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Presentation for $G \wr \mathrm{Sing}_2$*

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Abstract. For a group G and a subsemigroup S of the full transformation semigroup $\mathcal{T}_n$, the wreath product $G \wr S$ is defined to be the semidirect product $G^n \rtimes S$, with the coordinatewise action of S on G^n. The full wreath product $G \wr \mathcal{T}_n$ is isomorphic to the endomorphism monoid of the free G-act on n generators. Here, we are particularly interested in the case that $S = \mathrm{Sing}_2$ is the singular part of $\mathcal{T}_2$, consisting of all non-invertible transformations. Our main result is a presentation for $G \wr \mathrm{Sing}_2$ in terms of the idempotent generating set. It is also shown that the generating relations cannot be reduced.

Keywords: Wreath product; Semidirect product; Transformation semigroup; Presentation.

1. Introduction

The study of idempotents have long played an important role in algebraic and combinatorial semigroup theory. In 1966, Howie [15] showed that every semigroup S embeds in an idempotent generated (singular transformation) semigroup

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that may be taken to be finite if S is finite. On the other hand, every idempotent generated semigroup T is a homomorphic image of a free idempotent generated semigroup that has the same biordered set of idempotents as T. These kinds of semigroups are defined by means of presentation, which consists of a set of generators and a set of relations. A presentation for the symmetric group $\mathcal{S}_n$ were given in 1897 by Moore [19], and a presentation for the full transformation semigroup $\mathcal{T}_n$ was discovered in 1958 by Aizenstat [1]. Presentations for other semigroups (and other related objects) may be found in [4, 5, 7, 10, 12, 21]. In numerous cases, these semigroups contain $\mathcal{S}_n$ as their group of units, and the resulting presentations contain Moore's presentation for $\mathcal{S}_n$. When the group of units is removed and one considers the singular subsemigroup, a new method is needed. In 2010 a presentation for Sing_n, the singular part of the full transformation semigroup $\mathcal{T}_n$, is given by East [5] in terms of the generating set consisting of all idempotents of rank $n-1$.

It has long been known that the endomorphism monoid of a free G-act of finite rank n, or $\mathrm{End}F_n(G)$, is isomorphic to a full wreath product $G \wr \mathcal{T}_n$; and the maximal subgroup of $\mathrm{End}F_n(G)$ containing a rank r idempotent is isomorphic to $G \wr \mathcal{S}_r$. It is therefore very natural to study the structure of wreath products $G \wr \mathcal{T}_n$, or more generally, $G \wr S$ for an arbitrary subsemigroup S of $\mathcal{T}_n$. A general presentation for the endomorphism monoid $\mathrm{End}(A)$ of an arbitrary independence algebra A is not currently known. But for a special subclass of such algebras, the above-mentioned free G-acts of finite rank, such a presentation can be described using results of Lavers [18] on general products of monoids, since (as noted above) these endomorphism monoids are isomorphic to wreath products of the form $G \wr \mathcal{T}_n$. In [9] a presentation for the singular part of the full wreath product $M \wr \mathcal{T}_n$ was considered for arbitrary monoid M. In this article we are interested in the problem of finding a presentation for the wreath products $G \wr S$ for an arbitrary group G and an arbitrary subsemigroup $S \subseteq \mathcal{T}_n$, particularly in the case that $S = \mathrm{Sing}_2$. This kind of problem can be quite difficult in the case that S does not contain the identity transformation (as happens when $S = \mathrm{Sing}_n$, for example), since many articles on presentations for semigroup constructions (including wreath and semidirect products) focus on the case of monoids [11, 16, 18, 23]. Notable exceptions that are not restricted to monoids have concentrated on constructions that do not capture the kind of wreath and semidirect products that arise from endomorphisms of G-acts [3, 22].

The article is organized as follows. In Section 2, we establish and gather some background results on (transformation) semigroups and presentations. In Section 3, we state our main result by giving a presentation for a wreath product $G \wr \mathrm{Sing}_2$ of a group G and the singular part of $\mathcal{T}_2$. Finally, in Section 4, we further remark that the generating relations of the presentation cannot be reduced. This method is valid when we have a presentation for the singular part of the full wreath product $G \wr \mathcal{T}_2$, where G is a group. Finding a reduction for the generating relations of a presentation for the singular part of a full wreath product $M \wr \mathcal{T}_n$ is a matter for further study, which is not discussed in [9].

2. Preliminaries

Let S be a semigroup, and write S^1 for the monoid obtained by adjoining an identity 1 to S, if necessary. Unless otherwise specified, we will generally write 1 for the identity element of any monoid. For any subset $A \subseteq S$, we write $E(A) = \{a \in A \,|\, a^2 = a\}$ for the set of idempotents in A, $\langle A\rangle$ for the subsemigroup of S generated by A and $\mathrm{FG}(A)$ for the free group generated by A. For more background on semigroups, see [14].

The *rank* of a semigroup, denoted $\mathrm{rank}(S)$, is the smallest size of a generating set for S, see [14]. Let X be an *alphabet* (a set whose elements are called *letters*), and denote by X^+ the free semigroup on X. We denote the *empty word* (over any alphabet) by 1. If R is a binary relation on X^+, we denote by $R^\sharp$ the smallest congruence on X^+ generated by R. To say that a semigroup S has *semigroup presentation* $\langle X \,|\, R\rangle$ is to say that $S \cong X^+/R^\sharp$, or equivalently, if there is an epimorphism $\varphi : X^+ \to S$ with $\ker\varphi = R^\sharp$. If such an epimorphism exists, then S has *presentation* $\langle X \,|\, R\rangle$ *via* φ. If we replace A^+ by A^*, we obtain a monoid presentation for a semigroup S. The elements of R are generally referred to as *relations*, and a relation $(w_1, w_2) \in R$ will usually be displayed as an equation $w_1 = w_2$.

It is well-known that a group presentation for a group G is usually defined to be a pair $\langle X \,|\, R\rangle$, where X is the set of generators of a free group $FG(X)$, and $R = \{u_i v_i^{-1} : i \in I\} \subseteq FG(X)$. In the case that G has a group presentation $\langle X \,|\, R\rangle$ then G has a monoid presentation $\langle X \cup X^{-1} \,|\, R'\rangle$, where $X^{-1} = \{x^{-1} | \in X\}$ is a set in one-one correspondence with X, $R' = R \cup \{xx^{-1} = \varepsilon = x^{-1}x | \, x \in X\}$ and ε is the empty word in $FG(X)$.

Proposition 2.1. *If G is a group which has a monoid presentation $\langle K \,|\, W\rangle$ then G^n has a monoid presentation $\langle H \,|\, R\rangle$, where $H = \{\tau_{i,g} \,|\, 1 \le i \le n,\ g \in K\}$ and $R = \{\tau_{i,1} = 1,\ \tau_{i,g}\tau_{j,h} = \tau_{j,h}\tau_{i,g},\ \tau_{i,g}\tau_{i,h} = \tau_{i,gh} \mid 1 \le i, j \le n,\ i \ne j,\ g, h \in K, \tau_{i,g}, \tau_{j,h} \in W\}$.*

Proof. Define a homomorphism from $\mathrm{FG}(H)$ to G^n induced by φ: $H \to G^n$, $\tau_{i,g} \mapsto (1, \ldots, 1, \underset{i}{g}, 1, \ldots, 1)$. Denote $R^\sharp$ by ρ. Notice that

$$\begin{aligned}(\tau_{i,g}\tau_{j,h})\varphi = (\tau_{i,g}\varphi)(\tau_{j,h}\varphi) &= (\ldots, \underset{i}{g}, \ldots)(\ldots, \underset{j}{h}, \ldots)\\ &= (\ldots, \underset{j}{h}, \ldots, \underset{i}{g}, \ldots)\\ &= (\ldots, \underset{j}{h}, \ldots)(\ldots, \underset{i}{g}, \ldots) = (\tau_{j,h}\tau_{i,g})\varphi,\end{aligned}$$

and that

$$\begin{aligned}(\tau_{i,g}\tau_{i,h})\varphi = (\tau_{i,g}\varphi)(\tau_{i,h}\varphi) &= (\ldots, \underset{i}{g}, \ldots)(\ldots, \underset{i}{h}, \ldots)\\ &= (\ldots, \underset{i}{gh}, \ldots) = \tau_{i,gh}\varphi.\end{aligned}$$

We see that $\rho \subseteq \ker(\varphi)$. Furthermore, for all $(g_1, \dots, g_n) \in G^n$, we have

$$\begin{aligned}(g_1, \dots, g_n) &= (g_1, 1, \dots, 1)(1, g_2, 1, \dots, 1)\dots(1, \dots, 1, g_n)\\ &= (\tau_{1,g_1}\tau_{2,g_2}\dots\tau_{n,g_n})\varphi.\end{aligned}$$

Therefore there is a well-defined epimorphism $\overline{\varphi}$: $\mathrm{FG}(H)/\rho \to G^n$, $\tau_{i,g}\rho \mapsto (1, \dots, 1, \underset{i}{g}, 1, \dots, 1)$. To get that G^n and $\mathrm{FG}(H)/\rho$ are isomorphic, it suffices to show that $\overline{\varphi}$ is also one-one.

For all $(\tau_{i_1,g_1}\tau_{i_2,g_2}\dots\tau_{i_l,g_l})\rho \in \mathrm{FG}(H)/\rho$, rearranging the order of $\tau_{i,g}$s by using $(\tau_{i,g}\rho)(\tau_{j,h}\rho) = (\tau_{j,h}\rho)(\tau_{i,g}\rho)$ and combining 'like terms' by using $(\tau_{i,g}\rho)(\tau_{i,h}\rho) = \tau_{i,gh}\rho$, we get a 'normal form' $(\tau_{1,h_1}\tau_{2,h_2}\dots\tau_{n,h_n})\rho$ for $(\tau_{i_1,g_1}\tau_{i_2,g_2}\dots\tau_{i_l,g_l})\rho$. If $[(\tau_{i_1,g_1}\tau_{i_2,g_2}\dots\tau_{i_l,g_l})\rho]\varphi = (1, 1, \dots, 1)$, or equivalently, $[(\tau_{1,h_1}\tau_{2,h_2}\dots\tau_{n,h_n})\rho]\varphi = (1, 1, \dots, 1)$, then $(h_1, h_2, \dots, h_n) = (1, 1, \dots, 1)$, which gives $h_1 = 1$, $h_2 = 1$, $\dots$, $h_n = 1$. Therefore, $(\tau_{i_1,g_1}\tau_{i_2,g_2}\dots\tau_{i_l,g_l})\rho = (1, 1, \dots, 1)\rho = 1_{\mathrm{FG}(H)/\rho}$ whence $\overline{\varphi}$ is one-one. ■

The (full) transformation semigroup on a set X is the semigroup $\mathcal{T}_X$ of all transformation on X (i.e. all functions from X to itself) under the operation of composition [17]. Transformation semigroups are ubiquitous in semigroup theory because of Cayley's Theorem which states that every semigroup S embeds in some transformation semigroup $\mathcal{T}_X$. If S is a group, the Cayley representation maps S into the symmetric group $\mathcal{S}_X \subseteq \mathcal{T}_X$, which is the group of units of $\mathcal{T}_X$, and consists of all permutations of X, that is, $\mathcal{S}_n = \{\alpha \in \mathcal{T}_n \mid \mathrm{rank}\,(\alpha) = n\}$. If S does not possess an identity element, the Cayley representation maps S into $\mathcal{T}_X \setminus \mathcal{S}_X$, the set of all non-invertible (i.e. *singular*) transformations on X. The set $\mathcal{T}_X \setminus \mathcal{S}_X$ is a subsemigroup (the so called singular subsemigroup) of $\mathcal{T}_X$ if and only if X is finite.

For an integer $n \geq 0$, we write $\mathbf{n} = \{1, \dots, n\}$ and $\mathcal{T}_n$ for the *full transformation semigroup of degree* n, which consists of all transformations of $\mathbf{n}$ (i.e. all maps $\mathbf{n} \to \mathbf{n}$) under composition. (When $n = 0$, $\mathbf{n} = \emptyset$ and $\mathcal{T}_0$ consists only of the empty function $\emptyset \to \emptyset$.) For $\alpha \in \mathcal{T}_n$ and $i \in \mathbf{n}$, we write $i\alpha$ for the image of i under α, so that transformations are composed left-to-right. For $\alpha \in \mathcal{T}_n$, define

$$\begin{aligned}&\mathrm{im}\,(\alpha) = \{i\alpha \mid i \in \mathbf{n}\}, \qquad \ker(\alpha) = \{(i, j) \in \mathbf{n} \times \mathbf{n} \mid i\alpha = j\alpha\},\\ &\mathrm{rank}\,(\alpha) = |\,\mathrm{im}\,(\alpha)|.\end{aligned}$$

Recall that *Green's relations* are defined by

$$\begin{aligned}&\alpha\,\mathcal{L}\,\beta \iff S^1a = S^1b, \qquad \alpha\,\mathcal{R}\,\beta \iff aS^1 = bS^1,\\ &\alpha\,\mathcal{J}\,\beta \iff S^1aS^1 = S^1bS^1,\end{aligned}$$

where S^1 denotes S with an identity element adjoined (unless S already has one); hence, these three relations record when two elements of S generate the same right, left, and two-sided principal ideals, respectively. Furthermore, we let $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$, while $\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$ is the join of the equivalences $\mathcal{R}$ and $\mathcal{L}$. As is well known, for finite semigroups we always have $\mathcal{D} = \mathcal{J}$, while in general the inclusions $\mathcal{H} \subseteq \mathcal{R}, \mathcal{L} \subseteq \mathcal{D} \subseteq \mathcal{J}$ hold. It is well known (see [14, Exercise 2.6.16]) that, for $\alpha, \beta \in \mathcal{T}_n$,

$$\alpha \,\mathcal{L}\, \beta \iff \mathrm{im}\,(\alpha) = \mathrm{im}\,(\beta), \qquad \alpha \,\mathcal{R}\, \beta \iff \ker(\alpha) = \ker(\beta),$$
$$\alpha \,\mathcal{D}\, \beta \iff \mathrm{rank}\,(\alpha) = \mathrm{rank}\,(\beta).$$

Among all the $\mathcal{D}$-classes, D_{n-1}, the $\mathcal{D}$-class whose elements are all of rank $n-1$, plays a key role, as explained in Theorem 2.2.

A famous result of Howie [15] states that Sing_n is generated by its idempotents: in fact, by its idempotents of rank $n-1$. The latter are precisely the maps ε_{ij} (for $i, j \in \mathbf{n}$ with $i \neq j$) defined by

$$k\varepsilon_{ij} = \begin{cases} k & \text{if } k \neq j, \\ i & \text{if } k = j. \end{cases}$$

We will write $\mathcal{X} = \{\varepsilon_{ij} \,|\, i, j \in \mathbf{n}, i \neq j\}$ for the set of all rank $(n-1)$ idempotents from $\mathcal{T}_n$. It is easy to check that for all $i, j, k, l \in \mathbf{n}$ with $i \neq j$ and $k \neq l$,

$$\varepsilon_{ij} \,\mathcal{L}\, \varepsilon_{kl} \iff j = l \quad \text{and} \quad \varepsilon_{ij} \,\mathcal{R}\, \varepsilon_{kl} \iff \{i, j\} = \{k, l\}.$$

The next result, as shown in [15, Theorem I], states the fact that $\mathrm{Sing}_n = \langle \mathcal{X} \rangle$ if $n \geq 2$.

Theorem 2.2. *Every element of* Sing_n *is a product of idempotents whose rank is* $n-1$.

Note that $\mathrm{Sing}_n = \emptyset$ if $n \leq 1$. Note also that $\mathrm{Sing}_2 = \{\varepsilon_{12}, \varepsilon_{21}\}$ is a right-zero semigroup.

A presentation for Sing_n was given in [6], in terms of the idempotent generating set. Define an alphabet

$$X = \{e_{ij} \,|\, i, j \in \mathbf{n},\ i \neq j\},$$

an epimorphism

$$\phi : \ X^+ \to \mathrm{Sing}_n : \ e_{ij} \mapsto \varepsilon_{ij},$$

and let R be the set of relations

$$\begin{aligned}
e_{ij}^2 = e_{ij} &= e_{ji}e_{ij} && \text{for distinct } i, j \\
e_{ij}e_{kl} &= e_{kl}e_{ij} && \text{for distinct } i, j, k, l \\
e_{ik}e_{jk} &= e_{ik} && \text{for distinct } i, j, k \\
e_{ij}e_{ik} = e_{ik}e_{ij} &= e_{jk}e_{ij} && \text{for distinct } i, j, k \\
e_{ki}e_{ij}e_{jk} &= e_{ik}e_{kj}e_{ji}e_{ik} && \text{for distinct } i, j, k \\
e_{ki}e_{ij}e_{jk}e_{kl} &= e_{ik}e_{kl}e_{li}e_{ij}e_{jl} && \text{for distinct } ni, j, k, l.
\end{aligned}$$

The next result is [6, Theorem 6].

Theorem 2.3. *For $n \geq 2$, the semigroup* Sing_n *has presentation* $\langle X \mid R \rangle$ *via* ϕ.

Let S be a semigroup and M a monoid with identity 1. Suppose that S has a left action on M by monoid endomorphisms; that is, there is a homomorphism φ: $S \to \mathrm{End}^*(M)$, $s \mapsto \varphi_s$, where $\mathrm{End}^*(M)$ denotes the monoid of endomorphisms of M with right-to-left composition. For $s \in S$ and $a \in M$, we write $s \cdot a = \varphi_s(a)$. So

$$s \cdot 1 = 1, \quad s \cdot (t \cdot a) = (st) \cdot a, \quad s \cdot (ab) = (s \cdot a)(s \cdot b) \quad \text{for all } s, t \in S \text{ and } a, b \in M.$$

The *semidirect product* $M \rtimes S = M \rtimes_\varphi S$ has underlying set $M \times S = \{(a, s) \mid a \in M,\ s \in S\}$, and product defined by

$$(a, s)(b, t) = (a(s \cdot b), st) \quad \text{for all } s, t \in S \text{ and } a, b \in M.$$

The fact that S acts by *monoid* endomorphisms ensures that S may be identified with the subsemigroup $\{(1, s) \mid s \in S\}$ of $M \rtimes S$. If S is a monoid acting monoidally on M (i.e. $1 \cdot a = a$ for all $a \in M$), then $\{(a, 1) \mid a \in M\}$ is an isomorphic copy of M inside $M \rtimes S$. However, this article is mostly concerned with the case that S is not a monoid, in which case $M \rtimes S$ does not contain such a canonical copy of M. A motivating example of the semidirect product are the *wreath products*.

Let S be a subsemigroup of the full transformation semigroup $\mathcal{T}_n$, and let G be an arbitrary group. Then S has a natural left action on G^n (the direct product of n copies of G) given by

$$\alpha \cdot (a_1, \ldots, a_n) = (a_{1\alpha}, \ldots a_{n\alpha}) \quad \text{for } \alpha \in S \text{ and } a_1, \ldots, a_n \in G.$$

The resulting semidirect product $G^n \rtimes S$ is the *wreath product* of G by S, denoted by $G \wr S$. Multiplication in $G \wr S$ obeys the rule

$$((a_1, \ldots, a_n), \alpha)((b_1, \ldots, b_n), \beta) = ((a_1 b_{1\alpha}, \ldots, a_n b_{n\alpha}), \alpha\beta).$$

When $S = \mathcal{T}_n$, we obtain the *full* wreath product $G \wr \mathcal{T}_n$. When $S = \mathrm{Sing}_n = \mathcal{T}_n \backslash \mathcal{S}_n$, we obtain the *singular* wreath product $G \wr \mathrm{Sing}_n$. If $G = \{1\}$, then $G \wr S \cong S$ for any S. On the other hand, if $S = \{1\}$, where $1 \in \mathcal{T}_n$ denotes the identity map, then $G \wr S \cong G^n$. The remainder of the article concerns only singular wreath products $G \wr \mathrm{Sing}_n$. Because Sing_n is empty for $n \leq 1$, we will assume that $n \geq 2$ whenever we make a statement about Sing_n.

Our main result is a presentation for $G \wr \mathrm{Sing}_2$, in terms of the idempotent generating set. One natural idea is gluing Proposition 2.1 and Theorem 2.3. However, that Sing_n does not contain the identity transformation pulls back the possibility.

The remaining part of this section is to recall the definitions of reduction system and its properties. As far as possible we follow the standard notation and terminology, as may be found in [2].

Let A be a set of objects and $\to$ a binary relation on A. We call the structure $(A, \to)$ a *reduction system* and the relation $\to$ a *reduction relation*. The reflexive, transitive closure of $\to$ is denoted by $\xrightarrow{*}$, while $\xleftrightarrow{*}$ denotes the smallest equivalence relation on A which contains $\to$. We denote the equivalence class of an element $x \in A$ by $[x]$. An element $x \in A$ is said to be *irreducible* if there is no $y \in A$ such that $x \to y$; otherwise, x is *reducible*. For any $x, y \in A$, if $x \xrightarrow{*} y$ and y is irreducible, then y is a *normal form* of x. A reduction system $(A, \to)$ is *noetherian* if there is no infinite sequence $x_0, x_1, \dots \in A$ such that for all $i \geq 0$, $x_i \to x_{i+1}$.

We say that a reduction system $(A, \to)$ is *confluent* if whenever $w, x, y \in A$ are such that $w \xrightarrow{*} x$ and $w \xrightarrow{*} y$, then there is a $z \in A$ such that $x \xrightarrow{*} z$ and $y \xrightarrow{*} z$, as described by the figure on the left in Figure 1, and $(A, \to)$ is *locally confluent* if whenever $w, x, y \in A$ are such that $w \to x$ and $w \to y$, then there is a $z \in A$ such that $x \xrightarrow{*} z$ and $y \xrightarrow{*} z$, as described by figure on the right in Figure 1.

$$\begin{tikzcd}
& w \arrow[dl, "*"'] \arrow[dr, "*"] & \\
x \arrow[dr, "*"] & & y \arrow[dl, "*"'] \\
& z &
\end{tikzcd}
\qquad
\begin{tikzcd}
& w \arrow[dl] \arrow[dr] & \\
x \arrow[dr, "*"] & & y \arrow[dl, "*"'] \\
& z &
\end{tikzcd}$$

Figure 1: confluence and local confluence

Proposition 2.4. [2] *Let $(A, \to)$ be a reduction system. Then the following statements hold:*

(i) *If $(A, \to)$ is noetherian and confluent, then for each $x \in A$, $[x]$ contains a unique normal form.*

(ii) *If $(A, \to)$ is noetherian, then it is confluent if and only if it is locally confluent.*

Let E be a biordered set. Recall that the *free idempotent generated semigroup over E* [24] is a free object $\mathrm{IG}(E)$ in the category of semigroups that are generated by E, given by the presentation:

$$\mathrm{IG}(E) = \langle \overline{E} \,|\, \overline{e}\,\overline{f} = \overline{ef},\ e, f \in E,\ \{e, f\} \cap \{ef, fe\} = \emptyset \rangle,$$

where $\overline{E} = \{\overline{e} \,|\, e \in E\}$. Recall also that we denote the free semigroup on $\overline{E}$ by $\overline{E}^+$.

Proposition 2.5. [24] *Let E be a biordered set, and let R be the relation on $\overline{E}^+$ defined by*

$$R = \{(\overline{e}\,\overline{f}, \overline{ef}) \,|\, (e, f) \textit{ is a basic pair}\}.$$

Then $(\overline{E}^+, \to)$ *forms a noetherian reduction system, where* $\to$ *is defined by*

$$u \to v \iff (\exists\,(l,r) \in R)(\exists\,x, y \in \overline{E}^+)\ u = xly \text{ and } v = xry.$$

It is worth remarking that the smallest equivalence relation containing $\to$ on $\overline{E}^+$ is exactly the congruence generated by R.

3. Presentation for $G \wr \mathrm{Sing}_2$

In this section, we begin with a simple description for idempotents of $G \wr \mathrm{Sing}_n$, and proceed to the main topic of the paper: finding a presentation for $G \wr \mathrm{Sing}_2$.

Recall from Theorem 2.2 that Sing_n is generated by its idempotents of rank $n-1$, and from [9, Theorem 5.12] that $G \wr \mathrm{Sing}_n$ is generated by idempotents whose underlying (idempotent) transformation has rank $n-1$. We first study idempotents in $G^n \rtimes D_{n-1}$.

For $i, j \in \mathbf{n}$ with $i \neq j$, and for $g \in G$, we define

$$\varepsilon_{ij;g} = ((1, \ldots, 1, \underset{i}{g}, 1, \ldots, 1), \varepsilon_{ij}) \in G \wr \mathrm{Sing}_n\,.$$

As usual, we also identify $\varepsilon_{ij} \in \mathrm{Sing}_n$ with $\varepsilon_{ij;1} \in G \wr \mathrm{Sing}_n$.

Proposition 3.1. *Idempotents in* $G^n \rtimes D_{n-1}$ *must be of the form* $\varepsilon_{ij;g}$.

Proof. Suppose that $((g_1, \ldots, g_n), \varepsilon) \in G^n \rtimes D_{n-1}$. It is obvious that

$$\begin{aligned}
&((g_1, \ldots, g_n), \varepsilon)^2 = ((g_1, \ldots, g_n), \varepsilon)\\
\iff& \varepsilon^2 = \varepsilon, \text{ and } g_i g_{i\varepsilon} = g_i \text{ for } i \in \mathbf{n}\\
\iff& \varepsilon^2 = \varepsilon, \text{ and } g_{i\varepsilon} = 1 \text{ for } i \in \mathbf{n}\\
\iff& \varepsilon^2 = \varepsilon, \text{ and } g_j = 1 \text{ for } j \in \mathrm{im}\,(\varepsilon).
\end{aligned}$$

Since $\mathrm{rank}\,(\varepsilon) = |\,\mathrm{im}\,(\varepsilon)| = n-1$, we must have at least $n-1$ places in $(g_1, \ldots, g_n)$ with $g_j = 1$. This together with the fact that idempotents in D_{n-1} are of the form ε_{kl} gives that idempotents in $G^n \rtimes D_{n-1}$ must be of the form $((1, \ldots, 1, \underset{i}{g}, 1, \ldots, 1), \varepsilon_{kl})$.

Furthermore, we have

$$\begin{aligned}
&((1, \ldots, 1, \underset{i}{g}, 1, \ldots, 1), \varepsilon_{kl})^2\\
=& \begin{cases}
((1, \ldots, 1, \underset{i}{g^2}, 1, \ldots, 1), \varepsilon_{kl}) & \text{if } \ i \neq k, l,\\
((1, \ldots, 1, \underset{i}{g^2}, 1, \ldots, 1, \underset{l}{g}, 1, \ldots, 1), \varepsilon_{kl}) & \text{if } \ i = k,\\
((1, \ldots, 1, \underset{i}{g}, 1, \ldots, 1), \varepsilon_{kl}) & \text{if } \ i = l.
\end{cases}
\end{aligned}$$

It follows directly that

$$((1,\dots,1,\underset{i}{g},1,\dots,1),\varepsilon_{kl}) \in E(G^n \rtimes D_{n-1}) \iff i = l \text{ or } g = 1.$$

If $i = l$, then $((1,\dots,1,\underset{i}{g},1,\dots,1),\varepsilon_{kl}) = \varepsilon_{ki;g}$; if $g = 1$, then $((1,\dots,1,\underset{i}{g},1,\dots,1),\varepsilon_{kl})$ is in fact $\varepsilon_{kl;1}$. ∎

We are now ready to state our main result. Define an alphabet

$$Z = \{e_{ij;g} \mid i,j \in \mathbf{n},\ i \neq j,\ g \in G\},$$

in one-one correspondence with the generating set of $G \wr \mathrm{Sing}_2$ from [9, Thm. 5.12] and Proposition 3.1. Let P be the set of relations

$$e_{ij;g}e_{ij;h} = e_{ij;g} = e_{ji;g^{-1}}e_{ij;g} \qquad \text{for } g,h \in G \text{ and distinct } i,j \tag{1}$$
$$e_{ij;1}e_{ji;g}e_{ij;h} = e_{ji;1}e_{ij;gh} \qquad \text{for } g,h \in G \text{ and distinct } i,j. \tag{2}$$

By [9, Remark 4.8] and Proposition 3.1, we may define an epimorphism

$$\varphi : Z^+ \to G \wr \mathrm{Sing}_2,\ e_{ij;g} \mapsto \varepsilon_{ij;g}.$$

The main goal of this paper is to provide a proof of the following result.

Theorem 3.2. *The semigroup $G \wr \mathrm{Sing}_2$ has presentation $\langle Z \mid P\rangle$ via φ.*

We now collect the needed parts to deduce the principal result of this paper. Let $\sim$ be the congruence on Z^+ generated by P.

Lemma 3.3. *We have the inclusion $\sim \subseteq \ker(\varphi)$.*

Proof. This follows by a simple check that each relation from P holds as an equation in $G \wr \mathrm{Sing}_2$ when the letters $e_{ij;g}$ are replaced by the maps $\varepsilon_{ij;g}$ as appropriate. ∎

Corollary 3.4. *Every $(e_{j_1i_1;g_1}e_{j_2i_2;g_2}\cdots e_{j_ti_t;g_t})\varphi$ has a simplified preimage in one of the following forms:*

(i) $e_{ji;g_1}e_{ij;g_2}\cdots e_{ji;g_{2n+1}}$;
(ii) $e_{ji;g_1}e_{ij;g_2}\cdots e_{ij;g_{2n}}$.

Proof. This follows directly from the fact that $(e_{ji;g}e_{ji;h})\varphi = e_{ji;g}\varphi$. ∎

The following lemma tells us how the product of $e_{ig;g}$s behaves under φ.

Lemma 3.5. *We have*

$$(e_{ji;g_1}e_{ij;g_2}\cdots e_{ji;g_n})\varphi = ((g_1g_2\underset{i}{\cdots}g_n, g_2\underset{j}{\cdots}g_n), \varepsilon_{ji}), \tag{3}$$

$$(e_{ji;g_1}e_{ij;g_2}\cdots e_{ij;g_n})\varphi = ((g_1g_2\underset{i}{\cdots}g_n, g_2\underset{j}{\cdots}g_n), \varepsilon_{ij}). \tag{4}$$

Proof. The assertions can be checked using proof by mathematical induction. First, that they are true for $n = 2$ and $n = 3$ may be easily checked. We then assume that they are true for $n = k$. For $n = k + 1$,

$$\begin{aligned}
(e_{ji;g_1}e_{ij;g_2}\cdots e_{ji;g_k}e_{ij;g_{k+1}})\varphi &= (e_{ji;g_1}e_{ij;g_2}\cdots e_{ji;g_k})\varphi(e_{ij;g_{k+1}}\varphi)\\
&= ((g_1g_2\underset{i}{\cdots}g_k, g_2\underset{j}{\cdots}g_k), \varepsilon_{ji})((\underset{i}{1}, \underset{j}{g_{k+1}}), \varepsilon_{ij})\\
&= ((g_1g_2\cdots\underset{i}{g_k}g_{k+1}, g_2\cdots\underset{j}{g_k}g_{k+1}), \varepsilon_{ij}),\\
(e_{ji;g_1}e_{ij;g_2}\cdots e_{ij;g_k}e_{ji;g_{k+1}})\varphi &= (e_{ji;g_1}e_{ij;g_2}\cdots e_{ij;g_k})\varphi(e_{ji;g_{k+1}}\varphi)\\
&= ((g_1g_2\underset{i}{\cdots}g_k, g_2\underset{j}{\cdots}g_k), \varepsilon_{ij})((\underset{i}{g_{k+1}}, \underset{j}{1}), \varepsilon_{ji})\\
&= ((g_1g_2\cdots\underset{i}{g_k}g_{k+1}, g_2\cdots\underset{j}{g_k}g_{k+1}), \varepsilon_{ji}),
\end{aligned}$$

which implies that they are also true. It follows that equations (3) and (4) are true for all $n \in \mathbf{n}$. ■

The next proposition plays a key role in the proof of Theorem 3.2.

Proposition 3.6. *We have* $\ker\varphi \subseteq P^\sharp$.

Proof. Assume that two elements in Z^+ are $\ker\varphi$-related. By Corollary 3.4 and Lemma 3.5, they have simplified forms in one of the following cases:

(i) $(e_{ji;g_1}e_{ij;g_2}\cdots e_{ji;g_{2n+1}})\varphi = (e_{ji;h_1}e_{ij;h_2}\cdots e_{ji;h_{2m+1}})\varphi$;

(ii) $(e_{ji;g_1}e_{ij;g_2}\cdots e_{ji;g_{2n+1}})\varphi = (e_{ij;h_1}e_{ji;h_2}\cdots e_{ji;h_{2m}})\varphi$.

It is worth remarking that $e_{ji;g_1}e_{ij;g_2}\cdots e_{ji;g_n}$ and $e_{ji;g_1}e_{ij;g_2}\cdots e_{ij;h_m}$ will never be $\ker\varphi$-related since their images under φ are $(\ldots, \varepsilon_{ji})$ and $(\ldots, \varepsilon_{ij})$, respectively.

For Case (i), we have

$$((g_1g_2\underset{i}{\cdots}g_{2n+1}, g_2\cdots\underset{j}{g_{2n+1}}), \varepsilon_{ji}) = ((h_1h_2\underset{i}{\cdots}h_{2m+1}, h_2\cdots\underset{j}{h_{2m+1}}), \varepsilon_{ji}),$$

which gives $g_1g_2\cdots g_{2n+1} = h_1h_2\cdots h_{2m+1}$ and $g_2\cdots g_{2n+1} = h_2\cdots h_{2m+1}$, whence $g_1 = h_1$. Then

$$\begin{aligned}
e_{ji;g_1}e_{ij;g_2}\cdots e_{ji;g_{2n+1}} &\sim e_{ji;g_1}e_{ji;1}e_{ij;g_2}e_{ji;g_3}\cdots e_{ji;g_{2n+1}} && \text{by (1)}\\
&\sim e_{ji;g_1}e_{ij;1}e_{ji;g_2g_3}e_{ij;g_4}\cdots e_{ji;g_{2n+1}} && \text{by (2)}\\
&\sim e_{ji;g_1}e_{ji;1}e_{ij;g_2g_3g_4}\cdots e_{ji;g_{2n+1}} && \text{by (2)}\\
&\sim \cdots \sim e_{ji;g_1}e_{ij;1}e_{ji;g_2\cdots g_{2n+1}}, && \text{by (2)}
\end{aligned}$$

while

$$\begin{aligned} e_{ji;h_1}e_{ij;h_2}\cdots e_{ji;h_{2m+1}} &\sim e_{ji;g_1}e_{ji;1}e_{ij;h_2}e_{ji;h_3}\cdots e_{ji;h_{2m+1}} && \text{by } h_1=g_1 \text{ and (1)} \\ &\sim e_{ji;g_1}e_{ij;1}e_{ji;h_2h_3}\cdots e_{ji;h_{2m+1}} && \text{by (2)} \\ &\sim \cdots \sim e_{ji;g_1}e_{ij;1}e_{ji;h_2\cdots h_{2m+1}} && \text{by (2)} \\ &= e_{ji;g_1}e_{ij;1}e_{ji;g_2\cdots g_{2n+1}}. \end{aligned}$$

For Case (ii), we have

$$((g_1g_2 \underset{i}{\cdots} g_{2n+1}, g_2 \underset{j}{\cdots} g_{2n+1}), \varepsilon_{ji}) = ((h_1h_2 \underset{j}{\cdots} h_{2m}, h_2 \underset{i}{\cdots} h_{2m}), \varepsilon_{ji}),$$

which implies $g_1g_2\cdots g_{2n+1} = h_2\cdots h_{2m}$ and $g_2\cdots g_{2n+1} = h_1h_2\cdots h_{2m}$. It follows that $g_1h_1 = 1$, whence $h_1 = g_1^{-1}$. Then

$$\begin{aligned} e_{ij;h_1}e_{ji;h_2}\cdots e_{ji;h_{2m}} &\sim e_{ij;g_1^{-1}}e_{ij;1}e_{ji;h_2}e_{ij;h_3}\cdots e_{ji;h_{2m}} && \text{by } h_1=g_1^{-1} \text{ and (1)} \\ &\sim e_{ij;g_1^{-1}}e_{ji;1}e_{ij;h_2h_3}\cdots e_{ji;h_{2m}} && \text{by (2)} \\ &\sim e_{ij;g_1^{-1}}e_{ij;1}e_{ji;h_2h_3h_4}\cdots e_{ji;h_{2m}} && \text{by (2)} \\ &\sim \cdots \sim e_{ij;g_1^{-1}}e_{ij;1}e_{ji;h_2\cdots h_{2m}} && \text{by (2)} \\ &\sim e_{ij;g_1^{-1}}e_{ji;h_2\cdots h_{2m}}, && \text{by (1)} \end{aligned}$$

while

$$\begin{aligned} e_{ji;g_1}e_{ij;g_2}\cdots e_{ji;g_{2n+1}} &\sim e_{ij;g_1^{-1}}e_{ij;1}e_{ji;g_1}e_{ij;g_2}e_{ji;g_3}\cdots e_{ji;g_{2n+1}} && \text{by (1)} \\ &\sim e_{ij;g_1^{-1}}e_{ji;1}e_{ij;g_1g_2}e_{ji;g_3}\cdots e_{ji;g_{2n+1}} && \text{by (2)} \\ &\sim e_{ij;g_1^{-1}}e_{ij;1}e_{ij;g_1g_2g_3}\cdots e_{ji;g_{2n+1}} && \text{by (2)} \\ &\sim \cdots \sim e_{ij;g_1^{-1}}e_{ij;1}e_{ji;g_2\cdots g_{2n+1}} && \text{by (2)} \\ &\sim e_{ij;g_1^{-1}}e_{ji;g_2\cdots g_{2n+1}} && \text{by (1)} \\ &= e_{ij;g_1^{-1}}e_{ji;h_2\cdots h_{2m}}. \end{aligned}$$

This completes the proof of the proposition. ∎

Proof of Theorem 3.2. It remains only to show that $\ker\varphi = P^\sharp$. By Lemma 3.3, we have $P^\sharp \subseteq \ker\varphi$. Proposition 3.6 establishes that $\ker\varphi \subseteq P^\sharp$, and so the proof is completed. ∎

Remark 3.7. [9] provides presentations for $M \wr \mathrm{Sing}_n$ with M a monoid in terms of certain natural generating sets. In order to obtain a presentation in terms of the idempotent generating set, they had to use the technique of Tietze transformation, i.e. deducing a presentation for $M \wr \mathrm{Sing}_n$ in terms of a very large generating set to the above-mentioned simpler presentation. For the case in which M is a group and $n = 2$, a direct proof can be achieved, as shown in this paper. We further remark that relations (1) and (2) cannot be reduced.

4. Can (2) Be Reduced?

It is aimed in this section to prove that the generating relation (2) cannot be reduced. The following two lemmas will be of use later.

Lemma 4.1. *In $G \wr \mathrm{Sing}_2$, we have*

$$\begin{aligned} \varepsilon_{ij;g}\,\mathcal{L}\,\varepsilon_{kl;h} &\iff \varepsilon_{ij}\,\mathcal{L}\,\varepsilon_{kl} \iff j = l, \\ \varepsilon_{ij;g}\,\mathcal{R}\,\varepsilon_{kl;h} &\iff i = l,\ j = k,\ g = h^{-1}. \end{aligned}$$

Specifically, we have $\varepsilon_{ij;g}\,\mathcal{L}\,\varepsilon_{ij;h}$ and $\varepsilon_{ij;g}\,\mathcal{R}\,\varepsilon_{ji;g^{-1}}$ for all $g \in G$.

Proof. Straightforward. ∎

It is clear from Lemma 4.1 that relation (1) consists exactly of equations induced by Green's relations. To see that relation (2) cannot be implied from Green's relations, we need an auxiliary result. Assume that φ: $Z^+ \to G \wr \mathrm{Sing}_2$, $\overline{e} \mapsto e$ in the presentation $\langle Z \,|\, P\rangle$ given in Theorem 3.2.

Lemma 4.2. *Let ρ be a congruence on Z^+ such that $\rho = X^\sharp$, where*

$$X = \{(\overline{e}\overline{f}, \overline{e}) \,|\, e\,\mathcal{L}\,f\} \cup \{(\overline{e}\overline{f}, \overline{f}) \,|\, e\,\mathcal{R}\,f\}.$$

Then for any $\overline{e_1}\cdots\overline{e_n} \in Z^+$, there exists a unique $w = \overline{g_1}\cdots\overline{g_m} \in Z^+$ such that

$$\overline{e_1}\cdots\overline{e_n}\,\rho\,w, \text{ and } (\overline{g_i}, \overline{g_{i+1}}) \notin \mathcal{L},\ (\overline{g_i}, \overline{g_{i+1}}) \notin \mathcal{R} \text{ for } 1 \le i \le m-1.$$

Proof. By Props. 2.4 and 2.5, we just need to show that our reduction relation X is locally confluent. For this purpose, it is sufficient to consider an arbitrary word of length 3, say $\overline{e}\overline{f}\overline{g} \in Z^+$, where e, f and f, g are comparable. Clearly, there are four cases, namely, $e\,\mathcal{L}\,f\,\mathcal{L}\,g$, $e\,\mathcal{L}\,f\,\mathcal{R}\,g$, $e\,\mathcal{R}\,f\,\mathcal{L}\,g$ and $e\,\mathcal{R}\,f\,\mathcal{R}\,g$. Then we have the following four diagrams in Figure 2.

Thus (Z^+, X) is locally confluent, and so by Proposition 2.4 we conclude that every element in Z^+/ρ has a unique normal form. ∎

In $G \wr \mathrm{Sing}_2$, for $e = ((1,1), c_2)$, $f = ((1,t), c_1)$, $g = ((1,1), c_1)$, $h = ((t,1), c_2)$, we have

$$\begin{aligned} efe &= ((1,1), c_2)((1,t), c_1)((1,1), c_2) \\ &= ((1,1), c_2)((1,1), c_1)((t,1), c_2) = ((1,1), c_1)((t,1), c_2) = gh. \end{aligned}$$

$$
\begin{tikzcd}[column sep=small]
& \overline{e}\,\overline{f}\,\overline{g} \arrow[dl] \arrow[dr] & & & \overline{e}\,\overline{f}\,\overline{g} \arrow[dl] \arrow[dr] & \\
\overline{eg} \arrow[dr, "*"] & & \overline{e}\,\overline{f} \arrow[dl, "*"'] & \overline{eg} \arrow[dr, "*"] & & \overline{eg} \arrow[dl, "*"'] \\
& \overline{e} & & & \overline{eg} &
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
& \overline{e}\,\overline{f}\,\overline{g} \arrow[dl] \arrow[dr] & & & \overline{e}\,\overline{f}\,\overline{g} \arrow[dl] \arrow[dr] & \\
\overline{f}\,\overline{g} \arrow[dr, "*"] & & \overline{e}\,\overline{f} \arrow[dl, "*"'] & \overline{f}\,\overline{g} \arrow[dr, "*"] & & \overline{eg} \arrow[dl, "*"'] \\
& \overline{f} & & & \overline{g} &
\end{tikzcd}
$$

Figure 2: the local confluence of Z^+

Then $\overline{e}\,\overline{f}\,\overline{e}\,\rho\,\overline{g}\,\overline{h}$. It follows from Lemma 4.2 that it is impossible to have $\overline{e}\,\overline{f}\,\overline{k}\,\rho\,\overline{g}\,\overline{h}$ for some $\overline{k} \in Z^+$. And we deduce that relation (2) cannot be implied from (1).

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