces (G_1, ξ_{m_1}) and (G_2, ξ_{m_2}) such that $V(G_1) = V(G_2)$ are said to be equivalent if the m-derived graph of each sub graph in (G_1, ξ_{m_1}) equal to the m-derived graph of the same sub graph in (G_2, ξ_{m_2}). In other words, the two M-spaces (G_1, ξ_{m_1}) and (G_2, ξ_{m_2}) are equivalent if and only if $[V(H)]'_{m_1} = [V(H)]'_{m_2}$ for all $V(H) \subseteq V(G_1)$.

Example 2.12.

Let $G_1 = (V(G_1), E(G_1)), G_2 = (V(G_2), E(G_2)) : V(G_1) = V(G_2) = \{v_1, v_2, v_3\}$ and

$E(G_1) = \{(v_1, v_1), (v_2, v_1), (v_3, v_2), (v_3, v_3)\}$ and $E(G_2) = \{(v_1, v_2), (v_2, v_3), (v_3, v_3)\}$.

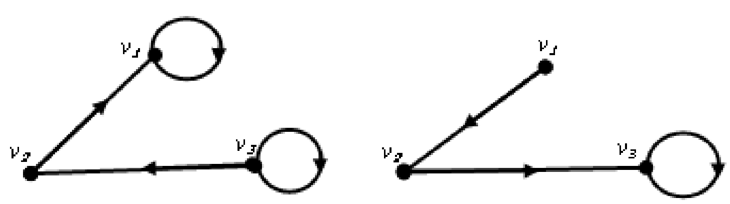

Figure 2.4: Graphs G_1 and G_2 given in Example 2.12

Then ξ_{m_1} Induced by G_1 is given by:

$\xi_{m_1}(v_1) = \{\{v_1\}, \{v_1, v_2\}\}, \xi_{m_1}(v_2) = \{\{v_1\}, \{v_3\}\}$ and $\xi_{m_1}(v_3) = \{\{v_2, v_3\}, \{v_3\}\}$.

Also, ξ_{m_2} induced by G_2 is given by:

$\xi_{m_2}(v_1) = \{\{v_2\}, \phi\}\}, \xi_{m_2}(v_2) = \{\{v_3\}, \{v_1\}\}$ and $\xi_{m_2}(v_3) = \{\{v_3\}, \{v_2, v_3\}\}$.

The two M-space (G_1, ξ_{m_1}) and (G_2, ξ_{m_2}) are equivalent.

Example 2.13:

Let: $G_1 = (V(G_1), E(G_1)), G_2 = (V(G_2), E(G_2)) : V(G_1) = V(G_2) = \{v_1, v_2, v_3\}, E(G_1) = \{(v_1, v_1), (v_1, v_2), (v_2, v_1), (v_2, v_3), (v_3, v_1)\}, E(G_2) = \{(v_1, v_2), (v_1, v_3), (v_2, v_2), (v_2, v_3), (v_3, v_1)\}$.

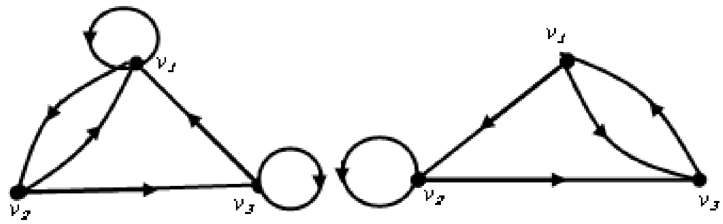

Figure 2.5: Graphs G_1 and G_2 Given in Example 2.13.

Then ξ_{m_1} Induced by $G_1$is given by:

$\xi_{m_1}(v_1)=\{\{v_1,v_2\},V(G_1)\}, \xi_{m_1}(v_2)=\{\{v_1,v_3\},\{v_1\}\}$ and$\xi_{m_1}(v_3)=\{\{v_1,v_3\},\{v_2,v_5\}\}$.

Also ξ_{m_2} induced by G_2 is given by:

$\xi_{m_2}(v_1)=\{\{v_2,v_3\},\{v_3\}\}, \xi_{m_2}(v_2)=\{\{v_2,v_3\},\{v_1,v_2\}\}$ and$\xi_{m_2}(v_3)=\{\{v_1\},\{v_1,v_2\}\}$.

Let $H\subseteq G_1,G_2$; $V(H)=\{v_1,v_3\}$, then$[V(H)]^{`}_{m_1}=\{v_2\}$ and$[V(H)]^{`}_{m_2}=\{v_1,v_2,v_3\}$. Accordingly, there exists $H\subseteq G_1,G_2$,namely $V(H)=\{v_1,v_3\}$ such that $[V(H)]^{`}_{m_1} \neq [V(H)]^{`}_{m_2}$and hence the two M-spaces (G_1,ξ_{m_1}) and (G_2,ξ_{m_2}) are not equivalent.

Definition 2.14.

An M-space $(G,\xi_m)$is called reflexive (resp.serial, symmetric, transitive, and equivalence)if ξ_m is induced by a reflexive (resp.serial, symmetric, transitive, and equivalence) graph.

Example 2.15.

Let $G=(V(G),E(G))$:$V(G) =\{v_1,v_2,v_3,v_4\}$, $E(G)=\{(v_1,v_1),(v_1,v_2),(v_2,v_2),(v_2,v_3),(v_3,v_3),(v_3,v_1),(v_3,v_4),(v_4,v_4)\}$.

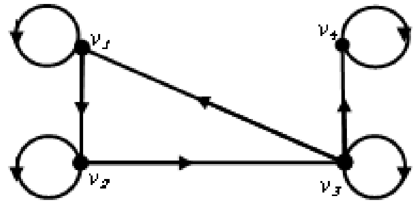

Figure 2.6: Graph G given in Example 2.15

Hence ξ_m is defined by $\xi_m(v_1) = \{\{v_1, v_2\}, \{v_1, v_3\}\}$, $\xi_m(v_2) = \{\{v_2, v_3\}, \{v_1, v_2\}\}$, $\xi_m(v_3) = \{\{v_1, v_3,v_4\}, \{v_2, v_3\}\}$ and$\xi_m(v_4) = \{\{v_4\}, \{v_3, v_4\}\}$.

Clearly, (G, ξ_m) is reflexive M-space.

Example 2.16.

Let $G=(V(G),E(G))$:$V(G)=\{v_1,v_2,v_3,v_4\}$,$E(G)=\{(v_1,v_1),(v_1,v_2),(v_2,v_4),(v_3,v_3),(v_3,v_2),(v_4,v_3)\}$.

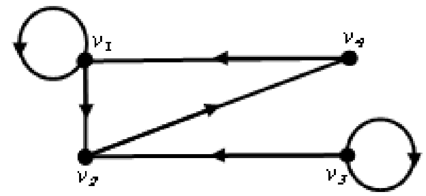

Figure 2.7: Graph *G* given in Example 2.16

Hence ξ_m is defined by $\xi_m(v_1) = \{\{v_1, v_2\}, \{v_1, v_4\}\}$, $\xi_m(v_2) = \{\{v_4\}, \{v_1, v_3\}\}$, $\xi_m(v_3) = \{\{v_2, v_3\}, \{v_3\}\}$ and$\xi_m(v_4) = \{\{v_4\}, \{v_2\}\}$.Clearly, (G, ξ_m) is serial*M*-space.

Example 2.17.

Let $G=(V(G),E(G))$:$V(G)=\{v_1,v_2,v_3,v_4\}$,$E(G)=\{(v_1,v_1),(v_1,v_2),(v_2,v_1),(v_2,v_3),(v_3,v_2),(v_3,v_3),(v_3,v_4),(v_4,v_3)\}$.

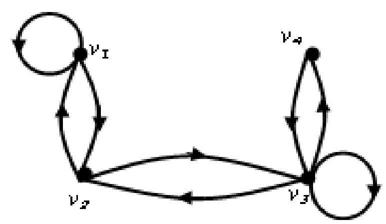

Figure 2.8: Graph *G* given in Example 2.17

Hence ξ_m is defined by $\xi_m(v_1) = \{\{v_1, v_2\}\}$, $\xi_m(v_2) = \{\{v_1, v_3\}\}$, $\xi_m(v_3) = \{\{v_2, v_3,v_4\}\}$ and$\xi_m(v_4) = \{\{v_3\}\}$.Clearly, (G, ξ_m) is symmetric *M*-space.

Example 2.18.

Let $G=(V(G),E(G))$:$V(G)=\{v_1,v_2,v_3,v_4\}$,$E(G)=\{(v_1,v_1),(v_1,v_2),(v_2,v_1),(v_2,v_2),(v_3,v_1),(v_3,v_2),(v_3,v_4),(v_4,v_1),(v_4,v_2)\}$.

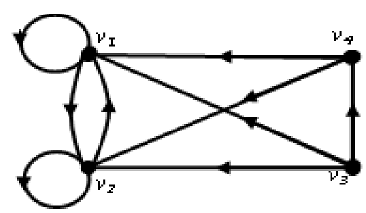

Figure 2.9: Graph *G* given in Example 2.18

Hence ξ_m is defined by $\xi_m(v_1) = \{\{v_1, v_2\}, \{v_1, v_2, v_3, v_4\}\}$, $\xi_m(v_2) = \{\{v_1, v_2\}, \{v_1, v_2, v_3, v_4\}\}$, $\xi_m(v_3) = \{\{v_1, v_2,v_4\},\phi\}$ and$\xi_m(v_4) = \{\{v_1, v_2\}, \{v_3\}\}$.Clearly, (G, ξ_m) is transitive*M*-space.

Example 2.19.

Let $G=(V(G),E(G))$:$V(G)=\{v_1,v_2,v_3,v_4\}$,$E(G)=\{(v_1,v_1),(v_1,v_2),(v_1,v_3),(v_2,v_1),(v_2,v_2),(v_2,v_3),(v_3,v_1),(v_3,v_2),(v_3,v_3),(v_4,v_4)\}$.

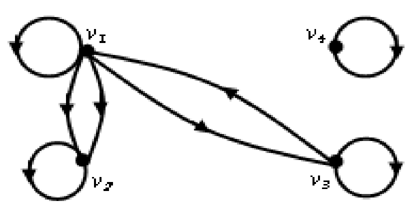

Figure 2.10: Graph *G* given in Example 2.19

Hence ξ_m is defined by $\xi_m(v_1) = \{\{v_1, v_2, v_3\}\}$, $\xi_m(v_2) = \{\{v_1, v_2, v_3\}, \{v_1, v_2\}\}$, $\xi_m(v_3) = \{\{v_1, v_3\}, \{v_1v_2, v_3\}\}$ and$\xi_m(v_4) = \{\{v_4\}\}$.Clearly, (G, ξ_m) is equivalence M-space.

Lemma 2.20.

In an reflexive M-space each vertex contained in each one of its mixed degrees.

Proof: Let (G, ξ_m) be a reflexive M-space. So ξ_m is induced by a reflexive graph G and hence $v \in OD(v)$ for all $v \in V(G)$. Since G is reflexive, then $G^{-1}$is also reflexive and so $v \in ID(v)$ for all $v \in V(G)$.Consequently $v \in MD(v)$ for all $v \in V(G)$.

Theorem 2.21. Two reflexive M-spaces (G_1, ξ_{m_1}) and (G_2, ξ_{m_2}) such that $V(G_1)=V(G_2)=V(G)$ are equivalent if and only if for each mixed degree $M_1D(v)$ of a vertex $v \in V(G)$ there exists $M_2D(v)$ which is contained in $M_1D(v)$ and vice versa.

Proof. Let$(G_1, \xi_{m_1})$and (G_2, ξ_{m_2}) be two equivalent reflexive M-spaces and $v \in V(G)$.Suppose that $M_1D(v)$ is mixed degree of v and since$(G_1, \xi_{m_1})$is reflexive M-space, then by Lemma(2.19),we have $v \in M_1D(v)$. Putting $V(H)=V(G)-M_1D(v)$,hence $M_1D(v) \cap V(H)=\phi$ and so $v \notin V(H)$ and $v \notin [V(H)]^{`}_{m_1}$. Since $(G_1, \xi_{m_1})$ and $(G_2, \xi_{m_2})$ are equivalent then the $m$-derived graphs of $H$ are the same in both $M$-spaces, i.e.$[V(H)]^{`}_{m_1}=[V(H)]^{`}_{m_2}$ and hence $v \in [V(H)]^{`}_{m_2}$. Accordingly, there exists $M_2D(v)$ such that $M_2D(v) \cap [V(H)-\{v\}]=\phi$ and since $v \notin V(H)$ then $M_2D(v) \cap V(H)=\phi$,therefore $M_2D(v) \subseteq V(G) - V(H)=M_1D(v)$, i.e.$M_2D(v) \subseteq M_1D(v)$.Similarly, because of the symmetry of the condition, for every mixed degree $M_2D(v)$ there exists a mixed degree $M_1D(v)$ which is contained in $M_2D(v)$.Consequently, the condition of the theorem is necessary.

Conversely, suppose that the condition of the theorem is satisfied and let $V(H) \subseteq V(G)$.If $v \notin [V(H)]^{`}_{m_1}$, then there is $M_1D(v)$ such that $M_1D(v) \cap [V(H)-\{v\}]=\phi$.But, by the condition of theorem, there exists $M_2D(v)$such that $M_2D(v) \subseteq M_1D(v)$, and so $M_2D(v) \cap [V(H)-\{v\}]=\phi$ which implies $v \notin [V(H)]^{`}_{m_2}$,and hence $[V(H)]^{`}_{m_2} \subseteq [V(H)]^{`}_{m_1}$. Similarly, we can show that $[V(H)]^{`}_{m_1} \subseteq [V(H)]^{`}_{m_2}$. As a consequence we see that $[V(H)]^{`}_{m_1}=[V(H)]^{`}_{m_2}$ for all $V(H) \subseteq V(G)$ and therefore the two M-space are equivalent.

The following example illustrates the idea of Theorem (2.20),

Example 2.22.

Let$G_1=(V(G_1),E(G_1))$:$V(G_1)=\{v_1,v_2,v_3\}$,$E(G_1)=\{(v_1,v_1),(v_1,v_2),(v_2,v_2),(v_3,v_2),(v_3,v_3)\}$ and $G_2=(V(G_2),E(G_2))$: $V(G_2)=\{v_1,v_2,v_3\}$,$E(G_2)=\{(v_1,v_1),(v_2,v_1),(v_2,v_2),(v_3,v_1)\}$.

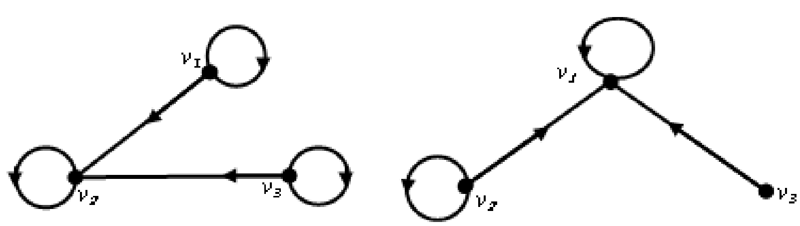

Figure 2.11: Graphs G_1 and G_2 given in Example 2.22

Then ξ_{m_1} induced by G_1 is given by$\xi_{m_1}(v_1)= \{\{v_1, v_2\}, \{v_1\}\}$, $\xi_{m_1}(v_2) = \{\{v_2\}, \{v_1, v_2, v_3\}\}$ and$\xi_{m_1}(v_3) = \{\{v_2, v_3\}, \{v_3\}\}$.

Also, ξ_{m_2} induced by G_2 is given by$\xi_{m_2}(v_1) = \{\{v_1\}, \{v_1, v_2\}\}$,$\xi_{m_2}(v_2) = \{\{v_1, v_2\}, \{v_2\}\}$ and$\xi_{m_2}(v_3) = \{\{v_1\}, \phi\}$.

Obviously, the two reflexive M-spaces (G_1, ξ_{m_1}) and (G_2, ξ_{m_2}) are equivalent since the condition of Theorem(2.21), is satisfied.

3. *M*-Closed Graph and *m*-Closure Operators

In this section, we introduce the notions of m-closed graphs and m-closure operators and we study some of their properties.

Definition 3.1.

In an M-space (G, ξ_m), a graph which contains all its limit vertices is called m-closed. The family $F_{\xi m}$ of all m-closed graphs of an M-space is defined by:

$\mathscr{F}_{\xi m}=\{V(H)\subseteq V(G);[V(H)]^{`}_{m}\subseteq V(H)\}$.

Theorem 3.2.

In an M-space, the intersection of any family of m-closed graphs is m-closed.

Proof. Let (G, ξ_m) be an M-space such that $K\in G$ and $V(K)=\bigcap_i (V(H_i); i \in I$, be the intersection of the m-closed graphs $H_i\subseteq G, i \in I$. Hence $K\subseteq H_i$ for all $i \in I$ which implies $[V(K)]^{`}_{m}\subseteq[V(H_i)]^{`}_{m}$ for all $i \in I$. But $[V(H_i)]^{`}_{m}\subseteq V(H_i)$ for all i $\in I$ since $H_i$ is $m$-closed and so $[V(K)]^{`}_{m}\subseteq V(H_i)$ for all $i \in I$ thus, $[V(K)]^{`}_{m}\subseteq\bigcap_i(V(H_i))=V(K)$ and hence K is m-closed.

If follows from definition of an m-closed graph that the empty graph ϕ is m-closed ($\phi^{`}_{m}=\phi\subseteq\phi$) and the whole $M$-space $G$ is also $m$-closed ($G^{`}_{m}\subseteq G$). Consequently, for every $H\subseteq G$ there exists at least one m-closed graph, namely G, containing H.

Remark 3.3. The union of two m-closed graphs contained in an M-space need not be m-closed as shown in the following example.

Example 3.4.

Let $G=(V(G),E(G))$: $V(G)=\{v_1,v_2,v_3,v_4,v_5\}$, $E(G)=\{(v_1,v_1),(v_1,v_5),(v_2,v_3),(v_2,v_4),(v_3,v_1),(v_3,v_3),(v_5,v_2),(v_5,v_4),(v_5,v_5)\}$

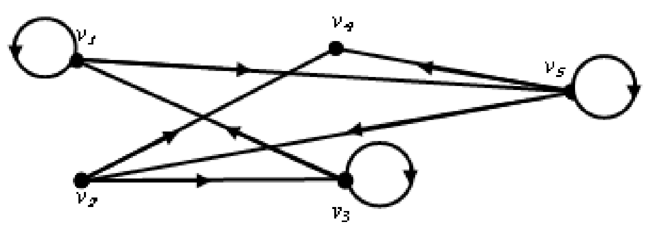

Figure 3.1: Graph *G* given in Example 3.4

$\xi_m(v_1) = \{\{v_1, v_5\}, \{v_1, v_3\}\}$, $\xi_m(v_2) = \{\{v_3, v_4\}, \{v_5\}\}$, $\xi_m(v_3) = \{\{v_1, v_3\}, \{v_2, v_3\}\}$, $\xi_m(v_4) = \{\phi, \{v_2, v_5\}\}$, $\xi_m(v_5) = \{\{v_2, v_4, v_5\}, \{v_1, v_5\}\}$.

Accordingly, the family $\mathscr{F}_{\xi m}$ of all m-closed graphs of this M-space is given by

$\mathscr{F}_{\xi m}=\{V(G), \phi, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_1,v_3\}, \{v_1,v_5\}, \{v_2,v_3\}, \{v_2,v_4\}, \{v_2,v_5\}, \{v_3,v_4\}, \{v_2,v_3,v_4\}, \{v_2,v_4,v_5\}, \{v_1,v_2,v_3,v_5\}\}$.

Obviously,thegraphs$H=(V(H),E(H))$:$V(H)=\{v_1\}$,$E(H)=\{(v_1,v_1)\}$and$K=(V(K),\ E(K))$:$V(K)=\{v_2\}$,$E(K)=\phi$are m-closed but their union$H\cup K=(V(H\cup K),E(H\cup K))$:$V(H\cup K)=\{v_1,v_2\}$, $E(H\cup K)=\{(v_1,v_1)\}$ is not m-closed.

Theorem 3.5. If (G,ξ_m) is an M-space and $H\subseteq G$ is m-closed graph, then every graph contained in H and containing $[V(H)]^{`}_m$is m-closed

Proof. Let (G,ξ_m) be an M-space and $H,K\subseteq G$ such that H is m-closed graph and$[V(H)]^{`}_m\subseteq V(K)\subseteq V(H)$.Since$V(K)\subseteq V(H)$then$[V(K)]^{`}_m\subseteq[V(H)]^{`}_m$andso $[V(K)]^{`}_m\subseteq V(K)$and therefor K is m-closed.

Corollary 3.6. The m-derived graph of an m-closed graph is m-closed.

Proof: The proof is an immediate consequence of the above theorem.

Definition 3.7. Let H be a sub graph of an M-space (G,ξ_m). The intersection of all m-closed graphs containing $H$is called the m-closure of H and is denoted by $Cl_m(V(H))$, i.e.

$Cl_m(V(H)) = \cap\ \{V(K) \in F_{\xi m};\ V(H) \subseteq V(K)\}$.

The operator Cl_m: $P(V(G)) \to P(V(G))$ is called m-closure operator.

By Theorem (3.2), $Cl_m(V(H))$ is m-closed graph for all $H\subseteq G$. Moreover, it is the smallest m-closed graph containing $V(H)$.H is m-closed if and only if $V(H)=Cl_m(V(H))$ and in particular, $Cl_m(Cl_m(V(H)))=Cl_m(V(H))$.

Example 3.8.

In Example (3.4), let $H\subseteq G$, $H = (V(H),\ E(H))$: $V(H)=\{v_1,v_2,v_3\}$,$E(H)=\{(v_1,v_1),(v_2,v_3),(v_3,v_3),(v_3,v_1)\}$

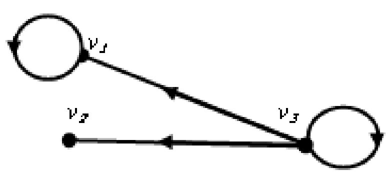

Figure 3.2: Sub Graph *H* of a Graph *G* given in Example 3.8

So, $Cl_m(V(H))=\{v_1,v_2,v_3,v_5\}$.

Proposition 3.9. If (G,ξ_m) is an M-space and$H\subseteq G$, then $V(H)\cup[V(H)]^{`}_m\subseteq Cl_m(V(H))$

Proof. Let (G,ξ_m) be an M-space and $H\subseteq G$. Since $V(H)\subseteq Cl_m(V(H))$then$[V(H)]^{`}_m\subseteq[Cl_m(V(H))]^{`}_m$. But $[Cl_m(V(H))]^{`}_m\subseteq Cl_m(V(H))$because $Cl_m(V(H))$ is $m$-closed and so $[V(H)]^{`}_m\subseteq Cl_m(V(H))$. Accordingly $V(H)\cup[V(H)]^{`}_m\subseteq Cl_m(V(H))$.

Remark 3.10. If $(G,\xi_m)$is an M-space $H\subseteq G$,then the relation$V(H)\cup[V(H)]^{`}_m=Cl_m(V(H))$ is not necessarily true.

The next example is employed as a counter example to show the above remark.

Example 3.11.

Let $G=(V(G),E(G))$:$V(G)=\{v_1,v_2,v_3,v_4,v_5\}$, $E(G)=\{(v_1,v_1),(v_2,v_3),(v_2,v_4),(v_3,v_4),(v_3,v_5),(v_4,v_1),(v_4,v_4),(v_5,v_2),(v_5,v_5)\}$

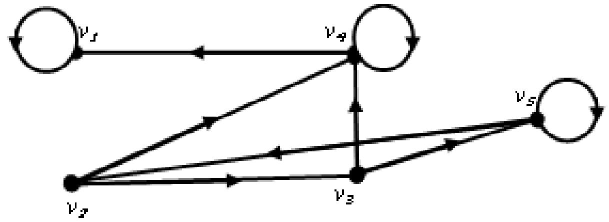

Figure 3.3: Graph *G* given in Example 3.11

So, $\xi_m$is given by $\xi_m(v_1) = \{\{v_1\}, \{v_1, v_4\}\}, \xi_m(v_2) = \{\{v_3, v_4\}, \{v_5\}\}, \xi_m(v_3) = \{\{v_4, v_5\}, \{v_2\}\}, \xi_m(v_4) = \{\{v_1, v_4\}, \{v_2, v_3, v_4\}\}$ and$\xi_m(v_5) = \{\{v_2, v_5\}, \{v_3, v_5\}\}$. Hence, we have

$\mathcal{F}_{\xi m}=\{V(G), \phi, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_1,v_4\}, \{v_1,v_5\}, \{v_3,v_4\}, \{v_1,v_3,v_4\}, \{v_2,v_3,v_5\}, \{v_2,v_3,v_4,v_5\}\}$.

Let $H\subseteq G$, $H=(V(H),E(H))$:$V(H)=\{v_2,v_4\},E(H)=\{(v_2,v_4),(v_4,v_4)\}$, then $[V(H)]_m^{`} = \{v_3\}$ and $Cl_m(V(H)) = \{v_2, v_3, v_4, v_5\}$. Obviously, $V(H)\cup [V(H)]_m^{`}\neq Cl_m(V(H))$

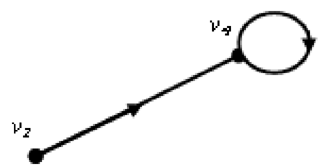

Figure 3.4: Sub graph *H* of a graph *G* given in Example 3.11

Proposition 3.12: If (G,ξ_m) is an *M*-space, then the *m*-closure operator Cl_m: $P(V(G))\rightarrow P(V(G))$ possesses the following properties for all $H, K\subseteq G$:

(a) $Cl_m(\phi)=\phi$,

(b) $Cl_m(V(G))=V(G)$,

(c) $V(H)\subseteq Cl_m(V(H))$,

(d) If $H\subseteq K$ then $Cl_m(V(H)) \subseteq Cl_m(V(K))$,

(e) $Cl_m (Cl_m(V(H))=Cl_m(V(H))$,

(f) $Cl_m(V(H) \cap V(K))\subseteq Cl_m(V(H))\cap Cl_m(V(K))$ and

(g) $Cl_m(V(H)\cup V(K))\supseteq Cl_m(V(H))\cup Cl_m(V(K))$.

Proof: Straightforward.

Remark 3.13. Let (G,ξ_m)be an *M*-space, then the following proposition are not true in general for every $H,K\subseteq G$:

(a) $Cl_m(V(H)\cap V(K))=Cl_m(V(H))\cap Cl_m(V(K))$ and

(b) $Cl_m(V(H)\cup V(K))=Cl_m(V(H))\cup Cl_m(V(K))$.

The following example illustrates Remark (3.13),

Example 3.14.

According to Example (3.11), we have

(a) Let $H=(V(H),E(H))$:$V(H)=\{v_2, v_4\}$, $E(H)=\{(v_2, v_4),(v_4, v_4)\}$then $Cl_m(V(H))=\{v_2, v_3, v_4, v_5\}$and $K=(V(K), E(K))$:$V(K)=\{v_2, v_5\}$, $E(K)=\{(v_5, v_2),(v_5, v_5)\}$ then $Cl_m(V(K))=\{v_2, v_3, v_5\}$. But, $H\cap K = (V(H)\cap V(K),E(H)\cap E(K))$: $V(H)\cap V(K) = \{v_2\}$, $E(H)\cap E(K)= \phi$ such that $Cl_m(H \cap K) =Cl_m(V(H) \cap V(K)) = \{v_2\}$ and so $Cl_m(V(H) \cap V(K))\neq Cl_m(V(H)) \cap Cl_m(V(K))$.

(b) Let $H=(V(H),E(H))$:$V(H)=\{v_4\}$, $E(H)=\{(v_4, v_4)\}$ then $Cl_m(V(H))=\{v_4\}$and $K=(V(K), E(K))$: $V(K)=\{v_5\}$, $E(K)=\{(v_5, v_5)\}$then $Cl_m(V(K))=\{v_5\}$. But, $H\cup K = (V(H)\cup V(K),E(H)\cup E(K))$: $V(H)\cup V(K) = \{v_4, v_5\}$, $E(H)\cup E(K) = \{(v_4, v_4), (v_5, v_5)\}$ such that $Cl_m(H \cup K) =Cl_m(V(H) \cup V(K)) = \{v_2, v_3, v_4, v_5\}$ and so $Cl_m(V(H) \cup V(K))\neq Cl_m(V(H)) \cup Cl_m(V(K))$.

4. *m*-OPEN GRAPHS AND *m*-INTERIOR OPERATOR

In this section we introduce the notions of m-open graphs, m-interior operators, m-boundary graphs and we study some of their properties. Also, the M-subspace is defined and some of its properties are investated.

Definition 4.1: The complement of an m-closed graph with respect to the M-space (G,ξ_m) in which it is contained is called m-open graph. The family $\Omega_{\xi m}$ of all m-open graphs is defined by

$\Omega_{\xi m}=\{V(O)\subseteq V(G);V(O)=V(G) -V(H)$ such that $V(H)\in \mathcal{F}_{\xi m}\}$.

In an M-space (G,ξ_m),since the m-derived graph is uniquely defined it follows that the family $F_{\xi m}$ of all m-closed graphs of this M-space is also uniquely defined. Accordingly, the corresponding family$\Omega_{\xi m}$ of all m-open graphs is also uniquely defined. As a consequence, the families of m-open graphs in two equivalents M-spaces are identical.

Theorem 4.2: In an M-space, the union of any family of m-open graphs is m-open.

Proof. Let(G,ξ_m) be an M-space such that $H\subseteq G$and$V(H)=\cup_i V(H_i)$ be the union of the m-open graphs $H_i\subseteq G,i \in I$.Hence $V(G) -V(H)=V(G)-\cup_i V(H_i)=\cap_i[V(G) -V(H_i)]$.Putting $V(K_i)=[V(G) -V(H_i)]$ we have $V(G) -V(H_i)=\cap_i V(K_i)$ where K_i, $i \in I$,is m-closed graph. Hence byTheorem (3.2),$V(G) -V(H_i)$ is m-closed and therefore H is m-open.

Remark 4.3: Obviously, the empty graph and the whole M-space G are m-open graphs.

Remark 4.4. The intersection of two m-open graphs contained in an M-space is not necessarily m-open graph as shown in the next example.

Example 4.5.

According to Example (3.11).We have $\Omega_{\xi m} = \{V(G), \phi,\{v_1\}, \{v_1, v_4\}, \{v_2, v_5\}, \{v_1, v_2,v_5\}, \{v_2, v_3, v_4\}, \{v_2, v_3, v_5\}, \{v_1, v_2, v_3, v_4\}, \{v_1, v_2, v_3, v_5\}, \{v_1, v_2, v_4, v_5\}, \{v_1, v_3, v_4, v_5\}, \{v_2, v_3, v_4, v_5\}\}$. Let$H=(V(H),E(H))$:$V(H)=\{v_2,v_5\}$,$E(H)=\{(v_5,v_2),(v_5,v_5)\}$ is m-open and$K=(V(K),E(K))$:$V(K)=\{v_2,v_3,v_4\}$, $E(K)=\{(v_2,v_3),(v_2,v_4),(v_3,v_4),(v_4,v_4)\}$is m-open.

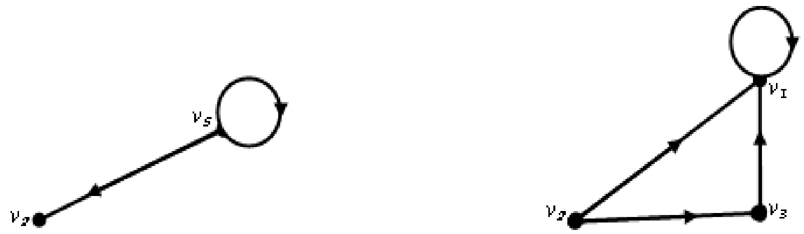

Figure 4.1: Sub Graphs *H* and *K* of a Graph *G* given in Example 4.5

But $H\cap K=(V(H\cap K),E(H\cap K)):V(H\cap K)=\{v_2\},E(H\cap K)=\phi$ is not m-open.

Corollary 4.6: If (G,ξ_m) is an M-space, then the family $\Omega_{\xi m}$ of all m-open graphs forms a supratopology on G.

Proof: The proof is immediately follows from Theorem (4.2), and Remark (4.3), and Remark (4.4).

Obviously, by Remark (4.4), the family $\Omega_{\xi m}$ of all m-open graphs in an M-space (G, ξ_m) need not be a topology on G.

Theorem 4.7: If (G,ξ_m) is an M-space and $H\subseteq G$, then H is m-open if and only if it contains at least one mixed degree of each of its vertices.

Proof. Let (G,ξ_m) be an M-space and H be an m-open graph contained in G and $v\in V(H)$. Suppose that for each mixed degree of $v,MD(v)$,we have $MD(v)\not\subseteq V(H)$, thus for each $MD(v),MD(v)\cap[V(G)-V(H)]\neq\phi$ which implies $v\in[V(G)-V(H)]^{`}_{m}$ .But $G-H$ is $m$-closed since $H$ is$m$-open and so $[V(G)-V(H)]^{`}_{m}\subseteq[V(G)-V(H)]$and hence $v\in[V(G)-V(H)]$. Therefore $v\notin V(H)$ which contradicts with $v\in V(H)$and consequently if $H\subseteq G$ is m-open and $v\in V(H)$,then there exists at least one mixed degree of v which is contained in $V(H)$. Conversely, let H contains at least one mixed degree of each of its vertices,i.e.for all $v\in V(H)$ there exists $MD(v)$ such that $MD(v)\subseteq V(H)$. Let $u\in[V(G)-V(H)]^{`}_{m}$ then $u\notin V(H)$.For if $u\in V(H)$ there would be a mixed degree of $u,MD(u)$,such that $MD(u)\subseteq V(H)$ and this would imply that $MD(u)\cap[V(G)-V(H)]=\phi$ and thus $u\notin[V(G)-V(H)]^{`}_{m}$which is impossible.Accordingly,$u\in[V(G)-V(H)]$and so $[V(G)-V(H)]^{`}_{m}\subseteq[V(G)-V(H)]$ which implies $G-H$ is m-closed and hence H is m-open.

Definition 4.8. Let (G,ξ_m)be an M-space and $H\subseteq G$, then the union of all m-open graphs contained in H is called the m-interior of H and denoted by $Int_{m}(V(H))$, i.e.

$Int_m(V(H))=\cup\{V(O)\in\Omega_{\xi m};V(O)\subseteq V(H)\}$.

The operator$Int_m:P(V(G))\longrightarrow P(V(G))$ is called the m-interior operator.

By Theorem (4.2), Intm$(V(H))$ is m-open graph for $H\subseteq G$. Furthermore, it is the largest m-open graph containing in H and $Int_m(V(H))\subseteq V(H)$ for all $H\subseteq G$. Consequently, H is m-open graph if and only if $V(H)=Int_m(V(H))$ and in particular, $Int_m(Int_m(V(H)))=Int_m(V(H))$.

Example 4.9. According to Example (4.5), let $H\subseteq G$; $H=(V(H),E(H)):V=\{v_1,v_3\},E(H)=\{(v_1,v_1)\}$

Figure 4.2: Sub Graph *H* of a Graph *G* given in Example 4.9

Then $Int_m(V(H))=\{v_1\}$.

Proposition4.10. If$(G,\xi_m)$is an M-space,then the m-interior operator$Int_m:P(V(G))\longrightarrow P(V(G))$ satisfies the following properties for all $H, K\subseteq G$:

(a) $Int_m(\phi)=\phi$,

(b) $Int_m(V(G))=V(G)$,

(c) $Int_m(V(H))\subseteq V(H)$,

(d) If $H\subseteq K$ then $Int_m(V(H))\subseteq Int_m(V(K))$,

(e) $Int_m(Int_m(V(H)))=Int_m(V(H))$,

(f) $Int_m(V(H)\cap V(K))\subseteq Int_m(V(H))\cap Int_m(V(K))$ and

(g) $Int_m(V(H)\cup V(K))\supseteq Int_m(V(H))\cup Int_m(V(K))$.

Proof: Straight forward.

Remark 4.11. Let (G,ξ_m) be an M-space, then the following properties are not true in general for every $H, K\subseteq G$:

(a) $Int_m(V(H)\cap V(K))=Int_m(V(H))\cap Int_m(V(K))$ and

(b) $Int_m(V(H)\cup V(K))=Int_m(V(H))\cup Int_m(V(K))$.

The following example is employed to show the above remark.

Example 4.12:

In Example (4.5), we obtain

(a) Let$H=(V(H),E(H))$:$V(H)=\{v_1,v_2,v_3,v_4\},E(H)=\{(v_1,v_1),(v_2,v_4),(v_2,v_3),(v_3,v_4),(v_4,v_1),(v_4,v_4)\}$ then $Int_m(V(H))=\{v_1,v_2,v_3,v_4\}andK=(V(K),E(K))$:$V(K)=\{v_2,v_5\},E(K)=\{(v_5,v_2),(v_5,v_5)\}$then$Int_m(V(K))=\{v_2,v_5\}$.$H\cap K=(V(H\cap K), E(H\cap K))$: $V(H\cap K)=\{v_2\}$.

$Int_m(V(H\cap K))=\phi$.

So, $Int_m(V(H)\cap V(K))\neq Int_m(V(H))\cap Int_m(V(K))$.

(b) Let$H=(V(H),E(H))$:$V(H)=\{v_1,v_3\},E(H)=\{(v_1,v_1)\}$then$Int_m(V(H))=\{v_1\}$and

$K=(V(K),E(K))$:$V(K)=\{v_1,v_2,v_4\},E(K)=\{(v_1,v_1),(v_2,v_4),(v_4,v_1),(v_4,v_4)\}$then $Int_m(V(K))=\{v_1,v_2,v_4\}$.$H\cup K=(V(H\cup K), E(H\cup K))$: $V(H\cup K)=\{v_1, v_2, v_3, v_4\}$. $Int_m(V(H\cup K))=\{v_1, v_2, v_3, v_4\}$

So, $Int_m(V(H)\cup V(K))\neq Int_m(V(H))\cup Int_m(V(K))$

Proposition 4.13. If $(G,\xi_m)$is an M-space and $H\subseteq G$, then

(a) $Int_m(V(H))= V(G)-[Cl_m(V(G)-V(H))]$ and

(b) $Cl_m(V(H))=V(G)-[Int_m(V(G)-V(H))]$.

Proof: Obvious

Definition 4.14. Let (G,ξ_m) be an M-space and $H\subseteq G$, then

$Bd_m(V(H))=Cl_m(V(H))-Int_m(V(H))$ is called the m-boundary of H and

$Ext_m(V(H))=V(G)-Cl_m(V(H))$ is called the m-exterior of H.

Definition 4.15. Let (G,ξ_m) be an M-space, $H\subseteq G$ and

$$\Omega^{H}_{\xi m}=\{V(H)\cap V(O)\ ;\ V(O)\in\Omega_{\xi m}\}.$$

The pair $(H,\Omega^{H}_{\xi m})$ is called an M-subspace of (G,ξ_m).

Theorem 4.16. If H is a subgraph of the M-space (G,ξ_m), then $\Omega^{H}_{\xi m}=\{V(H)\cap V(O):V(O)\in\Omega_{\xi m}\}$ is a supratoplogy on H.

Proof. Since $V(G)$ and ϕ are two members of $\Omega_{\xi m}$, then $H=H\cap G$ is a member of $\Omega^{H}_{\xi m}$ and $\phi=H\cap\phi\in\Omega^{H}_{\xi m}$. Nowlet $\{K_i\,|\,i\in I\}$ be a subclass of $\Omega^{H}_{\xi m}$, then by Definition(4.15) for each $i\in I$ there exists an m-open graph M_i such that $K_i=H\cap M_i$. Hence $\cup_i K_i=\cup_i(H\cap M_i)=H\cap(\cup_i M_i)$. But, by Theorem(4.2), $\cup_i M_i\in\Omega_{\xi m}$ then $\cup_i M_i\in\Omega^{H}_{\xi m}$. Consequently, $\Omega^{H}_{\xi m}$ is a supratopology on H.

Remark 4.17. Let (G,ξ_m) be an M-space and $H\subseteq G$, then $\Omega_{\xi m}$ need not be a topology on H. Also, on the contrary to the case oftopological subspace, if $H\subseteq G$ is an m-open graph then the relation $\Omega^{H}_{\xi m}\subseteq\Omega_{\xi m}$ is not true.

The following example shows Remark (4.17),

Example 4.18.

According to Example(3.4), we get

$\Omega_{\xi m}=\{V(G),\ \phi,\{v_4\},\ \{v_1,v_3\},\ \{v_1,v_5\},\{v_1,v_2,v_5\},\ \{v_1,v_3,v_4\},\ \{v_1,v_3,v_5\},\{v_1,v_4,v_5\},\ \{v_2,v_3,v_4\},\ \{v_2,v_4,v_5\},\{v_1,v_2,v_3,v_4\},\ \{v_1,v_2,v_3,v_5\},\ \{v_1,v_2,v_4,v_5\},\ \{v_1,v_3,v_4,v_5\},\{v_2,v_3,v_4,v_5\}\}$.

$\Omega^{H}_{\xi m}=\{V(H),\ \phi,\ \{v_1\},\ \{v_4\},\ \{v_1,v_2\},\ \{v_1,v_4\},\ \{v_2,v_4\}\}$. Obviously, $\Omega^{H}_{\xi m}\not\subseteq\Omega_{\xi m}$.

5. *m*-CONTINUITY AND *m*-HOMEOMORPHISM

The concept of continuity is a basic one in mathematics. In this section, the m-continuous (m-open and m-closed) functions are defined and some of their properties are investigated. Furthermore, the m-homeomorphism is defined and some of its properties are studied.

Definition 5.1. Let (G_1,ξ_m) and (G_2,ζ_m) be two M-spaces. A function f from G_1 into G_2 is said to be m-continuous if the inverse image of each m-open graph in G_2 is m-open in G_1, that is, if

$V(H)\in\Omega_{\zeta m}$ implies $f^{-1}(V(H))\in\Omega_{\xi m}$.

Example 5.2.

Let $G_1=(V(G_1),E(G_1))$: $V(G_1)=\{v_1,v_2,v_3,v_4,v_5\}$, $E(G_1)=\{(v_1,v_1),(v_1,v_5),(v_2,v_3),(v_2,v_4),(v_3,v_1),(v_3,v_3),(v_5,v_3),(v_5,v_4),(v_5,v_5)\}$

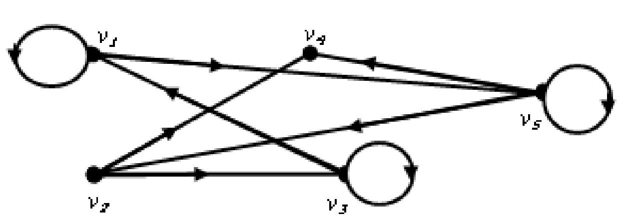

Figure 5.1: Graph G_1 given in Example 5.2

Hence, we get

$\xi_m(v_1)=\{\{v_1,v_5\},\{v_1,v_3\}\},\xi_m(v_2) = \{\{v_3,v_4\},\{v_5\}\},\xi_m(v_3)=\{\{v_1,v_3\},\{v_2,v_3\}\},\xi_m(v_4)=\{\phi,\{v_2,v_5\}\}$ and $\xi_m(v_5)=\{\{v_3,v_4,v_5\},\{v_1,v_5\}\}$.

So, the family $\Omega_{\xi m}$ of all m-open graphs of the M-space (G_1, ξ_m) is given by:

$\Omega_{\xi m}=\{V(G_1),\phi, \{v_4\}, \{v_1, v_3\}, \{v_1, v_5\}, \{v_1, v_2, v_5\}, \{v_1, v_3, v_4\}, \{v_1, v_3, v_5\}, \{v_1, v_4, v_5\}, \{v_2, v_3, v_4\}, \{v_2, v_4, v_5\},\{v_1, v_2, v_3, v_4\}, \{v_1, v_2, v_3, v_5\}, \{v_1, v_2, v_4, v_5\}, \{v_1, v_3, v_4, v_5\}, \{v_2,v_3,v_4,v_5\}\}$.

Also, let $G_2=(V(G_2),E(G_2))$: $V(G_2)=\{u_1,u_2,u_3,u_4,u_5\}$, $E(G_2)=\{(u_1,u_1),(u_2,u_3),(u_2,u_4),(u_3,u_4),(u_4,u_1),(u_4,u_4),(u_5,u_2),(u_5,u_5)\}$.

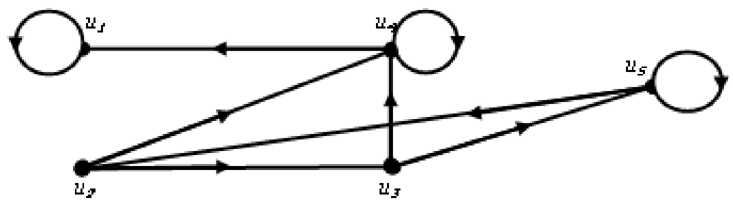

Figure 5.2: Graph G_2 given in Example 5.2.

So, ζ_m is defined by

$\zeta\zeta_m(u_1)=\{\{u_1\},\{u_1,u_4\}\},\zeta_m(u_2)=\{\{u_3,u_4\},\{u_5\}\},\zeta_m(u_3)=\{\{u_4,u_5\},\{u_2\}\},\zeta_m(u_4)=\{\{u_1,u_4\},\{u_2,u_3,u_4\}\}$ and $\zeta_m(u_5)=\{\{u_2,u_5\},\{u_3,u_5\}\}$.

Consequently, the family $\Omega_{\zeta m}$ of all m-open graphs of the M-space (G_2, ζ_m) is given by

$\Omega_{\zeta m}=\{V(G_2),\phi,\{u_1\}, \{u_1, u_4\}, \{u_2, u_5\}, \{u_1, u_2, u_5\}, \{u_2, u_3, u_4\}, \{u_2, u_3, u_5\}, \{u_1, u_2, u_3, u_4\}, \{u_1, u_2, u_3, u_5\}, \{u_1, u_2, u_4, u_5\}, \{u_1, u_3, u_4, u_5\}, \{u_2,u_3,u_4,u_5\}\}$.

Let $f:G_1\longrightarrow G_2$ and $g:G_1\longrightarrow G_2$ such that

$f(v_1)=u_2,f(v_2)=u_5,f(v_3)=u_3,f(v_4)=u_1,f(v_5)=u_5$ and $g(v_1)=u_4,g(v_2)=u_3,g(v_3)=u_1,g(v_4)=u_5,g(v_5)=u_2$.

Accordingly,the function f is m-continuous since the inverse image of each m-open graph in G_2 is m-open in G_1.But the function g is not m-continuous because $g^{-1}(\{u_1\})=\{u_3\}$ and $\{u_3\}$ is not m-open in G_1.

Some properties of m-continuous functions are investigated in the following theorem

Theorem 5.3. Let f be a function from an M-space (G_1, ξ_{m_1}) into an M-space (G_2,ξ_{m_2}),then the following statements are equivalent:

(a) f is m-continuous,

(b) The inverse image of each m-closed graph in G_2 is m-closed in G_1,

(c) $Cl_m(f^{-1}(V(K)))\subseteq f^{-1}(Cl_m(V(K)))$ for all $K\subseteq G_2$,

(d) $f(Cl_m(V(H)))\subseteq Cl_m(f(V(H))$ for all $H\subseteq G_1$,

(e) For each $v\in V(G)$ and each m-open graph $K\subseteq G_2$ Containing $f(v)$,there exists an m-open graph $H\subseteq G_1$ containing v such that $f(V(H))\subseteq V(K)$,

(f) $f([V(H)]^{\grave{}}_{m})\subseteq Cl_m(f(V(H)))$ for all $H\subseteq G_1$,

(g) $f^{-1}(Int_m(V(K)))\subseteq Int_m(f^{-1}(V(K)))$ for all $K\subseteq G_2$ and

(h) $Bd_m(f^{-1}(V(K)))\subseteq f^{-1}(Bd_m(V(K)))$ for all $K\subseteq G_2$.

Proof.(a) $\Longrightarrow$(b). Let $F\subseteq G_2$ be an m-closed graph, then $[V(G_2)-V(F)]$ is m-open in G_2. Since f is m-continuous, then $f^{-1}(V(G_2)-V(F))=f^{-1}(V(G_2))-f^{-1}(V(F))= V(G_1)-f^{-1}(V(F))$ is m-open in G_1 and hence $f^{-1}(V(F))$ is m-closed in G_1.

(b)$\Longrightarrow$(c). Let $K\subseteq G_2$, then $Cl_m(V(K))$ is m-closed in G_2 and since $V(K)\subseteq Cl_m(V(K))$, thus $f^{-1}(V(K))\subseteq f^{-1}(Cl_m(V(K)))$. But, by(b), $f^{-1}(Cl_m(V(K)))$ is m-closed in G_1 which containing $f^{-1}(V(K))$ andconsequently $Cl_m(f^{-1}(V(K)))\subseteq f^{-1}(Cl_m(V(K)))$.

(c)$\Longrightarrow$(d). Let $H\subseteq G_1$, then $f(H)\subseteq G_2$ and so by (c), we have $Cl_m(f^{-1}(f(V(H)))\subseteq f^{-1}(Cl_m(f(V(H)))$. Since $V(H)\subseteq f^{-1}(f(V(H)))$ then $Cl_m(V(H))\subseteq Cl_m(f^{-1}(f(V(H)))$ and hence $Cl_m(V(H))\subseteq f^{-1}(Cl_m(f(V(H)))$. Therefore $f(Cl_m(V(H)))\subseteq f(f^{-1}(Cl_m(f(V(H))))\subseteq Cl_m(f(V(H)))$. That is $f(Cl_m(V(H)))\subseteq Cl_m(f(V(H)))$.

(d)$\Longrightarrow$(a). Let $K\subseteq G_2$ be an m-open graph, then $F=(G_2-K)$ is m-closed graph in G_2 and so $f^{-1}(V(F))\subseteq V(G_1)$. By (d) we have $f(Cl_m(f^{-1}(V(F)))\subseteq Cl_m(f(f^{-1}(V(F)))$. Since $f(f^{-1}(V(F)))\subseteq V(F)$ then $Cl_m(f(f^{-1}(V(F)))\subseteq Cl_m(V(F))=V(F)$ and so $f(Cl_m(f^{-1}(V(F))\subseteq V(F)$ implies $f^{-1}(f(Cl_m(f^{-1}(V(F))))\subseteq f^{-1}(V(F))$.But $Cl_m(f^{-1}(V(F)))\subseteq f^{-1}(f(Cl_m(f^{-1}(V(F))))$ and so $Cl_m(f^{-1}(V(F)))\subseteq f^{-1}(V(F))$ and hence $Cl_m(f^{-1}(V(F)))=f^{-1}(V(F))$. Therefore $f^{-1}(V(F))$ in m-closed in G_1. Because $f^{-1}(V(F))=f^{-1}(V(G_2)-V(K))=V(G_1)-f^{-1}(V(K))$ then $G_1-f^{-1}(K)$ is m-closed in G_1 and then $f^{-1}(K)$ is m-open in G_1.

(a)$\Longrightarrow$(e). Let $v\in V(G_1)$ and $K\subseteq G_2$ be an m-open graph containing $f(v)$. Then, by (a), $H=f^{-1}(K)$ is an m-open graph in G_1 which containing v and hence $f(H)=f(f^{-1}(K))\subseteq K$. i.e., $f(H)\subseteq K$.

(e)$\Longrightarrow$(a). Let $K\subseteq G_2$ be an m-open graph and $f(v)\in V(K)$, then $v\in f^{-1}(V(K))$. By (e), there exists an m-open graph $H\subseteq G_1$ containing v such that $f(H_v)\subseteq K$ which implies $v\in V(H_v)\subseteq f^{-1}(f(H_v))\subseteq f^{-1}(V(K))$. Thus $\{v\}\subseteq V(H_v)\subseteq f^{-1}(V(K))$ and hence $\bigcup_{v\in f^{-1}(V(K))}\{v\}\subseteq\bigcup_{v\in f^{-1}(V(K))}V(H_v)\subseteq f^{-1}(V(K))$. But $f^{-1}(V(K))=\bigcup_{v\in f^{-1}(V(K))}\{v\}$ and so $f^{-1}(V(K))=\bigcup_{v\in f^{-1}(V(K))}V(H_v)$. Therefore $f^{-1}(V(K))$ is an m-open graph in G_1 because it is a union of m-open graphs and hence f is continuous.

(d)$\Longrightarrow$(f). Let $H\subseteq G_1$. Since $[V(H)]^{\grave{}}_{m}\subseteq Cl_m(V(H))$ and by (d) we have $f([V(H)]^{\grave{}}_{m})\subseteq f(Cl_m(V(H)))\subseteq Cl_m(f(V(H)))$. So $f([V(H)]^{\grave{}}_{m})\subseteq Cl_m(f(V(H)))$.

(f)$\Longrightarrow$(d). Let $K\subseteq G_2$ be an m-closed graph, then $V(K)=Cl_m(V(K))$ and thus $f^{-1}(V(K))=f^{-1}(Cl_m(V(K)))$. Since $f^{-1}(V(K))\subseteq V(G_1)$, then by (f), $f([f^{-1}(V(K))]^{\grave{}}_{m})\subseteq Cl_m(f(f^{-1}(V(K))))\subseteq Cl_m(V(K))=V(K)$, i.e., $f([f^{-1}(V(K))]^{\grave{}}_{m})\subseteq V(K)$ implies $[f^{-1}(V(K))]^{\grave{}}_{m}\subseteq f^{-1}(f([f^{-1}(V(K))]^{\grave{}}_{m}))\subseteq f^{-1}(V(K))$ and so $[f^{-1}(V(K))]^{\grave{}}_{m}\subseteq f^{-1}(V(K))$. Hence $f^{-1}(V(K))$ is m-closed graph in G_1.

(a)$\Longleftrightarrow$(g). Let $K\subseteq G_2$. Then $Int_m(V(K))\subseteq V(K)$ and so $f^{-1}(Int_m(V(K)))\subseteq f^{-1}(V(K))$. Since $Int_m(V(K))$ is m-open in G_2 and f is m-continuous, then $f^{-1}(Int_m(V(K)))$ is m-open in G_1. Now $f^{-1}(Int_m(V(K)))$ is m-open contained in $f^{-1}(V(K))$ so $f^{-1}(Int_m(V(K)))\subseteq Int_m(f^{-1}(V(K)))$. Conversely, suppose that K is an m-open graph in G_2 then $V(K)=Int_m(V(K))$ and so $f^{-1}(V(K))=f^{-1}(Int_m(V(K)))$. By (g), $f^{-1}(V(K))=f^{-1}(Int_m(V(K)))\subseteq Int_m(f^{-1}(V(K)))\subseteq f^{-1}(V(K))$ and hence $f^{-1}(V(K))=Int_m(f^{-1}(V(K)))$. Consequently, $f^{-1}(V(K))$ is m-open in G_1 and thus f is m-continuous.

(a)$\Longrightarrow$(h). Suppose that f is m-continuous and $K\subseteq G_2$, then $Cl_m(f^{-1}(V(K)))\subseteq f^{-1}(Cl_m(V(K))$ and $Int_m(f^{-1}(V(K)))\supseteq f^{-1}(Int_m(V(K)))$. So $[f^{-1}(V(K))]^{b}_{m}=[Cl_m(f^{-1}(V(K)))-Int_m(f^{-1}(V(K)))]\subseteq[f^{-1}(Cl_m(V(K)))-f^{-1}(Int_m(V(K)))]=[f^{-1}(Cl_m(V(K)))-Int_m(V(K))]=f^{-1}([V(K)]^{b}_{m})$. Accordingly, $[f^{-1}(V(K))]^{b}_{m}=f^{-1}([V(K)]^{b}_{m})$.

(h)⟹(b). Let Kbe an m-closed graph in G_2, then $Cl_m(V(K)) = V(K)$ and so $f^{-1}(Cl_m(V(K))) = f^{-1}(V(K))$. Since $[V(K)]^b_m \subseteq Cl_m(V(K))$ and by (h) we have $[f^{-1}(V(K))]^b_m \subseteq f^{-1}([V(K)]^b_m) \subseteq f^{-1}(Cl_m(V(K))) = f^{-1}(V(K))$, implies $[f^{-1}(V(K))]^b_m \subseteq f^{-1}(V(K))$. But $Int_m(f^{-1}(V(K))) \subseteq f^{-1}(V(K))$ and hence $[f^{-1}(V(K))]^b_m \cup Int_m(f^{-1}(V(K))) \subseteq f^{-1}(V(K))$,implies $Cl_m(f^{-1}(V(K))) \subseteq f^{-1}((V(K))$,implies $Cl_m(f^{-1}(V(K))) = f^{-1}(V(K))$. Therefore $f^{-1}(V(K))$ is m-closed in G_1.

Remark 5.4. Let $(G_1, \xi_m)$and (G_2, ζ_m) be M-space and $f: G_1 \longrightarrow G_2$,then the following statements are not necessarily equivalent:

(a) $f$is m-continuous.

(b) For each $v \in V(G)$ and each mixed degree $M \subseteq G_2$ of $f(v)$,there exists a mixed degree $N \subseteq G_1$ of v such that $f(N) \subseteq M$.

The next example illustrates Remark (5.4),

Example 5.5.

According to Example (5.2), let $v = v_1 \in V(G_1)$ and $M = \{u_3, u_4\} \subseteq V(G_2)$ which is a mixed degree of $f(v) = f(v_1) = u_2$. Obviously, there is no mixed degree $N \subseteq V(G_1)$ of such that $f(N) \subseteq M = \{u_3, u_4\}$.

Theorem 5.6. Let (G_1, ξ_m) and (G_2, ζ_m) be two M-spaces and $f: G_1 \longrightarrow G_2$ be an m-continuous function,then$f_H: H \longrightarrow G_2$ is an m-continuous where $H \subseteq G_1$is an M-subspace and f_H is the restriction of$f$to H.

Proof. Suppose that K is an m-open graph in G_2, i.e. $K \in \Omega_{\zeta m}$. Since f is m-continuous then $f^{-1}(K) \in \Omega_{\xi m}$ and so $H \cap f^{-1}(K) \in \Omega^H_{\xi m}$. But $f^{-1}_H(W) = H \cap f^{-1}(W)$ for all $W \subseteq G_2$and thus $f^{-1}_H(K) = H \cap f^{-1}(K)$. Therefor $f^{-1}_H(K) \in \Omega^H_{\xi m}$ and hence f_H is m-continuous.

Theorem 5.7. Let (G_1, ξ_m), $(G_2, \zeta_m)$and (G_3, η_m) be M-spaces. If $f: G_1 \longrightarrow G_2$ and $g: G_2 \longrightarrow G_3$ are m-continuous functions, then $g \circ f: G_1 \longrightarrow G_3$ is also m-continuous.

Proof. Let H be an m-open graph in W. Because g is m-continuous thus $g^{-1}(H)$ is m-open in G_2 and since f is m-continuous then $f^{-1}(g^{-1}(H))$ is m-open in G_1. But $(g \circ f)^{-1}(H) = f^{-1}(g^{-1}(H))$, so $(g \circ f)^{-1}(H)$ is m-open in G_1. Consequently, $g \circ f$is m-continuous.

Definition 5.8. Let (G_1, ξ_m) and (G_2, ζ_m) be two M-spaces.A function$f$from G_1 into G_2 is said to be m-open (m-closed) if the image of each m-open (m-closed) graph in G_1 is m-open (m-closed) in G_2.

In general,functions which are m-open need not be m-closed and vice versa as shown in the following example.

Example 5.9.

Let$G_1 = (V(G_1), E(G_1))$: $V(G_1) = \{v_1, v_2, v_3, v_4, v_5\}$, $E(G_1) = \{(v_1,v_2),(v_1,v_4),(v_2,v_2),(v_2,v_3),(v_2,v_4),(v_3,v_1),(v_3,v_4),(v_4,v_3),(v_4,v_5),(v_5,v_2),(v_5,v_5)\}$.

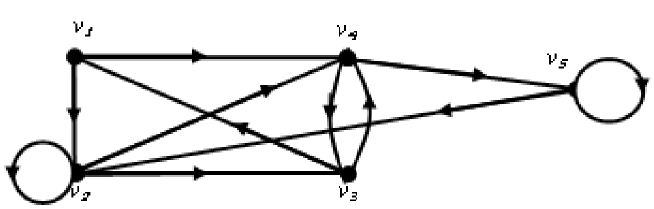

Figure 5.3: Graph G_1 given in Example 5.9

Hence, we get

$\xi_m(v_1)=\{\{v_2,v_4\},\{v_3\}\}$, $\xi_m(v_2)=\{\{v_2,v_3,v_4\},\{v_1,v_2,v_5\}\}$, $\xi_m(v_3)=\{\{v_1,v_4\},\{v_2,v_4\}\}$,

$\xi_m(v_4)=\{\{v_3,v_5\},\{v_1,v_2,v_3\}\}$ and $\xi_m(v_5)=\{\{v_2,v_5\},\{v_4,v_5\}\}$.

So, the families of m-open graphs and m-closed graphs of the M-space (G_1, ξ_m) are given respectively by

$\mathcal{F}_{\xi m}=\{V(G_1), \phi, \{v_1\},\{v_2\},\{v_5\}\}$.

$\Omega_{\xi m}=\{V(G_1), \phi, \{v_1, v_2, v_3, v_4\}, \{v_1, v_3, v_4, v_5\}, \{v_2,v_3,v_4,v_5\}\}$.

Also, let $G_2=(V(G_2),E(G_2))$: $V(G_2)=\{u_1,u_2,u_3,u_4,u_5\}$, $E(G_2)=\{(u_1,u_1),(u_1,u_5),(u_2,u_3),(u_2,u_4),(u_3,u_1),(u_3,u_3),(u_5,u_2),(u_5,u_4),(u_5,u_5)\}$

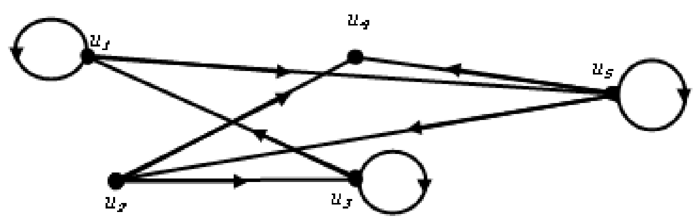

Figure 5.4: Graph G_2 given in Example 5.9

Thus, ζ_m is defined by

$\zeta_m(u_1)=\{\{u_1,u_5\},\{u_1,u_3\}\}$, $\zeta_m(u_2)=\{\{u_3,u_4\},\{u_5\}\}$, $\zeta_m(u_3)=\{\{u_1,u_3\},\{u_2,u_3\}\}$, $\zeta_m(u_4)=\{\phi,\{u_2,u_5\}\}$ and $\zeta_m(u_5)=\{\{u_2,u_4,u_5\},\{u_1,u_5\}\}$.

Consequently, the families of m-open graphs and m-closed graphs of the M-space (G_2, ζ_m) are given respectively by

$\mathcal{F}_{\zeta m}=\{V(G_2),\phi,\{u_1\},\{u_2\},\{u_3\},\{u_4\},\{u_5\},\{u_1,u_3\},\{u_1,u_5\},\{u_2,u_3\},\{u_2,u_4\},\{u_2,u_5\},\{u_3,u_4\},\{u_2,u_3,u_4\},\{u_2,u_4,u_5\},\{u_1,u_2,u_3,u_5\}\}$.

$\Omega_{\zeta m}=\{V(G_2),\phi,\{u_4\}, \{u_1, u_3\}, \{u_1, u_5\}, \{u_1, u_2, u_5\}, \{u_1, u_3, u_4\}, \{u_1, u_3, u_5\}, \{u_1, u_4, u_5\}, \{u_2, u_3, u_4\}, \{u_2, u_4, u_5\}, \{u_1, u_2, u_3, u_4\}, \{u_1, u_2, u_3, u_5\}, \{u_1, u_2, u_4, u_5\}, \{u_1, u_3, u_4, u_5\}, \{u_2,u_3,u_4,u_5\}\}$.

Let $f:G_1 \longrightarrow G_2$ and $g:G_1 \longrightarrow G_2$ and $h:G_1 \longrightarrow G_2$ such that

$f(v_1)=u_2,\ f(v_2)=u_2, f(v_3)=u_3, f(v_4)=u_4, f(v_5)=u_5$,

$g(v_1)=u_2, g(v_2)=u_4, g(v_3)=u_2, g(v_4)=u_4,\ g(v_5)=u_5$ and

$h(v_1)=u_1, h(v_1)=u_2, h(v_3)=u_1, h(v_4)=u_5, h(v_5)=u_3$.

Accordingly, the function f is m-open but not m-closed since $f(G_1) = \{u_2, u_3, u_4, u_5\}$ which is not m-closed graph in G_2. Moreover, f is not m-continuous since $f^{-1}(\{u_4\}) = \{v_4\}$ and $\{v_4\}$ is not m-open graph in G_1. On the other hand, the function g is m-closed but not m-open since $g(\{v_1, v_2, v_3, v_4\}) = \{u_2, u_4\}$ which is not m-open graph in G_2. Finally, the function h is m-open and m-closed.

Example 5.10.

According to Example (5.2), the function f is m-continuous but not m-open since $f(\{v_1,v_3\}) = \{u_1, u_3\}$ which is not m-open graph in G_2.

Theorem 5.11. Let f be a function from the M-space (G_1,ξ_m) into the M-space (G_2,ζ_m), then the following statements are equivalent:

(a) f is m-open,

(b) $f(Int_m(V(H))\subseteq Int_m(f(V(H)))$ for all $H\subseteq G_1$ and

(c) For each $v\in V(G)$ and each m-open graph $O\subseteq G_1$ containing v, there exists an m-open graph $K\subseteq G_2$ containing $f(v)$ such that $K\subseteq f(O)$.

Proof. (a)$\Longrightarrow$(b). Let $H\subseteq G_1$. Since $Int_m(V(H))\subseteq V(H)$ then $f(Int_m(V(H)))\subseteq f(V(H))$. But, $Int_m(V(H))$ is m-open graph in G_1 and f is m-open function. So, by (a), $f(Int_m(V(H)))$ is m-open in G_2 which contained in $f(V(H))$. Therefore, $f(Int_m(V(H)))\subseteq Int_m(f(V(K)))$.

(b)$\Longrightarrow$(a). Suppose that H is an m-open graph in G_1, then $V(H) = Int_m(V(H))$ and so $f(V(H)) = f(Int_m(V(H)))$. By (b), $f(Int_m(V(H))) \subseteq Int_m(f(V(H))$, then $f(V(H)) \subseteq Int_m(f(V(H)))$. But $Int_m(f(V(H)))\subseteq f(V(H))$ and thus $f(V(H)) = Int_m(f(V(H)))$. Accordingly, $f(H)$ is m-open graph in G_2 and hence f is m-open function.

(a)$\Longrightarrow$(c). Let $v\in V(G_1)$ and $H\subseteq G_1$ be an m-open graph such that $v\in V(H)$. Then, by (a), $K = f(H)$ is an m-open graph in G_2 which containing $f(v)$ and hence $K\subseteq f(H)$.

(c)$\Longrightarrow$(a). Let $H \subseteq G_1$ be an m-open graph and $v\in V(H)$, then $u = f(v)\in f(V(H))$. By (c), there exists an m-open graph $K_u\subseteq G_2$ containing u such that $K_u\subseteq f(V(H))$ which implies $u \in V(K_u)\subseteq f(V(H))$. Thus $\{u\} \subseteq K_v\subseteq f(V(H))$ and hence $\bigcup_{u \in f(V(H))}\{u\} \subseteq\bigcup_{u \in f(V(H))}K_u\subseteq f(V(H))$. But $f(V(H)) = \bigcup_{u\in f(V(H))}\{u\}$ and so $f(V(H)) = \bigcup_{u\in f(V(H))}K_u$. Therefore, $f(V(H))$ is an m-open graph in G_2 because it is a union of m-open graphs and hence f is m-open.

Remark 5.12. Let (G,ξ_m) and (G_2,ζ_m) be two M-space and $f:G_1\longrightarrow G_2$, then the following statements are not necessarily equivalent:

(a) f is m-open.

(b) For each $v\in V(G)$ and each mixed degree $M\subseteq V(G_1)$ of v, there exists a mixed degree $N\subseteq V(G_2)$ of $f(v)$ such that $N\subseteq f(M)$.

The following example illustrates Remark (5.12),

Example 5.13.

According to Example (5.9), let $v = v_3\in V(G_1)$ and $N = \{v_1, v_4\} \subseteq V(G_1)$ which is a mixed degree system of v_3. Obviously, there is no mixed degree system $M\subseteq V(G_2)$ of $f(v_3) = u_3$ such that $M\subseteq f(N) = \{u_2, u_4\}$.

Theorem 5.14. Let f be a function from the M-space (G_1,ξ_m) into the M-space (G_2,ζ_m), then f is m-closed if and only if $Cl_m(f(V(H)))\subseteq f(Cl_m(V(H)))$ for all $H\subseteq G_1$.

Proof. Suppose that f is m-closed and $H\subseteq G_1$. But $V(H)\subseteq Cl_m(V(H))$ which implies $f(V(H))\subseteq f(Cl_m(V(H)))$ and so $Cl_m(f(V(H)))\subseteq Cl_m(f(Cl_m(V(H))))$. Since $Cl_m(V(H))$ is m-closed in G_1 and f is m-closed, then $f(Cl_m(V(H)))$ is m-closed in G_2. Thus $f(Cl_m(V(H))) = Cl_m(f(Cl_m(V(H)))$ and hence $Cl_m(f(V(H)))\subseteq f(Cl_m(V(H)))$. Conversely, let H be an m-closed graph in G_1, then $V(H) =$

$Cl_m(V(H))$ and so $f(V(H)) = f(Cl_m(V(H)))$. Since $Cl_m(f(V(H)))\subseteq f(Cl_m(V(H)))$ thus $Cl_m(f(V(H)))\subseteq f(V(H))$. But $f(V(H))\subseteq Cl_m(f(V(H)))$ then $f(V(H)) = Cl_m(f(V(H)))$ and hence $f(V(H))$ is m-closed in G_2. Consequently, f is m-closed function.

Definition 5.15. Let (G_1,ξ_m) and (G_2,ζ_m) be two M-space. A function f from G_1 into G_2 is said to be an m-homeomorphism if

(a) f is bijective.

(b) f and f^{-1} are m-continuous.

The two M-spaces G_1 and G_2 are called m-homeomorphic.

Example 5.16.

Let $G_1=(V(G_1),E(G_1))$: $V(G_1)=\{v_1,v_2,v_3\}$, $E(G_1)=\{(v_1,v_2),(v_2,v_3),(v_3,v_3)\}$

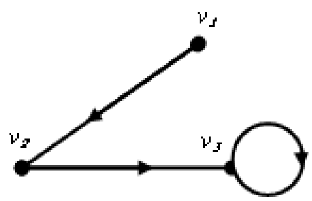

Figure 5.5: Graph G_1 given in Example 5.16

Then, ξ_m is given by

$\xi_m(v_1)=\{\{v_2\},\phi\}$, $\xi_m(v_2)=\{\{v_3\},\{v_1\}\}$ and $\xi_m(v_3)=\{\{v_3\},\{v_2,v_3\}\}$.

$\Omega_{\xi m}=\{V(G_1),\phi, \{v_1\}, \{v_3\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_2,v_3\}\}$.

Also, let $G_2=(V(G_2),E(G_2))$: $V(G_2)=\{u_1,u_2,u_3\}$, $E(G_2)=\{(u_1,u_1),(u_2,u_1),(u_3,u_2),(u_3,u_3)\}$.

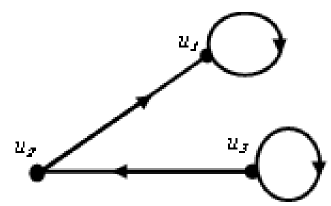

Figure 5.6: Graph G_2 given in Example 5.17

Thus, ζ_m is given by

$\zeta_m(u_1)=\{\{u_1\},\{u_1,u_2\}\}$, $\zeta_m(u_2)=\{\{u_1\},\{u_3\}\}$ and $\zeta_m(u_3)=\{\{u_2,u_3\},\{u_3\}\}$.

$\Omega_{\zeta m}=\{V(G_2),\phi,\{u_1\}, \{u_3\}, \{u_1, u_2\}, \{u_1, u_3\}, \{u_2,u_3\}\}$.

Let $f:G_1\longrightarrow G_2$ and $g:G_1\longrightarrow G_2$

$f(v_1)=u_3$, $f(v_2)=u_2$, $f(v_3)=u_1$ and

$g(v_1)=u_2$, $g(v_2)=u_1$, $g(v_3)=u_3$.

Accordingly, the function f is m-homeomorphism since f is bijective. Also, f and f^{-1} are m-continuous. But the function g is not m-homeomorphism since $g(\{v_1\}) = \{u_2\}$ and $\{u_2\}$ is not m-open graph in G_2 which implies g^{-1} is not m-continuous. Furthermore, $g^{-1}(\{u_1\}) = \{v_1\}$ and $\{v_1\}$ is not m-open graph in G_1 which implies g is not m-continuous.

Theorem 5.17. Let f be a bijective function from the M-space (G_1,ξ_m) onto the M-space (G_2,ζ_m), then the following statements are equivalent:

(a) f is an m-homeomorphism,

(b) f is m-continuous and m-open,

(c) f is m-continuous and m-closed and

(d) $Cl_m(f(V(H))=f(Cl_m(V(H)))$ for all $H \subseteq G_1$.

Proof: The proof is obvious.

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