

Fibrewise Near Separation Axioms

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Abstract

The purpose of this paper is to consider fibrewise near versions of the more important separation axioms of ordinary topology namely fibrewise near T_0 spaces, fibrewise near T_1 spaces, fibrewise near R_0 spaces, fibrewise near Hausdorff spaces, fibrewise near functionally Hausdorff spaces, fibrewise near regular spaces, fibrewise near completely regular spaces, fibrewise near normal spaces and fibrewise near functionally normal spaces. Also we give several results concerning it.

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1. Introduction and Preliminaries

To begin with we work in the category fibrewise sets over a given set, called the base set. If the base set is denoted by B then a fibrewise set over B consists of a set X together with a function $p : X \rightarrow B$, called the projection. For each point b of B the fibre over b is the subset $X_b = p^{-1}(b)$ of X ; fibres may be empty since we do not require p to be surjective, also for each subset B^* of B we regard $X_{B^*} = p^{-1}(B^*)$ as a fibrewise set over B^* with the projection determined by p , the

alternative notation $X|B^*$ is sometimes convenient. We regard the Cartesian product $B \times T$, for any set T , as a fibrewise set over B using the first projection.

Definition 1.1. [8] Let X and Y are fibrewise sets over B , with projections $p_X : X \rightarrow B$ and $p_Y : Y \rightarrow B$, respectively, a function $\varphi : X \rightarrow Y$ is said to be fibrewise if $p_Y \circ \varphi = p_X$, in other words if $\varphi(X_b) \subset Y_b$ for each point b of B .

Note that a fibrewise function $\varphi : X \rightarrow Y$ over B determines by restriction, a fibrewise function $\varphi_{B^*} : X_{B^*} \rightarrow Y_{B^*}$ over B^* for each B^* of B .

Given an indexed family $\{X_r\}$ of fibrewise sets over B the fibrewise product $\prod_B X_r$ is defined, as a fibrewise set over B , and comes equipped with the family of fibrewise projections $\pi_r : \prod_B X_r \rightarrow X_r$. Specifically the fibrewise product is defined as the subset of the ordinary product $\prod X_r$ in which the fibres are the products of the corresponding fibers of the factors X_r .

Definition 1.2. [8] Suppose that B is a topological space, the fibrewise topology on a fibrewise set X over B , mean any topology on X for which the projection p is continuous.

Remark 1.1. [8]

- (a) The coarsest such topology is the topology induced by p , in which the open sets of X are precisely the inverse image of the open sets of B ; this is called the fibrewise indiscrete topology.
- (b) The fibrewise topological space over B is defined to be a fibrewise set over B with a fibrewise topology.

We regard the topology product $B \times T$, for any topological space T , as a fibrewise topological spaces over B using the first projection. The equivalences in the category of fibrewise topological spaces are called fibrewise topological equivalences. If X is fibrewise topologically equivalent to $B \times T$, for some topological space T , we say that X is trivial, as a fibrewise topological spaces over B . In fibrewise topology the term neighborhood (briefly nbd) is used in precisely in the same sense as it is in ordinary topology, but the terms fibrewise basic may need some explanation, thus let X be fibrewise topological spaces over B , if x is a point of X_b , where $b \in B$, describe a family $N(x)$ of nbds of x in X as fibrewise basic if for each nbd U of x we have $X_W \cap V \subset U$, for some member V of $N(x)$ and nbd W of b in B . For example, in the case of the topological product $B \times T$, where T is a topological spaces, the family of Cartesian products $B \times N(t)$, where $N(t)$ runs through the nbds of t , is fibrewise basic for (b, t) . For other notions or notations which are not defined here we follow closely James [8], Engelking [6] and Bourbaki [4].

Definition 1.3. [8] A fibrewise function $\varphi : X \rightarrow Y$, where X and Y are fibrewise topological spaces over B is called:

- (a) Continuous if for each point $x \in X_b$, where $b \in B$, the inverse image of each open set of $\varphi(x)$ is an open set of x .
- (b) Open if for each point $x \in X_b$, where $b \in B$, the direct image of each open set of x is an open set of $\varphi(x)$.

Definition 1.4. [6] For every topological space X^* and any subspace X of X^* , the function $i_X : X \rightarrow X^*$ define by $i_X(x) = x$ is called embedded of the subspace X in the space X^* . Observe that i_X is continuous. Since $i_X^{-1}(U) = X \cap U$, where U is open set in X^* . And the embedded i_X is closed (resp. open) if and only if the subspace X is closed (resp. open).

Definition 1.5. [6] Suppose we are given a topological space X , a family $\{Y_s\}_{s \in S}$ of topological spaces and a family of continuous functions $\{\varphi_s\}_{s \in S}$, where $\varphi_s : X \rightarrow Y_s$ the function assigning to the point $x \in X$ the point $\{\varphi_s(x)\} \in \prod_{s \in S} Y_s$ is continuous; it is called the diagonal of the functions $\{\varphi_s\}_{s \in S}$ and is denoted by $\Delta_{s \in S} \varphi_s$ or by $\varphi_1 \Delta \varphi_2 \Delta \dots \Delta \varphi_k$ if $S = \{1, 2, \dots, k\}$.

Definition 1.6. A subset A of a topological space (X, τ) is called:

- (a) Regular open [14] (briefly R-open) if $A = \text{Int}(\text{Cl}(A))$.
- (b) Pre-open [10] (briefly P-open) if $A \subseteq \text{Int}(\text{Cl}(A))$.
- (c) Semi-open [9] (briefly S-open) if $A \subseteq \text{Cl}(\text{Int}(A))$.
- (d) γ -open [5] (= b-open [2]) (briefly γ -open) if $A \subseteq \text{Cl}(\text{Int}(A)) \cup \text{Int}(\text{Cl}(A))$.
- (e) α -open [12] (briefly α -open) if $A \subseteq \text{Int}(\text{Cl}(\text{Int}(A)))$.
- (f) β -open [1] (= semi-pre-open set [3]) (briefly β -open) if $A \subseteq \text{Cl}(\text{Int}(\text{Cl}(A)))$.

The complement of an R-open (resp. P-open, S-open, γ -open, α -open, β -open) is called R-closed (resp. P-closed, S-closed, γ -closed, α -closed, β -closed). The family of all R-open (resp. P-open, S-open, γ -open, α -open, β -open) are larger than τ (except R-open) and closed under forming arbitrary union.

Definition 1.9. A function $\varphi : X \rightarrow Y$ is said to be R-closed [13] (resp. P-closed [10], S-closed [9], γ -closed [5], α -closed [11], β -closed [1]) if the image of each closed set in X is R-closed (resp. S-closed, P-closed, γ -closed, α -closed, β -closed) in Y .

2. Fibrewise Near T_0 , Near T_1 and Near Hausdorff Spaces

Before introduce the definitions of fibrewise near separation axioms we introduce the following definition:

Definition 2.1. A fibrewise function $\varphi : X \rightarrow Y$, where X and Y are fibrewise topological spaces over B is called:

- (a) i -irresolute if for each point $x \in X_b$, where $b \in B$, the inverse image of each i -open set of $\varphi(x)$ is an i -open set of x , where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.
- (b) i -biopen if for each point $x \in X_b$, where $b \in B$, the direct image of each i -open set of x is an i -open set of $\varphi(x)$, where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.
- (c) i -biclosed if for each point $x \in X_b$, where $b \in B$, the direct image of each i -closed set of x is an i -closed set of $\varphi(x)$, where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

Now we introduce the versions of fibrewise near T_0 and near T_1 spaces as follows:

Definition 2.2. Let X be fibrewise topological space over B . Then X is called fibrewise near T_0 (briefly i - T_0) if whenever $x_1, x_2 \in X_b$, where $b \in B$ and $x_1 \neq x_2$, either there exists an i -open set of x_1 which does not contains x_2 in X , or vice versa, where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

Remark 2.1.

- (a) X is fibrewise i - T_0 space if and only if each fiber X_b is i - T_0 space, where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.
- (b) Subspaces of fibrewise i - T_0 spaces are fibrewise i - T_0 spaces, where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.
- (c) The fibrewise topological products of fibrewise i - T_0 spaces with the family of fibrewise i -irresolute projections are fibrewise i - T_0 spaces, where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

Of course one can formulate a fibrewise version of the near T_1 (briefly i - T_1) space in a similar fashion as follows "Let X be fibrewise topological space over B . Then X is called fibrewise i - T_1 if whenever $x_1, x_2 \in X_b$, where $b \in B$ and $x_1 \neq x_2$, there exist an i -open sets U_1, U_2 in X such that $x_1 \in U_1, x_2 \notin U_1$ and $x_1 \notin U_2, x_2 \in U_2$ ", where $i \in \{R, P, S, \gamma, \alpha, \beta\}$, but it turns out that there is no real use for this in what we are going to do. Instead we make some use of another axiom "The axiom is that every near open set contains the closure of each of its points", and use the term near R_0 space. This is true for near T_1 spaces, of course, and for near regular spaces. Thinking of it as a weak form of near regularity. For example, indiscrete spaces are near R_0 space. The fibrewise version of the near R_0 axiom is as follows.

Definition 2.3. The fibrewise topological space X over B is called fibrewise near R_0 (briefly i - R_0) if for each point $x \in X_b$, where $b \in B$, and each i -open set V of x in X , there exist a nbd W of b in B such that the closure of $\{x\}$ in X_W is contained in V (i.e. $X_W \cap Cl\{x\} \subset V$), where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

For example $B \times T$ is fibrewise i - R_0 space for all i - R_0 spaces T , where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

Remark 2.2.

- (a) The nbds of x are given by a fibrewise basis it is sufficient if the condition in definition (2.3) is satisfied for all fibrewise basic nbds.

(b) If X is fibrewise i - R_0 space over B , then X_{B^*} is fibrewise i - R_0 space over B^* for each subspace B^* of B , where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

Subspaces of fibrewise i - R_0 spaces are fibrewise i - R_0 spaces, where $i \in \{R, P, S, \gamma, \alpha, \beta\}$. In fact we have

Proposition 2.1. Let $\varphi : X \rightarrow X^*$ be a fibrewise embedding, where X and X^* are fibrewise topological spaces over B . If X^* is fibrewise i - R_0 then so is X , where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

Proof. Let $x \in X_b$, where $b \in B$, and let V be an i -open set of x in X . Then $V = \varphi^{-1}(V^*)$, where V^* is an i -open set of $x^* = \varphi(x)$ in X^* . Since X^* is fibrewise i - R_0 there exists a nbd W of b such that $X^*_W \cap Cl\{x^*\} \subset V^*$. Then $X_W \cap Cl\{x\} \subset \varphi^{-1}(X^*_W \cap Cl\{x^*\}) \subset \varphi^{-1}(V^*) = V$, and so X is fibrewise i - R_0 , where $i \in \{R, P, S, \gamma, \alpha, \beta\}$, as asserted.

The class of fibrewise i - R_0 spaces is finitely multiplicative, where $i \in \{R, P, S, \gamma, \alpha, \beta\}$, in the following sense.

Proposition 2.2. Let $\{X_r\}$ be a finite family of fibrewise i - R_0 spaces over B . Then the fibrewise topological product $X = \prod_B X_r$ is fibrewise i - R_0 .

Proof. Let $x \in X_b$, where $b \in B$. Consider an i -open set $V = \prod_B V_r$ of x in X , where V_r is an i -open set of $\pi_r(x) = x_r$ in X_r for each index r . Since X_r is fibrewise i - R_0 there exists a nbd W_r of b in B such that $(X_r|_{W_r}) \cap Cl\{x_r\} \subset V_r$. Then the intersection W of the W_r is a nbd of b such that $X_W \cap Cl\{x\} \subset V$ and so $X = \prod_B X_r$ is fibrewise i - R_0 , where $i \in \{R, P, S, \gamma, \alpha, \beta\}$. The same conclusion holds for infinite fibrewise products provided each of the factors is fibrewise nonempty.

Proposition 2.3. Let $\varphi : X \rightarrow Y$ be a closed i -irresolute fibrewise surjection, where X and Y are fibrewise topological spaces over B . If X is fibrewise i - R_0 , then so is Y , where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

Proof. Let $y \in Y_b$, where $b \in B$, and let V be an i -open set of y in Y . Pick $x \in \varphi^{-1}(y)$. Then $U = \varphi^{-1}(V)$ is an i -open set of x . Since X is fibrewise i - R_0 there exists a nbd W of b such that $X_W \cap Cl\{x\} \subset U$. Then $Y_W \cap \varphi(Cl\{x\}) \subset \varphi(U) = V$. Since φ is closed, then $\varphi(Cl\{x\}) = Cl(\varphi\{x\})$. Therefore $Y_W \cap Cl(\varphi\{x\}) \subset V$ and so Y is fibrewise i - R_0 , where $i \in \{R, P, S, \gamma, \alpha, \beta\}$, as asserted.

Now we introduce the version of fibrewise near Hausdorff spaces as follows:

Definition 2.4. The fibrewise topological space X over B is called fibrewise near Hausdorff (briefly i -Hausdorff) if whenever $x_1, x_2 \in X_b$, where $b \in B$ and $x_1 \neq x_2$, there exist disjoint i -open sets U_1, U_2 of x_1, x_2 in X , where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

For example, $B \times T$ is fibrewise i -Hausdorff space for all i -Hausdorff spaces T , where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

Remark 2.3. If X is fibrewise i -Hausdorff space over B , then X_{B^*} is fibrewise i -Hausdorff over B^* for each subspace B^* of B . In particular the fibres of X are i -Hausdorff spaces. However a fibrewise topological spaces with i -Hausdorff fibres is not necessarily i -Hausdorff: for example take $X = B$ with B indiscrete, where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

Proposition 2.4. The fibrewise topological space X over B is fibrewise i -Hausdorff if and only if the diagonal embedding $\Delta : X \rightarrow X \times_B X$ is i -closed, where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

Proof. (\Leftarrow) Let $x_1, x_2 \in X_b$, where $b \in B$ and $x_1 \neq x_2$. Since $\Delta(X)$ is i -closed in $X \times_B X$, then (x_1, x_2) , being a point of the complement, admits a fibrewise product i -open set $V_1 \times_B V_2$ which does not meet $\Delta(X)$, and then V_1, V_2 are disjoint i -open sets of x_1, x_2 , where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

(\Rightarrow) The reverse direction is similar.

Subspaces of fibrewise i -Hausdorff spaces are fibrewise i -Hausdorff spaces, where $i \in \{R, P, S, \gamma, \alpha, \beta\}$. In fact we have

Proposition 2.5. Let $\varphi : X \rightarrow X^*$ be an embedding fibrewise function, where X and X^* are fibrewise topological spaces over B . If X^* is fibrewise i -Hausdorff then so is X , where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

Proof. Let $x_1, x_2 \in X_b$, where $b \in B$ and $x_1 \neq x_2$. Then $\varphi(x_1), \varphi(x_2) \in X^*_b$ are distinct, since X^* is fibrewise i -Hausdorff, there exist an i -open sets V_1, V_2 of $\varphi(x_1), \varphi(x_2)$ in X^* which are disjoint. Their inverse images $\varphi^{-1}(V_1), \varphi^{-1}(V_2)$ are i -open sets of x_1, x_2 in X which are disjoint and so X is fibrewise i -Hausdorff, where $i \in \{R, P, S, \gamma, \alpha, \beta\}$. Alternatively (2.4) can be used.

Proposition 2.6. Let $\varphi : X \rightarrow Y$ be an i -irresolvent fibrewise functions, where X and Y are fibrewise topological spaces over B . If Y is fibrewise i -Hausdorff, then the fibrewise graph $G : X \rightarrow X \times_B Y$ of φ is an i -closed embedding.

Proof. the fibrewise graph is defined in the same way as the ordinary graph, but with values in the fibrewise product, so that the diagram shown below is commutative.

$$\begin{array}{ccc}
 X & \xrightarrow{\quad G \quad} & X \times_B Y \\
 \varphi \downarrow & & \downarrow \varphi \times \text{id}_Y \\
 Y & \xrightarrow{\quad \Delta \quad} & Y \times_B Y
 \end{array}$$

Since $\Delta(Y)$ is i -closed in $Y \times_B Y$, by (2.4), so $G(X) = (\varphi \times \text{id}_Y)^{-1}(\Delta(Y))$ is i -closed in $X \times_B Y$, where $i \in \{R, P, S, \gamma, \alpha, \beta\}$, as asserted.

The class of fibrewise i -Hausdorff spaces is multiplicative, where $i \in \{R, P, S, \gamma, \alpha, \beta\}$, in the following sense.

Proposition 2.7. Let $\{X_r\}$ be a family of fibrewise i -Hausdorff spaces over B . Then the fibrewise topological product $X = \prod_B X_r$ with the family of fibrewise i -irresolute projection $\pi_r : X = \prod_B X_r \rightarrow X_r$ is fibrewise i -Hausdorff, where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

Proof. Let $x_1, x_2 \in X_b$, where $b \in B$ and $x_1 \neq x_2$. Then $\pi_r(x_1) \neq \pi_r(x_2)$ for some index r . Since X_r is fibrewise i -Hausdorff there exist an i -open sets V_1, V_2 of $\pi_r(x_1), \pi_r(x_2)$ in X_r which are disjoint. Since π_r is i -irresolute, then the inverse images $\pi_r^{-1}(V_1), \pi_r^{-1}(V_2)$ are disjoint i -open sets of x_1, x_2 in X , where $i \in \{R, P, S, \gamma, \alpha, \beta\}$, as required.

The near functionally version of the fibrewise near Hausdorff axiom is stronger than the non near functional version but its properties are fairly similar. Here and elsewhere we use I to denote the closed unit interval $[0, 1]$ in the real line \mathbb{R} .

Definition 2.5. The fibrewise topological space X over B is called fibrewise near functionally (briefly i -functionally) Hausdorff if whenever $x_1, x_2 \in X_b$, where $b \in B$ and $x_1 \neq x_2$, there exist a nbd W of b and disjoint i -open sets U, V of x_1, x_2 in X and a continuous function $\lambda : X_W \rightarrow I$ such that $X_b \cap U \subset \lambda^{-1}(0)$ and $X_b \cap V \subset \lambda^{-1}(1)$, where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

For example $B \times T$ is fibrewise i -functionally Hausdorff space for all i -functionally Hausdorff spaces T , where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

Remark 2.4. If X is fibrewise i -functionally Hausdorff space over B , then $X^* = X_{B^*}$ is fibrewise i -functionally Hausdorff over B^* for each subspace B^* of B . In particular the fibres of X are i -functionally Hausdorff spaces, where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

Subspaces of fibrewise i -functionally Hausdorff spaces are fibrewise i -functionally Hausdorff spaces, where $i \in \{R, P, S, \gamma, \alpha, \beta\}$. In fact we have

Proposition 2.8. Let $\varphi : X \rightarrow X^*$ be an embedding fibrewise function, where X and X^* are fibrewise topological spaces over B . If X^* is fibrewise i -functionally Hausdorff then so is X , where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

Moreover the class of fibrewise i -functionally Hausdorff spaces is multiplicative, where $i \in \{R, P, S, \gamma, \alpha, \beta\}$, as stated in.

Proposition 2.9. Let $\{X_r\}$ be a family of fibrewise i -functionally Hausdorff spaces over B . Then the fibrewise topological product $X = \prod_B X_r$ with the

family of fibrewise i -irresolute projection $\pi_r : X = \prod_B X_r \rightarrow X_r$ is fibrewise i -functionally Hausdorff, where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

The proofs of Propositions (2.8) and (2.9) are similar to those for the corresponding results in the non-functional case and will therefore be omitted.

3. Fibrewise Near Regular and Near Normal Spaces

We now proceed to consider the fibrewise versions of the higher near separation axioms, starting with near regularity and near completely regularity.

Definition 3.1. The fibrewise topological space X over B is called fibrewise near regular (briefly i -regular) if for each point $x \in X_b$, where $b \in B$, and for each i -open set V of x in X , there exists a nbd W of b in B and an open set U of x in X_W such that the closure of U in X_W is contained in V (i.e. $X_W \cap \text{Cl}(U) \subset V$), where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

For example, trivial fibrewise spaces with i -regular fibre are fibrewise i -regular, where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

Remark 3.1.

- (a) The nbds of x are given by a fibrewise basis it is sufficient if the condition in definition (3.1) is satisfied for all fibrewise basic nbds.
- (b) If X is fibrewise i -regular space over B , then X_{B^*} is fibrewise i -regular space over B^* for each subspace B^* of B , where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

Subspaces of fibrewise i -regular spaces are fibrewise i -regular spaces, where $i \in \{R, P, S, \gamma, \alpha, \beta\}$. In fact we have

Proposition 3.1. Let $\varphi : X \rightarrow X^*$ be a fibrewise embedding function, where X and X^* are fibrewise topological spaces over B . If X^* is fibrewise i -regular then so is X , where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

Proof. Let $x \in X_b$, where $b \in B$, and let V be an i -open set of x in X . Then $V = \varphi^{-1}(V^*)$, where V^* is an i -open set of $x^* = \varphi(x)$ in X^* . Since X^* is fibrewise i -regular there exists a nbd W of b and an open set U^* of x^* in X^*_W such that $X^*_W \cap \text{Cl}(U^*) \subset V^*$. Then $U = \varphi^{-1}(U^*)$ is an open set of x in X_W such that $X_W \cap \text{Cl}(U) \subset V$, and so X is fibrewise i -regular, where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

The class of fibrewise i -regular spaces is fibrewise multiplicative, where $i \in \{R, P, S, \gamma, \alpha, \beta\}$, in the following sense.

Proposition 3.2. Let $\{X_r\}$ be a finite family of fibrewise i -regular spaces over B . Then the fibrewise topological product $X = \prod_B X_r$ is fibrewise i -regular, where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

Proof. Let $x \in X_b$, where $b \in B$. Consider an i -open set $V = \prod_B V_r$ of x in X , where V_r is an i -open set of $\pi_r(x) = x_r$ in X_r for each index r . Since X_r is fibrewise i -regular there exists a nbd W_r of b in B and an open set U_r of x_r in $X_r|W_r$ such that the closure $(X_r|W_r) \cap Cl(U_r)$ of U_r in $X_r|W_r$ is contained in V_r . Then the intersection W of the W_r is a nbd of b and $U = \prod_B U_r$ is an open set of x in X_W such that the closure $X_W \cap Cl(U)$ of U in X_W is containing in V , and so $X = \prod_B X_r$ is fibrewise i -regular, where $i \in \{R, P, S, \gamma, \alpha, \beta\}$. The same conclusion holds for infinite fibrewise products provided each of the factors is fibrewise non-empty.

Proposition 3.3. Let $\varphi : X \rightarrow Y$ be an open, closed and i -irresolut fibrewise surjection, where X and Y are fibrewise topological spaces over B . If X is fibrewise i -regular, then so is Y , where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

Proof. Let $y \in Y_b$, where $b \in B$, and let V be an i -open set of y in Y . Pick $x \in \varphi^{-1}(y)$. Then $U = \varphi^{-1}(V)$ is an i -open set of x . Since X fibrewise i -regular there exists a nbd W of b and an open set U^* of x such that $X_W \cap Cl(U^*) \subset U$. Then $Y_W \cap \varphi(Cl(U^*)) \subset \varphi(U) = V$. Since φ is closed, then $\varphi(Cl(U^*)) = Cl(\varphi(U^*))$ and since φ is open, then $\varphi(U^*)$ is an open set of y . Thus Y is fibrewise i -regular, where $i \in \{R, P, S, \gamma, \alpha, \beta\}$ as asserted.

The near functionally version of the fibrewise near regularity axiom is stronger than the non near functionally version but its properties are fairly similar. In the ordinary theory the term completely regular is always used instead of functionally regular and we extend this usage to the fibrewise theory.

Definition 3.2. The fibrewise topological space X over B is called fibrewise near completely (briefly i -completely) regular if for each point $x \in X_b$, where $b \in B$ and for each i -open set V of x , there exist a nbd W of b and an open set U of x in X_W and a continuous function $\lambda : X_W \rightarrow I$ such that $X_b \cap U \subset \lambda^{-1}(0)$ and $X_W \cap (X_W - V) \subset \lambda^{-1}(1)$, where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

For example $B \times T$ is fibrewise i -completely regular space for all i -completely regular spaces T , where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

Remark 3.2.

- (a) The nbds of x are given by a fibrewise basis it is sufficient if the condition in definition (3.2) is satisfied for all fibrewise basic nbds.
- (b) If X is fibrewise i -completely regular space over B , then X_{B^*} is fibrewise i -completely regular space over B^* for each subspace B^* of B , where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

Subspaces of fibrewise i -completely regular spaces are fibrewise i -completely regular spaces, where $i \in \{R, P, S, \gamma, \alpha, \beta\}$. In fact we have

Proposition 3.4. Let $\varphi : X \rightarrow X^*$ be a fibrewise embedding, where X and X^* are fibrewise topological spaces over B . If X^* is fibrewise i -completely regular then so is X , where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

Proof. The proof is similar to the proof of proposition (3.1), so it is omitted.

The class of fibrewise i -completely regular spaces is finitely multiplicative, where $i \in \{R, P, S, \gamma, \alpha, \beta\}$, in the following sense.

Proposition 3.5. Let $\{X_r\}$ be a finite family of fibrewise i -completely regular spaces over B . Then the fibrewise topological product $X = \prod_B X_r$ is fibrewise i -completely regular, where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

Proof. Let $x \in X_b$, where $b \in B$. Consider a fibrewise i -open set $\prod_B V_r$ of x in X , where V_r is an i -open set of $\pi_r(x) = x_r$ in X_r for each index r . Since X_r is fibrewise i -completely regular there exists a nbd W_r of b and an open set U of x_r in X_r and a continuous function $\lambda_r : X_{W_r} \rightarrow I$ such that $(X_r)_b \cap U \subset \lambda_r^{-1}(0)$ and $X_{W_r} \cap (X_{W_r} - V_r) \subset \lambda_r^{-1}(1)$. Then the intersection W of the W_r is a nbd of b and $\lambda : X_W \rightarrow I$ is a continuous function where

$$\lambda(\xi) = \inf_{r=1, 2, \dots, n} \{\lambda_r(\xi_r)\} \text{ for } \xi = (\xi_r) \in X_W.$$

Since $X_b \cap \pi_r^{-1}(U) \subset \pi_r^{-1}((X_r)_b \cap U) \subset \pi_r^{-1}(\lambda_r^{-1}(0)) = (\lambda_r \circ \pi_r)^{-1}(0)$ and $X_W \cap \pi_r^{-1}(X_{W_r} - V_r) \subset \pi_r^{-1}(X_{W_r} \cap (X_{W_r} - V_r)) \subset \pi_r^{-1}(\lambda_r^{-1}(1)) = (\lambda_r \circ \pi_r)^{-1}(1)$, where $i \in \{R, P, S, \gamma, \alpha, \beta\}$, this proves the result. The same conclusion holds for infinite fibrewise products provided that each of the factors is fibrewise non-empty.

Proposition 3.6. Let $\varphi : X \rightarrow Y$ be an open, closed and i -irresolut fibrewise surjection, where X and Y are fibrewise topological spaces over B . If X is fibrewise i -completely regular, then so is Y , where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

Proof. Let $y \in Y_b$, where $b \in B$, and let V_y be an i -open set of y . Pick $x \in X_b$, so that $V_x = \varphi^{-1}(V_y)$ is an i -open set of x . Since X fibrewise i -completely regular there exists a nbd W of b and an open set U_x of x in X_W and a continuous function $\lambda : X_W \rightarrow I$ such that $X_b \cap U_x \subset \lambda^{-1}(0)$ and $X_W \cap (X_W - V_x) \subset \lambda^{-1}(1)$. Using proposition (1.3) in [8] we obtain a continuous function $\Omega : Y_W \rightarrow I$ such that $Y_b \cap U_y \subset \Omega^{-1}(0)$ and $Y_W \cap (Y_W - V_y) \subset \Omega^{-1}(1)$, where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

Now we introduce the version of fibrewise near normal space as follows:

Definition 3.3. The fibrewise topological space X over B is called fibrewise near normal (briefly i -normal) if for each point b of B and each pair H, K of disjoint closed sets of X , there exist a nbd W of b and a pair of disjoint i -open sets U, V of $X_W \cap H, X_W \cap K$ in X_W , where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

Remark 3.3. If X is fibrewise i -normal space over B , then X_{B^*} is fibrewise i -normal space over B^* for each subspace B^* of B , where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

Closed subspaces of fibrewise i -normal spaces are fibrewise i -normal, where $i \in \{R, P, S, \gamma, \alpha, \beta\}$. In fact we have

Proposition 3.7. Let $\varphi : X \rightarrow X^*$ be a closed fibrewise embedding, where X and X^* are fibrewise topological spaces over B . If X^* is fibrewise i -normal then so is X , where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

Proof. Let b be a point of B and let H, K be disjoint closed sets of X . Then $\varphi(H), \varphi(K)$ are disjoint closed sets of X^* . Since X^* is fibrewise i -normal there exists a nbd W of b and a pair of disjoint i -open sets U, V of $X^*_W \cap \varphi(H), X^*_W \cap \varphi(K)$ in X^*_W . Then $\varphi^{-1}(U)$ and $\varphi^{-1}(V)$ are disjoint i -open sets of $X_W \cap H, X_W \cap K$ in X_W , where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

Proposition 3.8. Let $\varphi : X \rightarrow Y$ be an i -biclosed continuous fibrewise surjection, where X and Y are fibrewise topological spaces over B . If X is fibrewise i -normal then so is Y , where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

Proof. Let b be a point of B and let H, K be disjoint closed sets of Y . Then $\varphi^{-1}(H), \varphi^{-1}(K)$ are disjoint closed sets of X . Since X is fibrewise i -normal there exists a nbd W of b and a pair of disjoint i -open sets U, V of $X_W \cap \varphi^{-1}(H)$ and $X_W \cap \varphi^{-1}(K)$. Since φ is i -biclosed, the sets $Y_W - \varphi(X_W - U), Y_W - \varphi(X_W - V)$ are i -open in Y_W , and form a disjoint pair of an i -open sets of $Y_W \cap H, Y_W \cap K$ in Y_W , where $i \in \{R, P, S, \gamma, \alpha, \beta\}$, as required.

Finally, we introduce the version of fibrewise near functionally normal space as follows:

Definition 3.4. The fibrewise topological space X over B is called fibrewise near functionally (briefly i -functionally) normal if for each point b of B and each pair H, K of disjoint closed sets of X , there exist a nbd W of b and a pair of disjoint i -open sets U, V and a continuous function $\lambda : X_W \rightarrow I$ such that $X_W \cap H \cap U \subset \lambda^{-1}(0)$ and $X_W \cap K \cap V \subset \lambda^{-1}(1)$ in X_W , where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

For example $B \times T$ is fibrewise i -functionally normal space whenever T is i -normal space, where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

Remark 3.4. If X is fibrewise i -functionally normal space over B , then X_{B^*} is fibrewise i -functionally normal space over B^* for each subspace B^* of B , where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

Closed subspaces of fibrewise i -functionally normal spaces are fibrewise i -functionally normal, where $i \in \{R, P, S, \gamma, \alpha, \beta\}$. In fact we have

Proposition 3.9. Let $\varphi : X \rightarrow X^*$ be a closed fibrewise embedding, where X and X^* are fibrewise topological spaces over B . If X^* is fibrewise i -functionally normal then so is X , where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

Proof. Let b be a point of B and let H, K be disjoint closed sets of X . Then $\varphi(H), \varphi(K)$ are disjoint closed sets of X^* . Since X^* is fibrewise i -functionally normal

there exists a nbd W of b and a pair of disjoint i -open sets U, V and a continuous function $\lambda : X^*_W \rightarrow I$ such that $X^*_W \cap \varphi(H) \cap U \subset \lambda^{-1}(0)$ and $X^*_W \cap \varphi(K) \cap V \subset \lambda^{-1}(1)$ in X^*_W . Then $\Omega = \alpha \circ \varphi$ is a continuous function $X_W \rightarrow I$ such that $X_W \cap H \cap \varphi^{-1}(U) \subset \Omega^{-1}(0)$ and $X_W \cap K \cap \varphi^{-1}(V) \subset \Omega^{-1}(1)$ in X_W , where $i \in \{R, P, S, \gamma, \alpha, \beta\}$, as required.

Proposition 3.10. Let $\varphi : X \rightarrow Y$ be an i -biopen, closed and continuous fibrewise surjection, where X and Y are fibrewise topological spaces over B . If X is fibrewise i -functionally normal, then so is Y , where $i \in \{R, P, S, \gamma, \alpha, \beta\}$.

Proof. Let b be a point of B and let H, K be disjoint closed sets of Y . Then $\varphi^{-1}(H), \varphi^{-1}(K)$ are disjoint closed sets of X . Since X is fibrewise i -functionally normal there exist a nbd W of b and a pair of disjoint i -open sets U, V and a continuous function $\lambda : X_W \rightarrow I$ such that $X_W \cap \varphi^{-1}(H) \cap U \subset \lambda^{-1}(0)$ and $X_W \cap \varphi^{-1}(K) \cap V \subset \lambda^{-1}(1)$ in X_W . Now a function $\Omega : Y_W \rightarrow I$ is given by

$$\Omega(y) = \sup_{x \in \varphi^{-1}(y)} \lambda(x); y \in Y_W$$

since φ is i -biopen and closed, as well as continuous, it follows that Ω is continuous. Since $Y_W \cap H \cap \varphi(U) \subset \Omega^{-1}(0)$ and $Y_W \cap K \cap \varphi(V) \subset \Omega^{-1}(1)$ in X_W , where $i \in \{R, P, S, \gamma, \alpha, \beta\}$. This proves the proposition.

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