

# Rough and Near Rough Probability in $G_m$ -Closure Spaces

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**Abstract** – The primary aim of this paper, is to introduce the rough probability from topological view. We used the  $G_m$ -topological spaces which result from the digraph on the stochastic approximation spaces to upper and lower distribution functions, the upper and lower mathematical expectations, the upper and lower variances, the upper and lower standard deviation and the upper and lower  $r^{th}$  moment. Different levels for those concepts are introduced, also we introduced some results based upon those concepts.

**Key words:**  $G_m$ -closure spaces, Rough sets, Near Rough Sets, Rough Probability, Near Rough Probability.

(2000) Math. Subject Classification: 54C05

## 1. Introduction

The theory of rough sets, proposed by Pawlak [15], is an extension of set theory for the study of intelligent systems characterized by insufficient and incomplete information. Using the concepts of lower and upper approximation in rough set theory, knowledge hidden in information systems may be unraveled and expressed in the form of decision rules. The notions of closure operator and closure system are very useful tools in several sections of mathematics. As an example, in algebra [5, 6], topology [7, 9, 10] and computer science theory [16, 20]. The purpose of the present work is to put a starting point for the application of abstract topological graph theory in the rough set analysis by using probability theory. Also, we shall integrate some ideas in terms of concept in topological graph theory. Topological graph theory is a branch of Mathematics, whose concepts exists not only in almost all branches of Mathematics, but also in many real life application. We believe that topological graph structure will be an important base for modification of knowledge extraction and processing.

## 2. Preliminaries

This section presents a review of some fundamental notions of  $G_m$ -closure spaces [2, 17, 18] and Pawlak's approach [14].

### 2.1. $G_m$ -Closure Spaces

In this section, we introduce the concepts of closure operators on digraphs, several known

topological property on the obtained  $G_m$ -closure spaces are studies.

**Definition 2.1.1.** [17, 18] Let  $G = (V(G), E(G))$  be a digraph,  $P(V(G))$  its power set of all subgraphs of  $G$  and  $Cl_G : P(V(G)) \rightarrow P(V(G))$  is a mapping associating with each subgraph  $H = (V(H), E(H))$  a subgraph  $Cl_G(V(H)) \subseteq V(G)$  called the closure subgraph of  $H$  such that:

$$Cl_G(V(H)) = V(H) \cup \{v \in V(G) - V(H) ; \overset{\rightarrow}{h}v \in E(G) \text{ for all } h \in V(H)\}$$

The operation  $Cl_G$  is called graph closure operator and the pair  $(G, \mathcal{F}_G)$  is called  $G$ -closure space, where  $\mathcal{F}_G$  is the family of elements of  $Cl_G$ . The dual of the graph closure operator  $Cl_G$  is the graph interior operator  $Int_G : P(V(G)) \rightarrow P(V(G))$  defined by  $Int_G(V(H)) = V(G) - Cl_G(V(G) - V(H))$  for all subgraph  $H \subseteq G$ . A family of elements of  $Int_G$  is called interior subgraph of  $H$  and denoted by  $\mathcal{I}_G$ . Clear that  $(G, \mathcal{I}_G)$  is a topological space. A subgraph  $H$  of  $G$ -closure space  $(G, \mathcal{I}_G)$  is called closed subgraph if  $Cl_G(V(H)) = V(H)$ . It is called open subgraph if its complement is closed subgraph, i.e.,  $Cl_G(V(G) - V(H)) = V(G) - V(H)$ , or equivalently  $Int_G(V(H)) = V(H)$ .

**Example 2.1.1.** Let  $G = (V(G), E(G))$  be a digraph such that:  $V(G) = \{v_1, v_2, v_3, v_4\}$ ,  $E(G) = \{(v_1, v_2), (v_1, v_3), (v_2, v_1), (v_2, v_3), (v_4, v_3)\}$ .

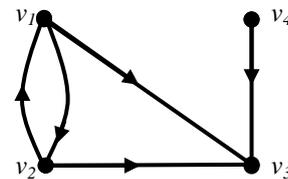


Figure 1: Graph  $G$  given in Example 2.1.1.

Table 2.1.1:  $Cl_G$  for all subgraph  $H \subseteq G$ .

$V(H)$	$Cl_G(V(H))$	$V(H)$	$Cl_G(V(H))$
$V(G)$	$V(G)$	$\{v_1, v_4\}$	$V(G)$
$\emptyset$	$\emptyset$	$\{v_2, v_3\}$	$\{v_1, v_2, v_3\}$
$\{v_1\}$	$\{v_1, v_2, v_3\}$	$\{v_2, v_4\}$	$V(G)$
$\{v_2\}$	$\{v_1, v_2, v_3\}$	$\{v_3, v_4\}$	$\{v_3, v_4\}$
$\{v_3\}$	$\{v_3\}$	$\{v_1, v_2, v_3\}$	$V(G)$
$\{v_4\}$	$\{v_3, v_4\}$	$\{v_1, v_2, v_4\}$	$V(G)$
$\{v_1, v_2\}$	$\{v_1, v_2, v_3\}$	$\{v_1, v_3, v_4\}$	$V(G)$
$\{v_1, v_3\}$	$\{v_1, v_2, v_3\}$	$\{v_2, v_3, v_4\}$	$V(G)$

$\mathcal{F}_G = \{V(G), \emptyset, \{v_3\}, \{v_3, v_4\}, \{v_1, v_2, v_3\}\}$ ,  $\mathcal{I}_G = \{V(G), \emptyset, \{v_4\}, \{v_1, v_2\}, \{v_1, v_2, v_4\}\}$ .

We obtain a new definition to construct topological closure spaces from  $G$ -closure spaces by redefine graph closure operator on the resultant subgraphs as a domain of the graph closure operator and stop when the operator transfers each subgraph to itself.

**Definition 2.1.2.** [17, 18] Let  $G = (V(G), E(G))$  be a digraph and  $Cl_{G_m} : P(V(G)) \rightarrow P(V(G))$  an operator such that:

- (a) It is called  $G_m$ -closure operator if  $Cl_{G_m}(V(H)) = Cl_G(Cl_G(\dots Cl_G(V(H))))$ ,  $m$ -times, for every subgraph  $H \subseteq G$ ,
- (b) it is called  $G_m$ -topological closure operator if  $Cl_{G_m+1}(V(H)) = Cl_{G_m}(V(H))$  for all subgraph  $H \subseteq G$ .

The space  $(G, \mathcal{F}_{G_m})$  is called  $G_m$ -closure space.

**Example 2.1.2.** Let  $G = (V(G), E(G))$  be a digraph such that:

$$V(G) = \{v_1, v_2, v_3, v_4\}, E(G) = \{(v_1, v_3), (v_2, v_1), (v_2, v_3), (v_3, v_4), (v_4, v_1)\}.$$

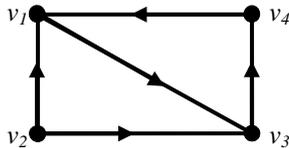


Figure 2: Graph  $G$  given in Example 2.1.2.

Table 2.1.2:  $Cl_G$  and  $Cl_{G_2}$  for all subgraph  $H \subseteq G$ .

$V(H)$	$Cl_G(V(H))$	$Cl_{G_2}(V(H))$
$V(G)$	$V(G)$	$V(G)$
$\phi$	$\phi$	$\phi$
$\{v_1\}$	$\{v_1, v_3\}$	$\{v_1, v_3, v_4\}$
$\{v_2\}$	$\{v_1, v_2, v_3\}$	$V(G)$
$\{v_3\}$	$\{v_3, v_4\}$	$\{v_1, v_3, v_4\}$
$\{v_4\}$	$\{v_1, v_4\}$	$\{v_1, v_3, v_4\}$
$\{v_1, v_2\}$	$\{v_1, v_2, v_3\}$	$V(G)$
$\{v_1, v_3\}$	$\{v_1, v_3, v_4\}$	$\{v_1, v_3, v_4\}$
$\{v_1, v_4\}$	$\{v_1, v_3, v_4\}$	$\{v_1, v_3, v_4\}$
$\{v_2, v_3\}$	$V(G)$	$V(G)$
$\{v_2, v_4\}$	$V(G)$	$V(G)$
$\{v_3, v_4\}$	$\{v_1, v_3, v_4\}$	$\{v_1, v_3, v_4\}$
$\{v_1, v_2, v_3\}$	$V(G)$	$V(G)$
$\{v_1, v_2, v_4\}$	$V(G)$	$V(G)$
$\{v_1, v_3, v_4\}$	$\{v_1, v_3, v_4\}$	$\{v_1, v_3, v_4\}$
$\{v_2, v_3, v_4\}$	$V(G)$	$V(G)$

$$\mathcal{F}_{G_2} = \{V(G), \phi, \{v_1, v_3, v_4\}\}, \mathcal{I}_{G_2} = \{V(G), \phi, \{v_2\}\}.$$

**Proposition 2.1.1.** [17] Let  $(G, \mathcal{F}_{G_m})$  be a  $G_m$ -closure space. If  $H$  and  $K$  are two subgraphs of  $G$  such that  $H \subseteq K \subseteq G$ , then

$$Cl_{G_m}(V(H)) \subseteq Cl_{G_m}(V(K)) \text{ and } Int_{G_m}(V(H)) \subseteq Int_{G_m}(V(K)).$$

**Proposition 2.1.2.** [17] Let  $(G, \mathcal{F}_{G_m})$  be a  $G_m$ -closure space. If  $H$  and  $K$  are two subgraphs of  $G$ , then

- (a)  $Cl_{G_m}(V(H) \cup V(K)) = Cl_{G_m}(V(H)) \cup Cl_{G_m}(V(K))$ .
- (b)  $Int_{G_m}(V(H) \cap V(K)) = Int_{G_m}(V(H)) \cap Int_{G_m}(V(K))$ .
- (c)  $Cl_{G_m}(V(H) \cap V(K)) \subseteq Cl_{G_m}(V(H)) \cap Cl_{G_m}(V(K))$ , and

$$(d) \text{ } Int_{G_m}(V(H)) \cup Int_{G_m}(V(K)) \subseteq Int_{G_m}(V(H) \cup V(K)).$$

**Definition 2.1.3.** [17] Let  $(G, \mathcal{F}_{G_m})$  be a  $G_m$ -closure space and  $H \subseteq G$ , the boundary of  $H$  is denoted by  $Bd_{G_m}(V(H))$  and is defined by

$$Bd_{G_m}(V(H)) = Cl_{G_m}(V(H)) - Int_{G_m}(V(H)).$$

By a similar way of definitions of regular open set [19], semi-open set [11], pre-open set [12],  $\gamma$ -open set [8] (b-open set [3]),  $\alpha$ -open set [13], and  $\beta$ -open set [1] (=semi-pre-open set [4]). We introduce the following definitions which are essential for our present study. In  $G_m$ -closure space  $(G, \mathcal{F}_{G_m})$  the subgraph  $H$  of  $G$  is called:

- (a) Regular open subgraph [17] (briefly  $R$ -osg) if  $V(H) = Int_{G_m}(Cl_{G_m}(V(H)))$ .
- (b) Semi-open subgraph [17] (briefly  $S$ -osg) if  $V(H) \subseteq Cl_{G_m}(Int_{G_m}(V(H)))$ .
- (c) Pre-open subgraph [17] (briefly  $P$ -osg) if  $V(H) \subseteq Int_{G_m}(Cl_{G_m}(V(H)))$ .
- (d)  $\gamma$ -open subgraph (briefly  $\gamma$ -osg) if  $V(H) \subseteq Cl_{G_m}(Int_{G_m}(V(H))) \cup Int_{G_m}(Cl_{G_m}(V(H)))$ .
- (e)  $\alpha$ -open subgraph [17] (briefly  $\alpha$ -osg) if  $V(H) \subseteq Int_{G_m}(Cl_{G_m}(Int_{G_m}(V(H))))$ .
- (f)  $\beta$ -open subgraph [17] (briefly  $\beta$ -osg) if  $V(H) \subseteq Cl_{G_m}(Int_{G_m}(Cl_{G_m}(V(H))))$ .

The complement of an  $R$ -osg (resp.  $S$ -osg,  $P$ -osg,  $\gamma$ -osg,  $\alpha$ -osg and  $\beta$ -osg) is called  $R$ -closed subgraph (briefly  $R$ -csg) (resp.  $S$ -csg,  $P$ -csg,  $\gamma$ -csg,  $\alpha$ -csg and  $\beta$ -csg).

The family of all  $R$ -osgs (resp.  $S$ -osgs,  $P$ -osgs,  $\gamma$ -osgs,  $\alpha$ -osgs and  $\beta$ -osgs) of  $(G, \mathcal{F}_{G_m})$  is denoted by  $RO_{G_m}(G)$  (resp.  $SO_{G_m}(G)$ ,  $PO_{G_m}(G)$ ,  $\gamma O_{G_m}(G)$ ,  $\alpha O_{G_m}(G)$  and  $\beta O_{G_m}(G)$ ). All of  $SO_{G_m}(G)$ ,  $PO_{G_m}(G)$ ,  $\gamma O_{G_m}(G)$ ,  $\alpha O_{G_m}(G)$  and  $\beta O_{G_m}(G)$  are larger than  $\mathcal{F}_{G_m}$  and closed under forming arbitrary union.

The family of all  $R$ -csgs (resp.  $S$ -csgs,  $P$ -csgs,  $\gamma$ -csgs,  $\alpha$ -csgs and  $\beta$ -csgs) of  $(G, \mathcal{F}_{G_m})$  is denoted by  $RC_{G_m}(G)$  (resp.  $SC_{G_m}(G)$ ,  $PC_{G_m}(G)$ ,  $\gamma C_{G_m}(G)$ ,  $\alpha C_{G_m}(G)$  and  $\beta C_{G_m}(G)$ ).

The near closure (resp. near interior and near boundary) of a subgraph  $H$  of  $G$  in a  $G_m$ -closure space  $(G, \mathcal{F}_{G_m})$  is denoted by  $Cl_{G_m}^j(V(H))$  (resp.  $Int_{G_m}^j(V(H))$  and  $Bd_{G_m}^j(V(H))$ ) and defined by

$$Cl_{G_m}^j(V(H)) = \cap \{V(F) ; V(F) \text{ is } j\text{-csg and } V(H) \subseteq V(F)\}.$$

$$\text{(resp. } Int_{G_m}^j(V(H)) = V(G) - Cl_{G_m}^j(V(G) - V(H)) \text{ and}$$

$$Bd_{G_m}^j(V(H)) = Cl_{G_m}^j(V(H)) - Int_{G_m}^j(V(H)) \text{ ) where } j \in \{R, S, P, \gamma, \alpha, \beta\}.$$

**Proposition 2.1.3.** [17] Let  $(G, \mathcal{F}_{G_m})$  be  $G_m$ -closure space, the implication  $\mathcal{F}_{G_m}$  and the families of near

open and near closed graphs are given by following statements.

- (a)  $RO_{G_m}(G) \subseteq \mathcal{T}_{G_m} \subseteq \alpha O_{G_m}(G) \subseteq SO_{G_m}(G) \subseteq \gamma O_{G_m}(G) \subseteq \beta O_{G_m}(G)$ ,
- (b)  $RO_{G_m}(G) \subseteq \mathcal{T}_{G_m} \subseteq \alpha O_{G_m}(G) \subseteq PO_{G_m}(G) \subseteq \gamma O_{G_m}(G) \subseteq \beta O_{G_m}(G)$ ,
- (c)  $RC_{G_m}(G) \subseteq \mathcal{F}_{G_m} \subseteq \alpha C_{G_m}(G) \subseteq SC_{G_m}(G) \subseteq \gamma C_{G_m}(G) \subseteq \beta C_{G_m}(G)$ ,
- (d)  $RC_{G_m}(G) \subseteq \mathcal{F}_{G_m} \subseteq \alpha C_{G_m}(G) \subseteq PC_{G_m}(G) \subseteq \gamma C_{G_m}(G) \subseteq \beta C_{G_m}(G)$ .

**2.2. Pawlak's Approach**

Consider the approximation space  $K = (X, R)$ , where  $X$  is a set called the universe and  $R$  is an equivalence relation. The order triple  $S = (X, R, p)$  is called the stochastic approximation space [14], where  $p$  is a probability measure. Any subset of  $X$  will called an event. The probability measure  $p$  has the following properties:

$p(\phi) = 0, p(X) = 1$  and if  $A = \cup_{i=1}^n A^i$  is an observable set in  $K$ , then

$$p(A) = \sum_{i=1}^n p(A^i).$$

It is clear that  $A$  is a union of disjoint sets, since  $R$  is an equivalence relation. Pawlak introduced the definitions of the lower and upper probabilities of an event  $A$  in the stochastic approximation space  $S = (X, R, p)$ . these definitions are:

- The lower probability of  $A$ , denoted by  $Lp(A)$ , is given by  $Lp(A) = p(L(A))$ .
- The upper probability of  $A$ , denoted by  $Up(A)$ , is given by  $Up(A) = p(U(A))$ .

Clearly,  $0 \leq Lp(A) \leq 1$  and  $0 \leq Up(A) \leq 1$ .

The probability measure  $p$  in the stochastic approximation space  $S = (X, R, p)$  satisfies the following properties [14]:

- (a)  $Lp(A) \leq p(A) \leq Up(A)$ .
- (b)  $Lp(\phi) = Up(\phi) = 0$ .
- (c)  $Lp(X) = Up(X) = 1$ .
- (d)  $Lp(A^c) = 1 - Up(A)$ .
- (e)  $Up(A^c) = 1 - Lp(A)$ .
- (f)  $Up(A \cup B) \leq Up(A) + Up(B) - Up(A \cap B)$ .
- (g)  $Lp(A \cup B) \geq Lp(A) + Lp(B) - Lp(A \cap B)$ .

**Definition 2.2.1.** [14]. Let  $A$  be an event in the stochastic approximation space  $S = (X, R, p)$ . The rough probability of  $A$ , denoted by  $p^*(A)$ , is given by:

$$p^*(A) = \langle Lp(A), Up(A) \rangle.$$

Clearly, the rough probability is the interval to which the probability of the unobservable event belongs. If  $A$  is an observable event in  $S$ , then the rough probability  $p^*(A)$  will be the same as the classical probability  $p(A)$ , that is,  $p^*(A)$  will be reduced to one point.

Moreover,

- If  $A$  is externally unobservable, then

$$p^*(A) = \langle Lp(A), 1 \rangle.$$

- If  $A$  is internally unobservable, then  $p^*(A) = \langle 0, Up(A) \rangle$ .
- If  $A$  is totally unobservable, then  $p^*(A) = \langle 0, 1 \rangle$ .

An exact value of the probability of an event  $A$  is given if it is observable. If  $A$  is roughly observable, a lower and upper values to the probability of  $A$  are given. In the case when the event  $A$  is internally (resp. externally) unobservable, only the upper (resp. lower) bound can be determined. But if  $A$  is totally unobservable, both the lower and upper bounds for the probability of  $A$  can be determined.

**3. Rough Probability in  $G_m$ -Closure Spaces**

In this section we study stochastic approximation spaces from topological view using  $G_m$ -closure spaces. We generalize the stochastic approximation space in the case of general graph. Since the approximation space  $G_m = (G, Cl_{G_m})$  with general graph  $G$  defines a uniquely  $G_m$ -closure space  $(G, \mathcal{F}_{G_m})$ , then the order triple  $S = (G, Cl_{G_m}, p)$  is called the stochastic approximation space, where  $Cl_{G_m}$  is a  $G_m$ -closure operator and  $p$  is the probability measure. We give this hypothesis in the following definition.

**Definition 3.1.** Let  $G_m = (G, Cl_{G_m})$  be an approximation space where  $G$  is a finite and nonempty universe graph,  $Cl_{G_m}$  is a general relation on  $G$ , and  $\mathcal{F}_{G_m}$  is the  $G_m$ -closure space associated to  $G_m$ . Then the order 4-triple  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$  is called a  $G_m$ -closure stochastic approximation space.

The probability measure  $p$  has the following properties:

$$p(\phi) = 0, p(G) = 1 \text{ and if } H = \cup_{i=1}^n H^i \text{ is an observable graph in } G_m, \text{ then}$$

$$p(H) = \sum_{i=1}^n p(H^i) - \sum_{i < j} p(H^i \cap H^j) + \sum_{i < j < k} p(H^i \cap H^j \cap H^k) - \dots \pm p(H^1 \cap \dots \cap H^n)$$

It is clear that  $H$  may be a union of joint graphs, since  $G$  is a general graph.

**3.1. Rough Probability**

There is no problem to find probability of an observable graph as it will be the same as the usual probability. The problem occurs when evaluation the probability of the unobservable graphs. In order to investigate this problem we obtain some rules to find lower and upper probabilities in  $G_m$ -closure stochastic approximation spaces with general graphs.

**Definition 3.2.** Let  $H$  be an event (subgraph) in the  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ . Then the lower (resp. upper) probability of  $H$  is given by:

$$Lp(H) = p(Int_{G_m}(V(H)))$$

$$(resp. Up(H) = p(Cl_{G_m}(V(H))))$$

Clearly,  $0 \leq Lp(H) \leq 1$  and  $0 \leq Up(H) \leq 1$ .

**Proposition 3.1.** Let  $H, K$  be two events in the  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m} p, \mathcal{F}_{G_m})$ . Then

- (a)  $Lp(H) \leq p(H) \leq Up(H)$ .
- (b)  $Lp(\phi) = Up(\phi) = 0$ .
- (c)  $Lp(G) = Up(G) = 1$ .
- (d)  $Lp(H^c) = 1 - Up(H)$ .
- (e)  $Up(H^c) = 1 - Lp(H)$ .
- (f)  $Up(H \cup K) \leq Up(H) + Up(K) - Up(H \cap K)$ .
- (g)  $Lp(H \cup K) \geq Lp(H) + Lp(K) - Lp(H \cap K)$ .

**Proof.** By using the properties of  $G_m$ -interior and  $G_m$ -closure, the proof is obvious.

**Definition 3.3.** Let  $H$  be an event in the  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m} p, \mathcal{F}_{G_m})$ . The rough probability of  $H$ , denoted by  $p^*(H)$ , is given by:

$$p^*(H) = \langle Lp(H), Up(H) \rangle.$$

**Example 3.1.** Consider the experiment of choosing one vertex from five vertices numbered from one to five. The collection of the five vertices form the outcome space. Hence, let  $G = (V(G), E(G))$  be a digraph such that  $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$  and  $E(G) = \{(v_1, v_2), (v_2, v_1), (v_4, v_5)\}$ . Let  $G_m = (G, Cl_m)$  be an approximation space and  $\mathcal{F}_{G_m}$  is the  $G_m$ -closure spaces associated to  $G_m$ . Thus

$$\mathcal{F}_{G1} = \{G, \phi, \{v_3\}, \{v_5\}, \{v_3, v_5\}, \{v_1, v_2\}, \{v_4, v_5\}, \{v_1, v_2, v_5\}, \{v_1, v_2, v_3\}, \{v_3, v_4, v_5\}, \{v_1, v_2, v_3, v_5\}, \{v_1, v_2, v_4, v_5\}\}.$$

$$\mathcal{I}_{G1} = \{G, \phi, \{v_3\}, \{v_4\}, \{v_1, v_2\}, \{v_4, v_5\}, \{v_3, v_4\}, \{v_1, v_2, v_3\}, \{v_3, v_4, v_5\}, \{v_1, v_2, v_4\}, \{v_1, v_2, v_3, v_4\}, \{v_1, v_2, v_4, v_5\}\}.$$

Define the random variable  $V$  to be the number on the chosen vertex. We can construct Table 3.1 which contains the lower and the upper probabilities of a random variable  $V = v$ . It is easy to see the following:

- Neither of the lower and upper probabilities summed to one.
- The value  $v_3$  of  $V$  has exact probability, since  $Lp(V) = Up(V) = 1/5$  at  $V = v_3$ .

Table 3.1: Lower and upper probabilities of a random variable  $V = v$ .

$V$	$v_1$	$v_2$	$v_3$	$v_4$	$v_5$
$Lp(V = v)$	0	0	1/5	1/5	0
$Up(V = v)$	2/5	2/5	1/5	2/5	1/5

If  $H$  is an observable event in  $S_m$ , the rough probability  $p^*(H)$  will be the same as the classical probability  $p(H)$ , that is,  $p^*(H)$  will be reduced to one point.

Moreover,

- If  $H$  is externally unobservable, then  $p^*(H) = \langle Lp(H), 1 \rangle$ .
- If  $H$  is internally unobservable, then  $p^*(H) = \langle 0, Up(H) \rangle$ .
- If  $H$  is totally unobservable, then  $p^*(H) = \langle 0, 1 \rangle$ .

An exact value of the probability of an event  $H$  is given if it is observable. If  $H$  is roughly observable, the lower and the upper values to the probability of  $H$  are given. In the case when the event  $H$  is internally (resp. externally) unobservable, only the upper (resp. lower) bound can be determined. But if  $H$  is totally unobservable, both the lower and upper bounds for the probability of  $H$  can be determined.

### 3.2. Rough Distribution Function

The distribution function of a random variable  $V$  gives the probability that  $V$  does not exceed  $v$ . We define the lower and upper distribution functions of a random variable  $V$ .

**Definition 3.4.** Let  $V$  be a random variable in the  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m} p, \mathcal{F}_{G_m})$ . The lower (resp. upper) distribution function of  $V$  is given by:

$$LF(v) = Lp(V \leq v) \text{ (resp. } UF(v) = Up(V \leq v) \text{)}.$$

**Definition 3.5.** Let  $V$  be a random variable in the  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m} p, \mathcal{F}_{G_m})$ . The rough distribution of  $V$ , denoted by  $F^*(v)$ , is given by:

$$F^*(v) = \langle LF(v), UF(v) \rangle.$$

**Example 3.2.** Consider the same experiment as in Example 3.1. The lower and upper distribution functions of  $V$  are

$$LF(v) = \begin{cases} 0 & -\infty < v < 3, \\ 1/5 & 3 \leq v < 4, \text{ and} \\ 2/5 & 4 \leq v < \infty \end{cases}$$

$$UF(v) = \begin{cases} 0 & -\infty < v < 1 \\ 2/5 & 1 \leq v < 2 \\ 4/5 & 2 \leq v < 3 \\ 1 & 3 \leq v < 4, \\ 7/5 & 4 \leq v < 5, \\ 8/5 & 5 \leq v < \infty \end{cases}$$

Therefore  $F^*(v_4) = \langle 2/5, 7/5 \rangle$ .

**Proposition 3.2.** Let  $V$  be a random variable in the  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m} p, \mathcal{F}_{G_m})$ . Then

$$LF(v) \leq F(v) \leq UF(v).$$

**Proof.** By using part (a) in Proposition 3.1, the proof is obvious.

### 3.3. Rough Expectation

The expectation of a random variable  $V$  is the average of all possible values of  $V$  weighted by their probabilities. We define the lower and upper expectations of a random variable  $V$ .

**Definition 3.6.** Let  $V$  be a random variable in the  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ . The lower (resp. upper) expectation of  $V$  is given by:

$$L\mu = LE(V) = \sum_{k=1}^n v_k Lp(V = v_k)$$

(resp.  $U\mu = UE(V) = \sum_{k=1}^n v_k Up(V = v_k)$ ).

**Definition 3.7.** Let  $V$  be a random variable in the  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ . The rough expectation of  $V$  is denoted by  $E^*(V)$  and is given by:

$$E^*(V) = \langle LE(V), UE(V) \rangle.$$

The rough expectation of  $V$  also denoted by  $\mu^* = \langle L\mu, U\mu \rangle$ .

**Example 3.3.** Consider the same experiment as in Example 3.1. Then the lower and upper expectations of  $V$  are

$$L\mu = LE(V) = 1.4, U\mu = UE(V) = 4.4.$$

Hence the rough mean (or rough expectation) of  $V$  is  $\mu^* = \langle 1.4, 4.4 \rangle$ .

### 3.4. Rough Variance and Rough Standard Deviation

We define the lower and upper variances of random variable  $V$ .

**Definition 3.8.** Let  $V$  be a random variable in the  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ . The lower (resp. upper) variance of  $V$  is given by:

$$LV(V) = LE(V - L\mu)^2$$

(resp.  $UV(V) = UE(V - U\mu)^2$ ).

**Definition 3.9.** Let  $V$  be a random variable in the  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ . The rough variance of  $V$  is denoted by  $V^*(V)$  and is given by:

$$V^*(V) = \langle LV(V), UV(V) \rangle.$$

**Definition 3.10.** Let  $V$  be a random variable in the  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ . The lower (resp. upper) standard deviation of  $V$  is given by:

$$L\sigma(V) = \sqrt{LV(V)} \text{ (resp. } U\sigma(V) = \sqrt{UV(V)} \text{ )}.$$

**Definition 3.11.** Let  $V$  be a random variable in the  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ . The rough standard deviation of  $V$  is denoted by  $\sigma^*(V)$  and is given by:

$$\sigma^*(V) = \langle L\sigma(V), U\sigma(V) \rangle.$$

**Example 3.4.** Consider the same experiment as in Example 3.1. Then the lower and upper variances of  $V$  are

$$LV(V) = 1.864, UV(V) = 7.456.$$

The rough variance of  $V$  is  $V^*(V) = \langle 1.864, 7.456 \rangle$ .

Finally, the rough standard deviation of  $V$  is  $\sigma^*(V) = \langle 1.365, 2.731 \rangle$ .

### 3.5. Rough Moments

We shall define the lower and upper moments of random variable  $V$ .

**Definition 3.12.** Let  $V$  be a random variable in the  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ . The lower (resp. upper)  $r^{\text{th}}$  moment of  $V$  about the lower mean  $L\mu$  (resp. upper mean  $U\mu$ ), also called the lower (resp. upper)  $r^{\text{th}}$  central moment, is defined as:

$$L_r\mu = LE(V - L\mu)^r = \sum_{k=1}^n (v_k - L\mu)^r Lp(V = v_k)$$

(resp.  $U_r\mu = UE(V - U\mu)^r = \sum_{k=1}^n (v_k - U\mu)^r Up(V = v_k)$ ) where  $r = 0, 1, 2, \dots$

The  $r^{\text{th}}$  lower (resp. upper) moment of  $V$  about origin is defined as

$$L_r\mu' = LE(V^r) \text{ (resp. } U_r\mu' = UE(V^r) \text{ ) where } r = 0, 1, 2, \dots$$

**Definition 3.13.** Let  $V$  be a random variable in the  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ . The rough  $r^{\text{th}}$  moment of  $V$  is denoted by  $\mu_r^*$  and is defined by:

$$\mu_r^*(V) = \langle L_r\mu, U_r\mu \rangle.$$

We shall introduce the definition of the moment generating function of a random variable  $V$ .

**Definition 3.14.** Let  $V$  be a random variable in the  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ . The lower (resp. upper) moment generating function of  $V$  is defined by:

$$LM_V(t) = LE(e^{tV}) = \sum_{k=1}^n e^{tv_k} Lp(V = v_k)$$

(resp.  $UM_V(t) = UE(e^{tV}) = \sum_{k=1}^n e^{tv_k} Up(V = v_k)$ )

**Definition 3.15.** Let  $V$  be a random variable in the  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ . The rough moment generating function of  $V$  is denoted by  $M_V^*(t)$  and is defined by:

$$M_V^*(t) = \langle LM_V(t), UM_V(t) \rangle.$$

**Example 3.5.** Consider the same experiment as in Example 3.1. From Table 3.1 it is easy to see the following:

- The lower  $r^{\text{th}}$  moment of  $V$  about the lower mean  $L\mu$  is

$$\begin{aligned}
 L_r\mu &= LE(V - L\mu)^r = \sum_{k=1}^5 (v_k - L\mu)^r Lp(V = v_k) \\
 &= 0 \left(1 - \frac{7}{5}\right)^r + 0 \left(2 - \frac{7}{5}\right)^r + \frac{1}{5} \left(3 - \frac{7}{5}\right)^r \\
 &+ \frac{1}{5} \left(4 - \frac{7}{5}\right)^r + 0 \left(5 - \frac{7}{5}\right)^r \\
 &= \left[ \frac{(8)^r + (13)^r}{(5)^{r+1}} \right], \text{ where } r = 0, 1, 2, \dots
 \end{aligned}$$

- The upper  $r^{th}$  moment of  $V$  about the upper mean  $U\mu$  is

$$\begin{aligned}
 U_r\mu &= UE(V - U\mu)^r = \sum_{k=1}^5 (v_k - U\mu)^r Up(V = v_k) \\
 &= \frac{2}{5} \left(1 - \frac{22}{5}\right)^r + \frac{2}{5} \left(2 - \frac{22}{5}\right)^r + \frac{1}{5} \left(3 - \frac{22}{5}\right)^r \\
 &+ \frac{22}{5} \left(4 - \frac{22}{5}\right)^r + \frac{1}{5} \left(5 - \frac{22}{5}\right)^r \\
 &= \left[ \frac{2[(-17)^r + (-12)^r + (-2)^r] + (-7)^r + (3)^r}{(5)^{r+1}} \right],
 \end{aligned}$$

where  $r = 0, 1, 2, \dots$

- The lower  $r^{th}$  moment of  $V$  about origin is

$$\begin{aligned}
 L_r\mu' &= LE(V^r) = 0 (1)^r + 0 (2)^r + \frac{1}{5} (3)^r + \frac{1}{5} (4)^r \\
 &+ 0 (5)^r \\
 &= \frac{1}{5} [(3)^r + (4)^r], \text{ where } r = 0, 1, 2, \dots
 \end{aligned}$$

- The upper  $r^{th}$  moment of  $V$  about origin is

$$\begin{aligned}
 U_r\mu' &= UE(V^r) = \frac{2}{5} (1)^r + \frac{2}{5} (2)^r + \frac{1}{5} (3)^r \\
 &+ \frac{2}{5} (4)^r + \frac{1}{5} (5)^r \\
 &= \frac{1}{5} [2(1 + (2)^r + (4)^r) + (3)^r + (5)^r],
 \end{aligned}$$

where  $r = 0, 1, 2, \dots$

- The lower moment generating function of  $V$  is

$$\begin{aligned}
 LM_V(t) &= LE(e^{tV}) = \sum_{k=1}^5 e^{tv_k} Lp(V = v_k) \\
 &= 0 e^t + 0 e^{2t} + \frac{1}{5} e^{3t} + \frac{1}{5} e^{4t} + 0 e^{5t} \\
 &= \frac{1}{5} [e^{3t} + e^{4t}].
 \end{aligned}$$

- The upper moment generating function of  $V$  is

$$\begin{aligned}
 UM_V(t) &= UE(e^{tV}) = \sum_{k=1}^5 e^{tv_k} Up(V = v_k) \\
 &= \frac{2}{5} e^t + \frac{2}{5} e^{2t} + \frac{1}{5} e^{3t} + \frac{2}{5} e^{4t} + \frac{1}{5} e^{5t} \\
 &= \frac{1}{5} [2(e^t + e^{2t} + e^{4t}) + e^{3t} + e^{5t}].
 \end{aligned}$$

#### 4. Near Rough Probability in $G_m$ -Closure Spaces

In this section, we introduce the near rough (briefly  $j$ -rough) probability for all  $j \in \{R, S, P, \gamma, \alpha, \beta\}$ .

##### 4.1. Near Rough Probability

We obtain some rules to find  $j$ -lower and  $j$ -upper probabilities in  $G_m$ -closure stochastic approximation spaces for all  $j \in \{R, S, P, \gamma, \alpha, \beta\}$ .

**Definition 4.1.** Let  $H$  be an event in the  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ . The  $j$ -lower (resp.  $j$ -upper) probability of  $H$  for all  $j \in \{R, S, P, \gamma, \alpha, \beta\}$  is given by:

$$\begin{aligned}
 L_j p(H) &= p(Int_{G_m}^j(V(H))) \\
 (\text{resp. } U_j p(H) &= p(Cl_{G_m}^j(V(H))).
 \end{aligned}$$

Clearly,  $0 \leq L_j p(H) \leq 1$ ,  $0 \leq U_j p(H) \leq 1$  for all  $j \in \{R, S, P, \gamma, \alpha, \beta\}$ .

**Proposition 4.1.** Let  $H$  be an event in the  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ . Then

- $L_j p(H) \leq p(H) \leq U_j p(H)$ ,
  - $L_j p(\phi) = U_j p(\phi) = 0$ ,
  - $L_j p(G) = U_j p(G) = 1$ ,
  - $L_j p(H^c) = 1 - U_j p(H)$ ,
  - $U_j p(H^c) = 1 - L_j p(H)$ ,
- for all  $j \in \{R, S, P, \gamma, \alpha, \beta\}$ .

**Proof.** By using the properties of  $G_{G_m}^j$ -interior and  $G_{G_m}^j$ -closure for all  $j \in \{R, S, P, \gamma, \alpha, \beta\}$ , the proof is obvious.

In general, part (f) and (g) in Proposition 3.1 do not satisfy in the case of  $j$ -rough probability for all  $j \in \{R, S, P, \gamma, \beta\}$ . Example 4.1 (resp. Example 4.2) illustrates that part (f) (resp. part (g)) in Proposition 3.1 does not satisfy in the case of  $P$ -rough (resp.  $\beta$ -rough) probability.

**Example 4.1.** Consider the same experiment as in Example 2.1.1.

If  $H = (V(H), E(H)); V(H) = \{v_1\}, E(H) = \phi$ , and  $K = (V(K), E(K)); V(K) = \{v_2\}, E(H) = \phi$ . Then  $U_p p(H \cup K) = 3/4$ ,  $U_p p(H) + U_p p(K) - U_p p(H \cap K) = 1/4 + 1/4 - 0 = 2/4$ . Thus  $U_p p(H \cup K) > U_p p(H) + U_p p(K) - U_p p(H \cap K)$ .

**Example 4.2.** Consider the same experiment as in Example 2.1.1.

If  $H = (V(H), E(H)); V(H) = \{v_2, v_3, v_4\}, E(H) = \{(v_2, v_3), (v_4, v_3)\}$ , and  $K = (V(K), E(K)); V(K) = \{v_1, v_3\}, E(H) = \{(v_1, v_3)\}$ . Then  $L_{\beta} p(H \cup K) = 1$ ,  $L_{\beta} p(H) + L_{\beta} p(K) - L_{\beta} p(H \cap K) = 3/4 + 2/4 - 0 = 5/4$ . Thus  $L_{\beta} p(H \cup K) < L_{\beta} p(H) + L_{\beta} p(K) - L_{\beta} p(H \cap K)$ .

**Definition 4.2.** Let  $H$  be an event in the  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ . The  $j$ -rough probability of  $H$  for all  $j \in \{R, S, P, \gamma, \alpha, \beta\}$ , denoted by  $p_j^*(H)$  and is given by:

$$p_j^*(H) = \langle L_j p(H), U_j p(H) \rangle.$$

If  $H$  is an  $j$ -observable event in  $S_m$ , the  $j$ -rough probability  $p_j^*(H)$  for all  $j \in \{R, S, P, \gamma, \alpha, \beta\}$  will be the same as the classical probability  $p(H)$ , that is,  $p_j^*(H)$  will be reduced to one point.

Moreover,

- If  $H$  is externally  $j$ -unobservable, then  $p_j^*(H) = \langle L_j p(H), 1 \rangle$ .
- If  $H$  is internally  $j$ -unobservable, then  $p_j^*(H) = \langle 0, U_j p(H) \rangle$ .
- If  $H$  is totally  $j$ -unobservable, then  $p_j^*(H) = \langle 0, 1 \rangle$ .

For all  $j \in \{R, S, P, \gamma, \alpha, \beta\}$ , the  $j$ -exact value of the probability of event  $H$  is given if it is  $j$ -observable. If  $H$  is roughly  $j$ -observable, the  $j$ -lower and the  $j$ -upper values to the probability of  $H$  are given. In the case when the event  $H$  is internally (resp. externally)  $j$ -unobservable, only the  $j$ -upper (resp.  $j$ -lower) bound can be determined. But if  $H$  is totally  $j$ -unobservable both the  $j$ -lower and  $j$ -upper bounds for the probability of  $H$  can be determined

**Proposition 4.2.** Let  $H$  be an event in the  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ . Then

$$Lp(H) \leq L_j p(H) \leq U_j p(H) \leq Up(H) \text{ for all } j \in \{S, P, \gamma, \alpha, \beta\}.$$

**Proof.** By using properties of  $G_m$ -interior,  $G_m^j$ -interior,  $G_m$ -closure and  $G_m^j$ -closure for all  $j \in \{S, P, \gamma, \alpha, \beta\}$ , the proof is obvious.

In general, the above Proposition need not be true in the case of  $j = R$  as illustrated in the following example.

**Example 4.3.** Consider the same experiment as in Example 2.1.2.

If  $H = (V(H), E(H)); V(H) = \{v_1, v_3, v_4\}, E(H) = \{(v_1, v_3), (v_3, v_4), (v_4, v_1)\}$ , and  $RO_{G_2}(G) = \{G, \phi\} = RC_{G_2}(G)$ . Then  $Lp(H) = 0, Up(H) = 3/4, L_{Rp}(H) = 0$  and  $U_{Rp}(H) = 1$ .

Therefore,  $L_{Rp}(H) = Lp(H)$  and  $Up(H) < U_{Rp}(H)$ .

**Proposition 4.3.** Let  $H$  be an event in the  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ . The implications between the lower probability and  $j$ -lower probability of  $H$  for all  $j \in \{S, P, \gamma, \alpha, \beta\}$  are given as follows:

- (a)  $Lp(H) \leq L_{\alpha p}(H) \leq L_{\beta p}(H) \leq L_{\gamma p}(H) \leq L_{\delta p}(H)$ ,
- (b)  $L_{\alpha p}(H) \leq L_{\beta p}(H) \leq L_{\gamma p}(H)$ .

**Proof.** By using Proposition 4.2 and Proposition 4.3 in [2], the proof is obvious.

**Proposition 4.4.** Let  $H$  be an event in the  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ . The implication between the upper probability and  $j$ -upper probability of  $H$  for all  $j \in \{S, P, \gamma, \alpha, \beta\}$  are given as follows:

- (a)  $U_{\beta p}(H) \leq U_{\gamma p}(H) \leq U_{\delta p}(H) \leq U_{\alpha p}(H) \leq Up(H)$ ,
- (b)  $U_{\gamma p}(H) \leq U_{\beta p}(H) \leq U_{\alpha p}(H)$ .

**Proof.** By using Proposition 4.2 and Proposition 4.3 in [2], the proof is obvious.

Figure 3 (resp. Figure 4) illustrates  $p(H), [Lp(H), Up(H)]$  and  $[L_j p(H), U_j p(H)]$  for a subgraph  $H$  of  $G$  for all  $j \in \{S, \gamma, \alpha, \beta\}$  (resp.  $j \in \{P, \gamma, \alpha, \beta\}$ ) in a  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ .

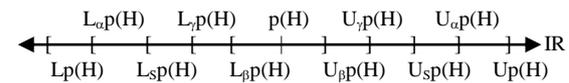


Figure 3:  $p(H), [Lp(H), Up(H)]$  and  $[L_j p(H), U_j p(H)]$  for a subgraph  $H$  of  $G$  for all  $j \in \{S, \gamma, \alpha, \beta\}$  in a  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ .

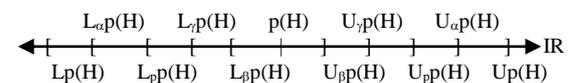


Figure 4:  $p(H), [Lp(H), Up(H)]$  and  $[L_j p(H), U_j p(H)]$  for a subgraph  $H$  of  $G$  for all  $j \in \{P, \gamma, \alpha, \beta\}$  in a  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ .

#### 4.2. Near Rough Distribution Function

In this section, we introduce the concept of near rough (briefly  $j$ -rough) distribution function of a random variable  $V$  for all  $j \in \{R, S, P, \gamma, \alpha, \beta\}$ . In the following definition, we define the  $j$ -lower and the  $j$ -upper distribution functions of a random variable  $V$  for all  $j \in \{R, S, P, \gamma, \alpha, \beta\}$ .

**Definition 4.3.** Let  $V$  be a random variable in the  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ . The  $j$ -lower (resp.  $j$ -upper) distribution function of  $V$  for all  $j \in \{R, S, P, \gamma, \alpha, \beta\}$  is given by:

$$L_j F(v) = L_j p(V \leq v) \text{ (resp. } U_j F(v) = U_j p(V \leq v) \text{)}.$$

**Definition 4.4.** Let  $V$  be a random variable in the  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ . The  $j$ -rough distribution of  $V$  for all  $j \in \{R, S, P, \gamma, \alpha, \beta\}$  is denoted by  $F_j^*(v)$  and is given by:

$$F_j^*(v) = \langle L_j F(v), U_j F(v) \rangle.$$

**Example 4.4.** Consider the same experiment as in example 3.1. Then

$RO_{G_1}(G) = \{V(G), \phi, \{v_3\}, \{v_1, v_2\}, \{v_4, v_5\}, \{v_1, v_2, v_3\}, \{v_3, v_4, v_5\}, \{v_1, v_2, v_4, v_5\}\}$ .

$RC_{G_1}(G) = \{V(G), \phi, \{v_1, v_2, v_4, v_5\}, \{v_3, v_4, v_5\}, \{v_1, v_2, v_3\}, \{v_4, v_5\}, \{v_1, v_2\}, \{v_3\}\}$ .

$PO_{G_1}(G) = \{V(G), \phi, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}, \{v_1, v_4, v_5\},$

$\{v_2, v_3, v_4\}, \{v_2, v_4, v_5\}, \{v_3, v_4, v_5\}, \{v_1, v_2, v_3, v_4\},$   
 $\{v_1, v_2, v_4, v_5\}, \{v_1, v_3, v_4, v_5\}, \{v_2, v_3, v_4, v_5\}\}.$   
 $PC_{G_1}(G) = \{V(G), \phi, \{v_2, v_3, v_4, v_5\}, \{v_1, v_3, v_4, v_5\},$   
 $\{v_1, v_2, v_4, v_5\}, \{v_1, v_2, v_3, v_5\}, \{v_3, v_4, v_5\}, \{v_2, v_4,$   
 $v_5\}, \{v_2, v_3, v_5\}, \{v_1, v_4, v_5\}, \{v_1, v_3, v_5\}, \{v_1, v_2, v_5\},$   
 $\{v_4, v_5\}, \{v_3, v_5\}, \{v_2, v_5\}, \{v_2, v_3\}, \{v_1, v_5\}, \{v_1, v_3\},$   
 $\{v_1, v_2\}, \{v_5\}, \{v_3\}, \{v_2\}, \{v_1\}\}.$

Define the random variable  $V$  to be the number on the chosen vertex. We can construct Table 4.1 which contains the  $j$ -lower and the  $j$ -upper probabilities of a random variable  $V = v$  for  $j \in \{R, P\}$ .

Table 4.1:  $j$ -Lower and  $j$ -upper probabilities of a random variable  $V = v$ , where  $j \in \{R, P\}$ .

$V$	$v_1$	$v_2$	$v_3$	$v_4$	$v_5$
$L_{RP}(V = v)$	0	0	1/5	0	0
$U_{RP}(V = v)$	2/5	2/5	1/5	2/5	2/5
$L_{PP}(V = v)$	1/5	1/5	1/5	1/5	0
$U_{PP}(V = v)$	1/5	1/5	1/5	2/5	1/5

Then

- The  $R$ -lower and  $R$ -upper distribution function of  $V$  are

$$L_R F(v) = \begin{cases} 0 & -\infty < v < 3, \\ 1/5 & 3 \leq v < \infty. \end{cases} \text{ and}$$

$$U_R F(v) = \begin{cases} 0 & -\infty < v < 1, \\ 2/5 & 1 \leq v < 2, \\ 4/5 & 2 \leq v < 3, \\ 1 & 3 \leq v < 4, \\ 7/5 & 4 \leq v < 5, \\ 9/5 & 5 \leq v < \infty. \end{cases}$$

- The  $P$ -lower and  $P$ -upper distribution function of  $V$  are

$$L_P F(v) = \begin{cases} 0 & -\infty < v < 1, \\ 1/5 & 1 \leq v < 2, \\ 2/5 & 2 \leq v < 3, \\ 3/5 & 3 \leq v < 4, \\ 4/5 & 4 \leq v < \infty. \end{cases} \text{ and}$$

$$U_P F(v) = \begin{cases} 0 & -\infty < v < 1, \\ 1/5 & 1 \leq v < 2, \\ 2/5 & 2 \leq v < 3, \\ 3/5 & 3 \leq v < 4, \\ 1 & 4 \leq v < 5, \\ 6/5 & 5 \leq v < \infty. \end{cases}$$

**Proposition 4.5.** Let  $V$  be a random variable in the  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ . Then

$$L F(V) \leq L_j F(V) \leq U_j F(V) \leq U F(V)$$

for all  $j \in \{S, P, \gamma, \alpha, \beta\}$ .

**Proof.** By using Proposition 4.2, the proof is obvious.

In general, the above proposition need not be true in the case of  $j = R$  as illustrated in the following example.

**Example 4.5.** In Example 4.4, we get

$$L F(v_5) = 2/5, U F(v_5) = 8/5, L_R F(v_5) = 1/5 \text{ and}$$

$$U_R F(v_5) = 9/5.$$

Therefore  $L_R F(v_5) < L F(v_5)$  and  $U F(v_5) < U_R F(v_5)$ .

**Proposition 4.6.** Let  $V$  be a random variable in the  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ . The implications between the lower distribution function and  $j$ -lower distribution function of  $V$  for all  $j \in \{S, P, \gamma, \alpha, \beta\}$  are given as follows:

- (a)  $L F(v) \leq L_\alpha F(v) \leq L_S F(v) \leq L_\gamma F(v) \leq L_\beta F(v)$ ,
- (b)  $L_\alpha F(v) \leq L_P F(v) \leq L_\gamma F(v)$ .

**Proof.** By using Proposition 4.3 and Proposition 4.5, the proof is obvious.

**Proposition 4.7.** Let  $V$  be a random variable in the  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ . The implications between the upper distribution function and  $j$ -upper distribution function of  $V$  for all  $j \in \{S, P, \gamma, \alpha, \beta\}$  are given as follows:

- (a)  $U_\beta F(v) \leq U_\gamma F(v) \leq U_S F(v) \leq U_\alpha F(v) \leq U F(v)$ ,
- (b)  $U_\gamma F(v) \leq U_P F(v) \leq U_\alpha F(v)$ .

**Proof.** By using Proposition 4.4 and Proposition 4.5, the proof is obvious.

Figure 5 (resp. Figure 6) illustrates  $F(v)$ ,  $[L F(v), U F(v)]$  and  $[L_j F(v), U_j F(v)]$  for a random variable  $V$  for all  $j \in \{S, \gamma, \alpha, \beta\}$  (resp.  $j \in \{P, \gamma, \alpha, \beta\}$ ) in a  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ .

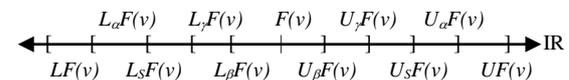


Figure 5:  $F(v)$ ,  $[L F(v), U F(v)]$  and  $[L_j F(v), U_j F(v)]$  for a random variable  $V$  for all  $j \in \{S, \gamma, \alpha, \beta\}$  in a  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ .

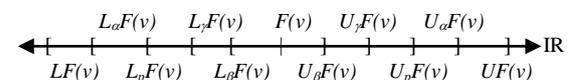


Figure 6:  $F(v)$ ,  $[L F(v), U F(v)]$  and  $[L_j F(v), U_j F(v)]$  for a random variable  $V$  for all  $j \in \{P, \gamma, \alpha, \beta\}$  in a  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ .

### 4.3. Near Rough Expectation

In this section, we introduce the near rough (briefly  $j$ -rough) expectation of a random variable  $V$  for all  $j \in \{R, S, P, \gamma, \alpha, \beta\}$ . We define the  $j$ -lower and the  $j$ -upper expectations of a random variable  $V$  for all  $j \in \{R, S, P, \gamma, \alpha, \beta\}$  in the following definition.

**Definition 4.5.** Let  $V$  be a random variable in the  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ . The  $j$ -lower (resp.  $j$ -upper) expectation of  $V$  for all  $j \in \{R, S, P, \gamma, \alpha, \beta\}$  is given by:

$$L_j\mu = L_jE(V) = \sum_{k=1}^n v_k L_jp(V = v_k)$$

(resp.  $U_j\mu = U_jE(V) = \sum_{k=1}^n v_k U_jp(V = v_k)$ ).

**Definition 4.6.** Let  $V$  be a random variable in the  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ . The  $j$ -rough expectation of  $V$  for all  $j \in \{R, S, P, \gamma, \alpha, \beta\}$  is denoted by  $E_j^*(V)$  and is given by:

$$E_j^*(V) = \langle L_jE(V), U_jE(V) \rangle.$$

The  $j$ -rough expectation of  $V$  also denoted by  $\mu_j^* = \langle L_j\mu, U_j\mu \rangle$  for all  $j \in \{R, S, P, \gamma, \alpha, \beta\}$ .

**Example 4.5.** Consider the same experiment as in Example 3.1. From Table 4.1, it is easy to see the following:

- Neither of the  $j$ -lower and the  $j$ -upper probabilities summed to one for  $j \in \{R, P\}$ .
- The value  $v_3$  of  $V$  has  $R$ -exact probability, since  $L_Rp(V) = U_Rp(V) = 1/5$  at  $V = v_3$ .
- The values  $v_1, v_2$  and  $v_3$  of  $V$  has  $P$ -exact probability, since

$$L_Pp(V) = U_Pp(V) = 1/5 \text{ at } V = v_1, v_2, v_3.$$

For  $j = R$  we get,

- The  $R$ -lower and  $R$ -upper expectation of  $V$  are  $L_R\mu = L_RE(V) = 0.6, U_R\mu = U_RE(V) = 5.4$ .
- The  $R$ -rough mean (or  $R$ -rough expectation) of  $V$  is

$$\mu_R^* = \langle 0.6, 5.4 \rangle.$$

For  $j = P$  we get,

- The  $P$ -lower and  $P$ -upper expectation of  $V$  are  $L_P\mu = L_PE(V) = 2, U_P\mu = U_PE(V) = 3.8$ .
- The  $P$ -rough mean (or  $P$ -rough expectation) of  $V$  is

$$\mu_P^* = \langle 2, 3.8 \rangle.$$

#### 4.4. Near Rough Variance and Near Rough Standard Deviation

In this section, we define the near rough (briefly  $j$ -rough) variance and the near rough (briefly  $j$ -rough) standard deviation of a random variable  $V$  for all  $j \in \{R, S, P, \gamma, \alpha, \beta\}$ .

**Definition 4.7.** Let  $V$  be a random variable in the  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ . The  $j$ -lower (resp.  $j$ -upper) variance of  $V$  for all  $j \in \{R, S, P, \gamma, \alpha, \beta\}$  is given by:

$$L_jV(V) = L_jE(V - L_j\mu)^2$$

(resp.  $U_jV(V) = U_jE(V - U_j\mu)^2$ ).

**Definition 4.8.** Let  $V$  be a random variable in the  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ . The  $j$ -rough variance of  $V$  for all

$j \in \{R, S, P, \gamma, \alpha, \beta\}$  is denoted by  $V_j^*(V)$  and is given by:

$$V_j^*(V) = \langle L_jV(V), U_jV(V) \rangle.$$

**Definition 4.9.** Let  $V$  be a random variable in the  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ . The  $j$ -lower (resp.  $j$ -upper) standard deviation of  $V$  for all  $j \in \{R, S, P, \gamma, \alpha, \beta\}$  is given by:

$$L_j\sigma(V) = \sqrt{L_jV(V)} \text{ (resp. } U_j\sigma(V) = \sqrt{U_jV(V)} \text{ )}.$$

**Definition 4.10.** Let  $V$  be a random variable in the  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ . The  $j$ -rough standard deviation of  $V$  for all  $j \in \{R, S, P, \gamma, \alpha, \beta\}$  is denoted by  $\sigma_j^*(V)$  and is given by:

$$\sigma_j^*(V) = \langle L_j\sigma(V), U_j\sigma(V) \rangle.$$

**Example 4.6.** Consider the same experiment as in Example 3.1. From Table 4.1, it is easy to see the following:

For  $j = R$  we get,

- The  $R$ -lower and  $R$ -upper variances of  $V$  are  $L_RV(V) = 1.152, U_RV(V) = 14.368$ .
- The  $R$ -rough variance of  $V$  is  $V_R^*(V) = \langle 1.152, 14.368 \rangle$ .
- The  $R$ -rough standard deviation of  $V$  is  $\sigma_R^*(V) = \langle 1.073, 3.791 \rangle$ .

For  $j = P$  we get,

- The  $P$ -lower and  $P$ -upper variances of  $V$  are  $L_PV(V) = 1.2, U_PV(V) = 2.648$ .
- The  $P$ -rough variance of  $V$  is  $V_P^*(V) = \langle 1.2, 2.648 \rangle$ .
- The  $P$ -rough standard deviation of  $V$  is  $\sigma_P^*(V) = \langle 1.095, 1.627 \rangle$ .

#### 4.5. Near Rough Moments

In this section we shall define the near rough (briefly  $j$ -rough) moment of random variable  $V$  for all  $j \in \{R, S, P, \gamma, \alpha, \beta\}$ .

**Definition 4.11.** Let  $V$  be a random variable in the  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ . The  $j$ -lower (resp.  $j$ -upper)  $r^{\text{th}}$  moment of  $V$  about the  $j$ -lower mean  $L_j\mu$  (resp.  $j$ -upper mean  $U_j\mu$ ) for all  $j \in \{R, S, P, \gamma, \alpha, \beta\}$ , also called the lower (resp. upper)  $r^{\text{th}}$  central moment, is defined as:

$$L_{jr}\mu = L_jE(V - L_j\mu)^r = \sum_{k=1}^n (v_k - L_j\mu)^r L_jp(V = v_k)$$

(resp.  $U_{jr}\mu = U_jE(V - U_j\mu)^r = \sum_{k=1}^n (v_k - U_j\mu)^r U_jp(V = v_k)$ ) where  $r = 0, 1, 2, \dots$

The  $r^{\text{th}}$   $j$ -lower (resp.  $j$ -upper) moment of  $V$  about origin is defined as

$$L_{jr}\mu' = L_jE(V^r) \text{ (resp. } U_{jr}\mu' = U_jE(V^r) \text{ ) where } r = 0, 1, 2, \dots$$

**Definition 4.12.** Let  $V$  be a random variable in the  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ . The  $j$ -rough  $r^{th}$  moment of  $V$  for all  $j \in \{R, S, P, \gamma, \alpha, \beta\}$  is denoted by  ${}_{r,j}\mu^*$  and is defined by:

$${}_{j,r}\mu^*(V) = \langle L_{j,r}\mu, U_{j,r}\mu \rangle.$$

In the following definition we shall define the  $j$ -lower and  $j$ -upper moment generating function of a random variable  $V$  for all  $j \in \{R, S, P, \gamma, \alpha, \beta\}$ .

**Definition 4.13.** Let  $V$  be a random variable in the  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ . The  $j$ -lower (resp.  $j$ -upper) moment generating function of  $V$  for all  $j \in \{R, S, P, \gamma, \alpha, \beta\}$  is defined by:

$$L_j M_V(t) = L_j E(e^{tV}) = \sum_{k=1}^n e^{vk} L_j p(V = v_k) \\ \text{(resp. } U_j M_V(t) = U_j E(e^{tV}) = \sum_{k=1}^n e^{vk} U_j p(V = v_k) \text{)}$$

**Definition 4.14.** Let  $V$  be a random variable in the  $G_m$ -closure stochastic approximation space  $S_m = (G, Cl_{G_m}, p, \mathcal{F}_{G_m})$ . The  $j$ -rough moment generating function of  $V$  for all  $j \in \{R, S, P, \gamma, \alpha, \beta\}$  is denoted by  $M_{jV}^*(t)$  and is defined by:

$$M_{jV}^*(t) = \langle L_j M_V(t), U_j M_V(t) \rangle.$$

**Example 4.5.** Consider the same experiment as in Example 3.1. From Table 4.1 it is easy to see the following:

- The  $P$ -lower  $r^{th}$  moment of  $V$  about the  $P$ -lower mean  $L_P\mu$  is

$$L_{P,r}\mu = L_P E(V - L_P\mu)^r = \sum_{k=1}^5 (v_k - L_P\mu)^r L_P p(V = v_k) \\ = \frac{1}{5} (1 - 2)^r + \frac{1}{5} (2 - 2)^r + \frac{1}{5} (3 - 2)^r \\ + \frac{1}{5} (4 - 2)^r + 0 = \frac{1}{5} [(-1)^r + 1 + 2^r], \text{ where } r \\ = 0, 1, 2, \dots$$

- The  $P$ -upper  $r^{th}$  moment of  $V$  about the  $P$ -upper mean  $U_P\mu$  is

$$U_{P,r}\mu = U_P E(V - U_P\mu)^r = \sum_{k=1}^n (v_k - U_P\mu)^r U_P p(V = v_k) \\ = \frac{1}{5} (1 - \frac{19}{5})^r + \frac{1}{5} (2 - \frac{19}{5})^r + \frac{1}{5} (3 - \frac{19}{5})^r \\ + \frac{2}{5} (4 - \frac{19}{5})^r + \frac{1}{5} (5 - \frac{19}{5})^r \\ = \frac{1}{5} [(-19)^r + (-9)^r + (-4)^r + 2 + (6)^r],$$

where  $r = 0, 1, 2, \dots$

- The  $P$ -lower  $r^{th}$  moment of  $V$  about origin is

$$L_{P,r}\mu' = L_P E(V^r) = \frac{1}{5} (1)^r + \frac{1}{5} (2)^r + \frac{1}{5} (3)^r \\ + \frac{1}{5} (4)^r + 0 = \frac{1}{5} [1 + (2)^r + (3)^r + (4)^r], \text{ where } r \\ = 0, 1, 2, \dots$$

- The  $P$ -upper  $r^{th}$  moment of  $V$  about origin is

$$U_{P,r}\mu' = U_P E(V^r) = \frac{1}{5} (1)^r + \frac{1}{5} (2)^r + \frac{1}{5} (3)^r \\ + \frac{2}{5} (4)^r + \frac{1}{5} (5)^r = \frac{1}{5} [1 + (2)^r + (3)^r + 2(4)^r + \\ (5)^r], \text{ where } r = 0, 1, 2, \dots$$

- The  $P$ -lower moment generating function of  $V$  is

$$L_P M_V(t) = L_P E(e^{tV}) = \sum_{k=1}^n e^{vk} L_P p(V = v_k) \\ = \frac{1}{5} e^t + \frac{1}{5} e^{2t} + \frac{1}{5} e^{3t} + \frac{1}{5} e^{4t} + 0 \\ = \frac{1}{5} [e^t + e^{2t} + e^{3t} + e^{4t}].$$

- The  $P$ -upper moment generating function of  $V$  is

$$U_P M_V(t) = U_P E(e^{tV}) = \sum_{k=1}^n e^{vk} U_P p(V = v_k) \\ = \frac{1}{5} e^t + \frac{1}{5} e^{2t} + \frac{1}{5} e^{3t} + \frac{2}{5} e^{4t} + \frac{1}{5} e^{5t} \\ = \frac{1}{5} [e^t + e^{2t} + e^{3t} + 2e^{4t} + e^{5t}].$$

## 5. Conclusions and Future work

The  $G_m$ -topological structure of the stochastic approximation spaces helped us for introducing different levels of rough probabilities, rough probability distributions, rough mathematical expectations, rough variances, and others. Our future work aims to apply the  $G_m$ -topological structure using information systems which considered from real life experiments.

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