Some Results on Fibrewise Topological Spaces

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Abstract

In this paper we define and study new concepts of functions on fibrewise topological spaces over B namely, fibrewise weakly (resp., closure, strongly) continuous functions which are analogous of weakly (resp., closure, strongly) continuous functions and the main result is : Let φ : X→Y be a fibrewise closure (resp., weakly, closure, strongly, strongly) continuous function, where Y is fibrewise topological space over B and X is a fibrewise set which has the induced fibrewise topology. If for each fibrewise topological space Z, a fibrewise function ψ : Z→X is weakly (resp., continuous, closure, closure, weakly), then the composition φoψ : Z→Y is weakly (resp., weakly, closure, strongly, continuous). Also, we define and study the concepts fibrewise ω-closed (resp., ω-coclosed, ω-biclosed, ω-open, ω-coopen, ω-biopen) topological spaces over B which is similar of definition of fibrewise closed (resp., open) topological spaces over B; also we state and prove several propositions concerning with these concepts.

Introduction

To being with we work in the fibrewise sets over a given set, called the base set. If the base set is denoted by B then a fibrewise set over B consists of a set X together with a function p : X→B, called the projection. For each point b of B the fibre over b is the subset $X_b=p^{-1}(b)$ of X, also for each subset B' of B we regard $X_{B'}=p^{-1}(B')$ as a fibrewise set over B' with the projection determined by p. Fibrewise sets over B constitute with the following definition of morphism. If X and Y are fibrewise sets over B, with projections p_X and p_Y , respectively, a function φ : X→Y is said to be fibrewise if $p_Y o\varphi=p_X$, in other words if $\varphi(X_b)\subset Y_b$ for each point b of B. Given an indexe family $\{X_r\}$ of fibrewise sets over B the fibrewise product $\prod_B X_r$ is defined, as a fibrewise set over B, and comes equipped with the family of fibrewise projection $\pi_r : \prod_B X_r \to X_r$, specifically the fibrewise product is defined as the subset of the ordinary product $\prod X_r$ in which the fibres are the products of the corresponding fibers of the factors X_r , so for each fibrewise set X over B the fibrewise functions $\varphi : X \to \prod_B X_r$ correspond precisely to the families of fibrewise functions $\{\varphi_r\}$, with $\varphi_r = \pi_r o \varphi : X \to X_r$. For example if $X_r = X$ for each index r the diagonal $\Delta : X \to \prod_B X$ is defined so that $\pi_r o \Delta = id_X$ for each r. If $\{X_r\}$ is as before, the fibrewise coproduct $\coprod_B X_r$ is also defined, as a fibrewise set over B, and comes equipped with the family of fibrewise insertions $\sigma_r : X_r \to \coprod_B X_r$, specifically the fibrewise coproduct coincides, as a set, with the ordinary coproduct (disjoint union) the fibres being the coproducts of the corresponding fibers of the summands X_r, so for each fibrewise set X over B the fibrewise functions $\psi : \coprod_B X_r \to X$ correspond precisely to the families of fibrewise functions $\{\psi_r\}$, where $\psi_r = \psi o \sigma_r : X_r \to X$. For example if $X_r = X$ for each index r the codiagonal $\nabla : \coprod_B X \to X$ is defined so that $\nabla o \sigma_r = id_X$ for each r. Now suppose that B is a topological space, by a fibrewise topology on a fibrewise set X over B, mean any topology on X for which the projection p is continuous. A fibrewise topological space over B is defined to be a fibrewise set over B with a fibrewise topology. All the above information we can find in (1).

The notation $X \times_B Y$ is used for the fibrewise product in the case of the family {X, Y} of two fibrewise sets and similarly for finite families generally. For a subset A of a topological space X, the closure of A is denoted by cl(A). For other notions or notations which are not defined here we follow closely Engelking (2).

Basic Definitions

Definition 2.1 (3, 4, and 5)

A function $\varphi : X \rightarrow Y$ is called weakly (resp., closure, strongly) continuous at a point $x \in X$ where X and Y are topological spaces, if given any open set V containing $\varphi(x)$ in Y, there exists an open set U containing x in X such that $\varphi(U) \subseteq cl(V)$ (resp., $\varphi(cl(U)) \subseteq cl(V)$, $\varphi(cl(U)) \subseteq V$).

If this condition is satisfied at each point $x \in X$, then φ is said to be weakly (resp., closure, strongly) continuous.

Definition 2.2 (1)

A fibrewise function $\varphi : X \rightarrow Y$ is called a fibrewise continuous where X and Y are fibrewise topological spaces over B, if for each $x \in X_b$, where $b \in B$ and every open set V of $\varphi(x)$ in Y, there exists an open set U containing x in X_b such that $\varphi(U) \subseteq V$.

Definition 2.3 (2)

A point x of a space X is called a condensation point of a set $A \subseteq X$ if every neighborhood of the point x contains an uncountable subset of A.

Definition 2.4 (6)

A subset of a space X is called ω-closed if it contains all its condensation points. The complement of a ω-closed set is called ω-open set.

Definition 2.5 (6)

A function $\varphi : X \rightarrow Y$ is called ω-closed function if it is maps closed sets onto ω-closed sets.

Definition 2.6 (1)

A fibrewise topological space X over B is called fibrewise closed if the projection p is closed.

Definition 2.7 (1)

A fibrewise topological space X over B is called fibrewise open if the projection p is open.

Fibrewise Weakly (resp., Closure, Strongly) Continuous Functions

The new concepts in this paper are given by the following definition.

Definition 3.1

A fibrewise function $\varphi : X \rightarrow Y$ is called a fibrewise weakly (resp., closure, strongly) continuous where X and Y are fibrewise topological spaces over B, if for each $x \in X_b$, where $b \in B$ and every open set V of $\varphi(x)$ in Y, there exists an open set U containing x in X_b such that $\varphi(U)\subseteq cl(V)$ (resp., $\varphi(cl(U))\subseteq cl(V)$, $\varphi(cl(U))\subseteq V$). Weakly continuous is denoted by w.c., closure continuous is denoted by c.c., and strongly continuous is denoted by s.c.

Let $\varphi : X\rightarrow Y$ be a fibrewise function where X is a fibrewise set and Y is a fibrewise topological space over B. If X has the induced topology, in the ordinary sense, which is a fibrewise topology, we have, as the induced fibrewise topology the following characterizations.

Proposition 3.2 (1)

Let $\varphi : X\rightarrow Y$ be a fibrewise function, where Y is a fibrewise topological space over B and X is a fibrwise set has the induced fibrewise topology. Then for each fibrewise topological space Z, a fibrewise function $\psi : Z\rightarrow X$ is continuous iff the composition $\varphi o\psi : Z\rightarrow Y$ is continuous.

Proposition 3.3

Let $\varphi : X\rightarrow Y$ be a fibrewise c.c. function, where Y is a fibrewise topological space over B and X is a fibrwise set has the induced fibrewise topology. If for each fibrewise topological space Z, a fibrewise function $\psi : Z\rightarrow X$ is w.c., then the composition $\varphi o\psi : Z\rightarrow Y$ is w.c.

Proof

Suppose that φ is c.c. and ψ is w.c.. Let $z\in Z_b$, where $b\in B$ and V open set of $(\varphi o\psi)(z)$ in Y, since φ is c.c., there exists an open set U containing x in X_b such that $\varphi(cl(U))\subseteq cl(V)$. Since ψ is w.c., then for every $z\in Z_b$ and every open set U of $\psi(z)=x$, there exists an open set W of z in Z_b such that $\psi(W)\subseteq cl(U)$, so $\varphi(\psi(W))\subseteq\varphi(cl(U))$ and $(\varphi o\psi)(W)\subseteq \varphi(cl(U))$, then we have $(\varphi o\psi)(W)\subseteq cl(V)$ and $\varphi o\psi$ is w.c.

Proposition 3.4

Let $\varphi : X\rightarrow Y$ be a fibrewise w.c. function, where Y is a fibrewise topological space over B and X is a fibrwise set has the induced fibrewise topology. If for each fibrewise topological space Z, a fibrewise function $\psi : Z\rightarrow X$ is continuous ,then the composition $\varphi o\psi : Z\rightarrow Y$ is w.c.

Proof

Suppose that φ is w.c. and ψ is continuous. Let $z\in Z_b$, where $b\in B$ and V open set of $(\varphi o\psi)(z)$ in Y, since φ is w.c., there exists an open set U containing x in X_b such that $\varphi(U)\subseteq cl(V)$. Since ψ is continuous, then for every $z \in Z_b$ and every open set U of $\psi(z)=x$, there exists an open set W of z in Z_b such that $\psi(W) \subseteq U$, so $\varphi(\psi(W)) \subseteq \varphi(U)$ and $(\varphi o \psi)(W) \subseteq \varphi(U)$, then we have $(\varphi o \psi)(W) \subseteq cl(V)$ and $\varphi o \psi$ is w.c.

Proposition 3.5

Let $\varphi : X \rightarrow Y$ be a fibrewise c.c. function, where Y is a fibrewise topological space over B and X is a fibrwise set has the induced fibrewise topology. If for each fibrewise topological space Z, a fibrewise function $\psi : Z \rightarrow X$ is c.c., then the composition $\varphi o \psi : Z \rightarrow Y$ is c.c.

Proof

Suppose that φ is c.c. and ψ is c.c.. Let $z \in Z_b$, where $b \in B$ and V open set of $(\varphi o \psi)(z)$ in Y, since φ is c.c., there exists an open set U containing x in X_b such that $\varphi(cl(U)) \subseteq cl(V)$. Since ψ is c.c., then for every $z \in Z_b$ and every open set U of $\psi(z)=x$, there exists an open set W of z in Z_b such that $\psi(cl(W)) \subseteq cl(U)$, so $\varphi(\psi(cl(W))) \subseteq \varphi(cl(U))$ and $(\varphi o \psi)(cl(W)) \subseteq \varphi(cl(U))$, then we have $(\varphi o \psi)(cl(W)) \subseteq cl(V)$ and $\varphi o \psi$ is c.c.

Proposition 3.6

Let $\varphi : X \rightarrow Y$ be a fibrewise s.c. function, where Y is a fibrewise topological space over B and X is a fibrwise set has the induced fibrewise topology. If for each fibrewise topological space Z, a fibrewise function $\psi : Z \rightarrow X$ is c.c., then the composition $\varphi o \psi : Z \rightarrow Y$ is s.c.

Proof: Suppose that φ is s.c. and ψ is c.c.. Let $z \in Z_b$, where $b \in B$ and V open set of $(\varphi o \psi)(z)$ in Y, since φ is s.c., there exists an open set U containing x in X_b such that $\varphi(cl(U)) \subseteq V$. Since ψ is c.c., then for every $z \in Z_b$ and every open set U of $\psi(z)=x$, there exists an open set W of z in Z_b such that $\psi(cl(W)) \subseteq cl(U)$, so $\varphi(\psi(cl(W))) \subseteq \varphi(cl(U))$ and $(\varphi o \psi)(cl(W)) \subseteq \varphi(cl(U))$, then we have $(\varphi o \psi)(cl(W)) \subseteq V$ and $\varphi o \psi$ is s.c.

Proposition 3.7

Let $\varphi : X \rightarrow Y$ be a fibrewise s.c. function, where Y is a fibrewise topological space over B and X is a fibrwise set has the induced fibrewise topology. If for each fibrewise topological space Z, a fibrewise function $\psi : Z \rightarrow X$ is w.c., then the composition $\varphi o \psi : Z \rightarrow Y$ is continuous.

Proof

Suppose that φ is s.c. and ψ is w.c.. Let $z\in Z_b$, where $b\in B$ and V open set of $(\varphi o\psi)(z)$ in Y, since φ is s.c., there exists an open set U containing x in X_b such that $\varphi(cl(U))\subseteq V$. Since ψ is w.c., then for every $z\in Z_b$ and every open set U of $\psi(z)=x$, there exists an open set W of z in Z_b such that $\psi(W)\subseteq cl(U)$, so $\varphi(\psi(W))\subseteq\varphi(cl(U))$ and $(\varphi o\psi)(W)\subseteq\varphi(cl(U))$, then we have $(\varphi o\psi)(W)\subseteq V$ and $\varphi o\psi$ is continuous.

Let us pass of general cases of propositions (3.3), (3.4), (3.5), (3.6), and (3.7) as follows:

Similarly in the case of families $\{\varphi_r\}$ of fibrewise c.c. (resp., w.c., c.c., s.c., s.c.,) functions, where $\varphi_r : X\rightarrow Y_r$ with Y_r fibrewise topological spaces over B for each r. In particular, given a family $\{X_r\}$ of fibrewise topological spaces over B, the fibrewise topological product $\prod_B X_r$ is defined to be the fibrewise product with the fibrewise topology induced by the family of c.c. (resp., w.c., c.c., s.c., s.c.,) projections $\pi_r : \prod_B X_r \rightarrow X_r$. If for each fibrewise topological space Z over B a fibrewise function $\theta : Z\rightarrow\prod_B X_r$ is w.c. (resp., continuous, c.c., c.c., w.c.,), then the composition $\pi_r o\theta : Z\rightarrow X_r$ is w.c. (resp., w.c., c.c., s.c., continuous). For example when X_r=X for each index r and the diagonal $\Delta : X\rightarrow\prod_B X$ is w.c. (resp., continuous, c.c., c.c., w.c.,), then the composition $\pi_r o\Delta=id_X$ is w.c. (resp., w.c., c.c., s.c., continuous).

Again if $\{X_r\}$ is a family of fibrewise topological spaces over B and $\psi : \coprod_B X_r \rightarrow X$ is a fibrewise c.c. (resp., w.c., c.c., s.c., s.c.,) function where X a fibrewise topology over B and $\coprod_B X_r$ is fibrewise topological coproduct at the set-theoretic level with the ordinary coproduct topology, also for each fibrewise topology X_r with the family of fibrewise insertions $\sigma_r : X_r\rightarrow\coprod_B X_r$ is w.c. (resp., continuous, c.c., c.c., w.c.,) then the composition $\psi_r = \psi o\sigma_r: X_r\rightarrow X$ is w.c. (resp., w.c., c.c., s.c., continuous). For example if X_r=X for each index r and the codiagonal $\nabla : \coprod_B X \rightarrow X$ is c.c. (resp., w.c., c.c., s.c., s.c.,), then the composition $\nabla o\sigma_r=id_X$ is w.c. (resp., w.c., c.c., s.c., continuous).

Fibrewise ω-closed and ω-open Topology

By a similar way of definition (2.5) we introduce the following definitions:

Definition 4.1

A function $\varphi : X \rightarrow Y$ is called ω-open function where X and Y are topological spaces if it is maps open sets onto ω-open sets.

Definition 4.2

A function $\varphi : X \rightarrow Y$ is called ω-coclosed (ω-biclosed) function where X and Y are topological spaces if it is maps ω-closed sets onto closed (ω-closed) sets.

Definition 4.3

A function $\varphi : X \rightarrow Y$ is called ω-coopen (ω-biopen) function where X and Y are topological spaces if it is maps ω-open sets onto open (ω-open) sets.

By a similar way of definition (2.6) we introduce the following definition:

Definition 4.4

A fibrewise topological space X over B is called fibrewise ω-closed (resp., ω-coclosed, ω-biclosed) if the projection p is ω-closed (resp., ω-coclosed, ω-biclosed).

Proposition 4.5

Let $\varphi : X \rightarrow Y$ be a closed fibrewise function, where X and Y are fibrewise topological spaces over B.

(a) If Y is fibrewise closed, then X is fibrewise closed. (1)

(b) If Y is fibrewise ω-closed, then X is fibrewise ω-closed.

(c) If Y is fibrewise ω-coclosed, then X is fibrewise closed.

(d) If Y is fibrewise ω-coclosed, then X is fibrewise ω-closed.

(e) If Y is fibrewise ω-biclosed, then X is fibrewise ω-closed.

Proof: The proofs of the five facts are similar; so, we will only prove the fact (b): Suppose that $\varphi : X \rightarrow Y$ is closed fibrewise function and Y is fibrewise ω-closed i.e., the projection $p_Y : Y \rightarrow B$ is ω-closed. To show that X is fibrewise ω-closed i.e., the projection $p_X : X \rightarrow B$ is ω-closed. Now let F be a closed subset of X_b, where $b \in B$, since φ is closed, then $\varphi(F)$ is closed subset of Y_b. Since p_Y is ω-closed, then $p_Y(\varphi(F))$ is ω-closed in B, but $p_Y(\varphi(F))=(p_Y o \varphi)(F)=p_X(F)$ is ω-closed in B. Thus p_X is ω-closed and X is fibrewise ω-closed.

Proposition 4.6

Let $\varphi : X \rightarrow Y$ be a ω-closed fibrewise function, where X and Y are fibrewise topological spaces over B.
(a) If Y is fibrewise ω-coclosed, then X is fibrewise closed.
(b) If Y is fibrewise ω-coclosed, then X is fibrewise ω-closed.
(c) If Y is fibrewise ω-biclosed, then X is fibrewise ω-closed.

Proof

The proof is similar to the proof of proposition (4.5), so it is omitted.

Proposition 4.7

Let $\varphi : X \rightarrow Y$ be a ω-coclosed fibrewise function, where X and Y are fibrewise topological spaces over B.
(a) If Y is fibrewise closed, then X is fibrewise ω-coclosed.
(b) If Y is fibrewise ω-closed, then X is fibrewise ω-biclosed.
(c) If Y is fibrewise ω-coclosed, then X is fibrewise ω-coclosed.
(d) If Y is fibrewise ω-biclosed, then X is fibrewise ω-biclosed.

Proof

The proof is similar to the proof of proposition (4.5), so it is omitted.

Proposition 4.8

Let $\varphi : X \rightarrow Y$ be a ω-biclosed fibrewise function, where X and Y are fibrewise topological spaces over B.
(a) If Y is fibrewise ω-coclosed, then X is fibrewise ω-coclosed.
(b) If Y is fibrewise ω-coclosed, then X is fibrewise ω-biclosed.
(c) If Y is fibrewise ω-biclosed, then X is fibrewise ω-biclosed.

Proof: The proof is similar to the proof of proposition (4.5), so it is omitted.

Proposition 4.9

Let X be a fibrewise topological space over B.

(a) Suppose that X_j is fibrewise closed for each member X_j of a finite covering of X. Then X is fibrewise closed. (1)
(b) Suppose that X_j is fibrewise ω-closed (resp., ω-coclosed, ω-biclosed) for each member X_j of a finite covering of X. Then X is fibrewise ω-closed (resp., ω-coclosed, ω-biclosed).

Proof

The proofs of the four facts are similar; so, we will only prove the case when X_j ω-closed: Let X be a fibrewise topological space over B, then the projection $p : X \rightarrow B$ exists. To show that p is ω-closed. Now, since X_j is fibrewise ω-closed, then the projection $p_j : X_j \rightarrow B$ is ω-closed for each member X_j of a finite covering of X. Let F be a closed subset of X, then $p(F)=\cup p_j(X_j\cap F)$ which is a finite union of ω-closed sets and hence p is ω-closed. Thus, X is fibrewise ω-closed.

Proposition 4.10

Let X be a fibrewise topological space over B. Then

(a) X is fibrewise closed iff for each fibre X_b of X and each open set U of X_b in X, there exists an open set O of b such that $X_O\subset U$. (1)

(b) X is fibrewise ω-closed iff for each fibre X_b of X and each open set U of X_b in X, there exists an ω-open set O of b such that $X_O\subset U$.

(c) X is fibrewise ω-coclosed iff for each fibre X_b of X and each ω-open set U of X_b in X, there exists an open set O of b such that $X_O\subset U$.

(c) X is fibrewise ω-biclosed iff for each fibre X_b of X and each ω-open set U of X_b in X, there exists an ω-open set O of b such that $X_O\subset U$.

Proof

The proofs of the four facts are similar; so, we will only prove the fact (b): (⇒) Suppose that X is fibrewise ω-closed i.e., the projection $p : X\rightarrow B$ is ω-closed. Now, let $b\in B$ and U open set of X_b in X, then $X\setminus U$ is closed in X, this implice $p(X\setminus U)$ is ω-closed in B, let $O=B\setminus p(X\setminus U)$, then O a ω-open set of b in B and $X_O=p^{-1}(O)=X\setminus p^{-1}(p(X\setminus U))\subset U$.

(⇐) Suppose that the assumption hold and $p : X\rightarrow B$. Now, let F be a closed subset of X and $b\in B\setminus p(F)$ and each open set U of fibre X_b in X. By assumption there exists a ω-open O of b such that $X_O\subset U$. It is easy to show that $O\subset B\setminus p(F)$, hence $B\setminus p(F)$ is ω-open in B and this implies p(F) is ω-closed in B and p is ω-closed. Thus X is fibrewise ω-closed.

By a similar way of definition (2.7) we introduce the following definition:

Definition 4.11

A fibrewise topological space X over B is called fibrewise ω-open (resp., ω-coopen, ω-biopen) if the projection p is ω-open (resp., ω-coopen, ω-biopen).

Proposition 4.12

Let $\varphi : X\rightarrow Y$ be an open fibrewise function, where X and Y are fibrewise topological spaces over B.

(a) If Y is fibrewise open, then X is fibrewise open. (1)

(b) If Y is fibrewise ω-open (resp., ω-coopen, ω-biopen), then X is fibrewise ω-open (resp., open, ω-open).

Proof

The proofs of the four facts are similar; so, we will only prove the case when Y is fibrewise ω-open: Suppose that $\varphi : X \rightarrow Y$ is open fibrewise function and Y is fibrewise ω-open i.e., the projection $p_Y : Y \rightarrow B$ is ω-open. To show that X is fibrewise ω-open i.e., the projection $p_X : X \rightarrow B$ is ω-open. Now let O is open subset of X_b, where $b \in B$, since φ is open, then φ(O) is open subset of Y_b, since p_Y is ω-open, then $p_Y(\varphi(O))$ is ω-open in B, but $p_Y(\varphi(O))=(p_Y o \varphi)(O)=p_X(O)$ is ω-open in B. Thus p_X is ω-open and X is fibrewise ω-open.

Proposition 4.13

Let $\varphi : X \rightarrow Y$ be a ω-open fibrewise function, where X and Y are fibrewise topological spaces over B. If Y is fibrewise ω-coopen (resp., ω-coopen, ω-biopen), then X is fibrewise open (resp., ω-open, ω-open).

Proof

The proof is similar to the proof of proposition (4.12), and therefore is omitted.

Proposition 4.14

Let $\varphi : X \rightarrow Y$ be a ω-coopen fibrewise function, where X and Y are fibrewise topological spaces over B. If Y is fibrewise open (resp., ω-open, ω-coopen, ω-biopen), then X is fibrewise ω-coopen (resp., ω-biopen, ω-coopen, ω-biopen).

Proof

The proof is similar to the proof of proposition (4.12), and therefore is omitted.

Proposition 4.15

Let $\varphi : X \rightarrow Y$ be a ω-biopen fibrewise function, where X and Y are fibrewise topological spaces over B. If Y is fibrewise ω-coopen (resp., ω-coopen, ω-biopen), then X is fibrewise ω-coopen (resp., ω-biopen, ω-biopen).

Proof

The proof is similar to the proof of proposition (4.12), and hence is omitted.

Proposition 4.16

(a) Let $\{X_r\}$ be a finite family of fibrewise open spaces over B. Then the fibrewise topological product $X=\prod_B X_r$ is also open. (1)

(b) Let $\{X_r\}$ be a finite family of fibrewise ω-open (resp., ω-coopen, ω-biopen) spaces over B. Then the fibrewise topological product $X=\prod_B X_r$ is also ω-open (resp., ω-coopen, ω-biopen).

Proof: The proofs of the four facts are similar; so, we will only prove the case when $\{X_r\}$ be a finite family of fibrewise ω-open: Suppose that $X=\prod_B X_r$ is a fibrewise topological space over B, then p : $X=\prod_B X_r \rightarrow B$ is exists. To show that p is ω-open. Now, since $\{X_r\}$ be a finite family of fibrewise ω-open spaces over B, then the projection $p_r : X_r \rightarrow B$ is ω-open for each r. Let O be an open subset of X, then $p(O)= p(\prod_B (X_r \cap O))=\prod_B p_r(X_r \cap O)$ which is a finite product of ω-open sets and hence p is ω-open. Thus, the fibrewise topological product $X=\prod_B X_r$ is a fibrewise ω-open.

In other words the class of fibrewise open (resp., ω-open, ω-coopen, ω-biopen) spaces is finitely multiplicative. In fact proposition (4.16) remains true for infinite families provided each member of the family is fibrewise nonempty in the sense that the projection is surjective.

Remark 4.17

If X is fibrewise open (resp., ω-open, ω-coopen, ω-biopen) then the second projection $\pi_2 : X\times_B Y \rightarrow Y$ is open (resp., ω-open, ω-coopen, ω-biopen) for all fibrewise topological spaces Y. because for every non-empty open (resp., open, ω-open, ω-open) set $W_1\times_B W_2 \subset X\times_B Y$, we have $\pi_2(W_1\times_B W_2)=W_2$ is open (resp., ω-open, open, ω-open). We use this in the proof of the following results.

Proposition 4.18

Let $\varphi : X\rightarrow Y$ be a fibrewise function, where X and Y are fibrewise topological spaces over B. Let $id\times\varphi : X\times_B X\rightarrow X\times_B Y$.

(a) If $id\times\varphi$ is open and that X is fibrewise open. Then φ it self is open. (1)

(b) If $id\times\varphi$ is open and that X is fibrewise open, Y is fibrewise ω-open. Then φ itself is ω-open.

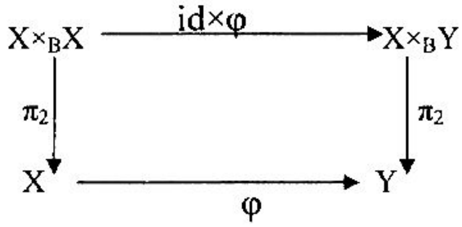
(c) If $id\times\varphi$ is ω-open and that X is fibrewise open, Y is fibrewise ω-coopen. Then φ itself is open.

(d) If $id\times\varphi$ is ω-open and that X is fibrewise open, Y is fibrewise ω-biopen. Then φ it self is ω-open.

(e)If $id\times\varphi$ is ω-open and that X is fibrewise ω-open, Y is fibrewise ω-biopen. Then φ it self is ω-biopen.
(f)If $id\times\varphi$ is ω-coopen and that X is fibrewise ω-coopen, Y is fibrewise open. Then φ it self is open.
(g)If $id\times\varphi$ is ω-biopen and that X is fibrewise ω-biopen, Y is fibrewise ω-biopen. Then φ it self is ω-biopen.

Proof

The proofs of the seven facts are similar; so, we will only prove the fact (b): Consider the following commutative diagram.



The projection on the left is surjective and ω-open, since Y is fibrewise ω-open, while the projection on the right is open, since X is fibrewise open. Therefore $\pi_r o(id\times\varphi)=\varphi o\pi_r$ is ω-open, and so φ is ω-open, by proposition (4.12,b) as asserted.

Our next three results apply equally to fibrewise closed (resp., ω-closed, ω-coclosed, ω-biclosed) and the fibrewise open (resp., ω-open, ω-coopen, ω-biopen) spaces.

Proposition 4.19

Let $\varphi : X\rightarrow Y$ be a continuous fibrewise surjection, where X and Y are fibrewise topological spaces over B.

(a) If X is fibrewise closed (resp., open), then Y is fibrewise closed (resp., open). (1)
(b) If X is fibrewise ω-closed (resp., ω-open), then Y is fibrewise ω-closed (resp., ω-open).
(c) If X is fibrewise ω-coclosed (resp., ω-coopen), then Y is fibrewise ω-coclosed (resp., ω-coopen).
(d) If X is fibrewise ω-biclosed (resp., ω-biopen), then Y is fibrewise ω-biclosed (resp., ω-biopen).

Proof

The proofs of the four facts are similar; so, we will only prove the fact (b): Suppose that $\varphi : X\rightarrow Y$ is continuous fibrewise surjection and X is fibrewise ω-closed (resp., ω-open) i.e., the projection $p_X : X\rightarrow B$

is ω-closed (resp., ω-open). To show that Y is fibrewise ω-closed (resp., ω-open) i.e., the projection $p_Y : Y \rightarrow B$ is ω-closed (resp., ω-open). Let G be a closed (resp., open) subset of Y_b, where $b \in B$. Since φ is continuous fibrewise , then $\varphi^{-1}(G)$ is closed (resp., open) subset of X_b. Since p_X is ω-closed (resp., ω-open), then $p_X(\varphi^{-1}(G))$ is ω-closed (resp., ω-open) in B, but $p_X(\varphi^{-1}(G))=(p_X o \varphi^{-1})(G)=p_Y(G)$ is ω-closed (resp., ω-open) in B. Thus p_Y is ω-closed (resp., ω-open) and Y is fibrewise ω-closed (resp., ω-open).

Proposition 4.20

Let X be a fibrewise topological space over B.

(a) Suppose that X is fibrewise closed (resp., open) over B. Then $X_{B'}$ is fibrewise closed (resp., open) over B` for each subspace B` of B. (1)

(b) Suppose that X is fibrewise ω-closed (resp., ω-open) over B. Then $X_{B'}$ is fibrewise ω-closed (resp., ω-open) over B` for each subspace B` of B.

(c) Suppose that X is fibrewise ω-coclosed (resp., ω-coopen) over B. Then $X_{B'}$ is fibrewise ω-coclosed (resp., ω-coopen) over B` for each subspace B` of B.

(d) Suppose that X is fibrewise ω-biclosed (resp., ω-biopen) over B. Then $X_{B'}$ is fibrewise ω-biclosed (resp., ω-biopen) over B` for each subspace B` of B.

Proof

The proofs of the four facts are similar; so, we will only prove the fact (b): Suppose that X is a fibrewise ω-closed (resp., ω-open) i.e., the projection $p : X \rightarrow B$ is ω-closed (resp., ω-open). To show that $X_{B'}$ is fibrewise ω-closed (resp., ω-open) over B` i.e., the projection $p_{B'} : X_{B'} \rightarrow B'$ is ω-closed (resp., ω-open). Now, let G be a closed (resp., open) subset of X, then $G \cap X_{B'}$ is closed (resp., open) in subspace $X_{B'}$ and $p_{B'}(G \cap X_{B'})=p(G \cap X_{B'})=p(G) \cap B'$ which is ω-closed (resp., ω-open) set in B`. Thus $p_{B'}$ is ω-closed (resp., ω-open) and $X_{B'}$ is fibrewise ω-closed (resp., ω-open) over B`.

Proposition 4.21

Let X be a fibrewise topological space over B.

(a)Suppose that X_{B_j} is fibrewise closed (resp., open) over B_j for each member B_j of an open covering of B. Then X is fibrewise closed (resp., open) over B. (1)

(b) Suppose that X_{Bj} is fibrewise ω-closed (resp., ω-open) over B_j for each member B_j of an open covering of B. Then X is fibrewise ω-closed (resp., ω-open) over B.
(c)Suppose that X_{Bj} is fibrewise ω-coclosed (resp., ω-coopen) over B_j for each member B_j of an open covering of B. Then X is fibrewise ω-coclosed (resp., ω-coopen) over B.
(d)Suppose that X_{Bj} is fibrewise ω-biclosed (resp., ω-biopen) over B_j for each member B_j of an open covering of B. Then X is fibrewise ω-biclosed (resp., ω-biopen) over B.

Proof

The poof of the four facts are similar; so, we will only prove the fact (b): Suppose that X is a fibrewise topological space over B, then the projection $p : X \rightarrow B$ exists. To show that p is ω-closed (resp., ω-open). Now, since X_{Bj} is fibrewise ω-closed (resp., ω-open) over B_j, then the projection $p_{Bj} : X_{Bj} \rightarrow B_j$ is ω-closed (resp., ω-open) for each member B_j of an open covering of B. Let G be a closed (resp., open) subset of X, then we have $p(G)=\cup p_B(X_{Bj} \cap G)$ which is a union of ω-closed (resp., ω-open) sets and hence p is ω-closed (resp., ω-open). Thus, X is fibrewise ω-closed (resp., ω-open) over B.

In fact the last propostion is also true for locally finite closed coverings by using theorem (1.1.11) and corollary (1.1.12) in (2).

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بعض النتائج عن الفضاءات التبولوجية الليفية

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الخلاصة

في هذا البحث عرفنا ودرسنا مفهوم جديد من الدوال على الفضاءات التبولوجية الليفية فوق المجموعة B سميناها، الدوال المستمرة الضعيفة (المغلقة، القوية) الليفية وهي مناظرة لمفاهيم الدوال المستمرة الضعيفة (المغلقة، القوية). واهم نتيجة توصلنا اليها: ليكن φ : X→Y دالة مستمرة مغلقة (ضعيفة، مغلقة، قوية، قوية) ،اذ Y فضاءا" تبولوجيا ليفيا" فوق المجموعة B وX مجموعة ليفية تمتلك تبولوجيا ليفية متولدة. اذا كان لكل فضاء تبولوجي ليفي Z، الدالة الليفية Z→X : ψ تكون مستمرة ضعيفة(مستمرة، مغلقة، مغلقة، ضعيفة)، فان الدالة المركبة Z→Y : $\varphi o \psi$ تكون مستمرة ضعيفة (ضعيفة، مغلقة، قوية، مستمرة) على الترتيب. كذلك عرفنا ودرسنا مفهوم الفضاءات الليفية المغلقة من النمط-ω (المقلوبه المغلقة من النمط-ω، الثنائية المغلقة من النمط-ω، المفتوحة من النمط-ω، المقلوبة المفتوحة من النمط-ω، الثنائية المفتوحة من النمط-ω) فوق المجموعة B التي هي بحد ذاتها مشابهة لمفهوم الفضاءات التبولوجية الليفية المغلقة (المفتوحة) فوق المجموعة B. كذلك اعطينا وبرهنا العديد من القضايا المتعلقة بهذه المفاهيم.