Duality of St-closed submodules and semi-extending modules

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Abstract. The main goal of this paper is to dualize the two concepts St-closed submodule and semi-extending module which were given by Ahmed and Abbas in 2015. These dualizations are called CSt-closed submodule and cosemi-extending module. Many important properties of these dualizations are investigated, as well as some others useful results which mentioned by those authors are dualized. Furthermore, the relationships of cosemi-extending and other related modules are considered.

Keywords: CSt-closed submodules, St-closed submodules, Cosemi-essential submodules, semi-essential submodules, Cosemi-extending modules, semi-extending modules.

1. Introduction

Throughout this article, R denotes a commutative ring with identity and all modules are unitary left R-module. "A submodule V of an R-module U is called essential if every non-zero submodule of an R-module U has a non-zero intersection with V" ([10], P.15). "A submodule V of U is called closed if V has no proper essential extensions inside U"([10], P.18). "A submodule V of U is called semi-essential if every non-zero prime submodule of U has a non-zero intersection with V" ([6]). The concept of St-closed submodules is strongly than closed submodules, where " a submodule V of U is said to be St-closed, if V has no proper semi-essential extensions inside U" ([8]). "A submodule V of U is called small in U (denoted by $V \ll U$), if for every proper submodule K of U, $V + K \neq U$" ([10], P.20). "A submodule W is called coessential of V in U (denoted by by $W \leq_{ce} V$ in U) if whenever $V/W \ll U/W$ then $V = W$" ([14]), and "V is called coclosed in U (simply $V \leq_{cc} U$) if A has no proper coessential submodule in U" ([14]). Hadi and Ibrahiem introduced P-small submodules as an extension to the concept of small submodules, where "a proper submodule V of an R-module U is called P-small (simply $V \ll_P U$) if $V + P \neq U$ for every prime submodule P of U" ([13]). "Let U be an R-module and $W \leq V \leq U$, if $\frac{V}{W} \ll_P \frac{U}{W}$ then W is called a cosemi-essential submodule of V in U" ([5]). "An R-module U is called extending if every closed submodule of U is a direct summand of U" ([16], P.118). Ahmed and Abbas introduced the concepts "semi-extending module" as a generalization of extending modules, where "a module U is called semi-extending if every St-closed submodule of U is a direct summand of U" ([7]).

In this paper, CSt-closed submodule and cosemi-extending modules are defined as dualizations of the classes of St-closed submodules and semi-extending module respectively. This paper consists of four sections, in section 2; we introduce the concept of CSt-closed submodule as a dualization of St-closed submodule, so we present the properties of this concept, some of them are dualizations of the results which appeared in [7, 8]. Beside, we add other useful results. Among them are the following: Let $U = U_1 \oplus U_2$ R-module, where U_1 and U_2 be R-modules and V be CSt-closed in U_1, then V is CSt-closed in U, see Proposition 2.9. Also; Let U be a finitely generated faithful and multiplication R-module, and V be a submodule of U, then $V \leq_{CSt} U$ if and only if $[V : U] \leq_{CSt} R$, see Proposition 2.13. Furthermore, we introduce the concept of Pr-supplemented and prove in Proposition 2.22 the following: Let U be a Noetherian (or multiplication) module, and $V \leq U$. Consider the following statements:

1. V is a Pr-supplement submodule of U.
2. V is CSt-closed submodule of U.
3. $Condition^*$: "For each submodule X of V; if $X \ll_P V$ then $X \ll_P U$". Then (1) ⇒ (2) ⇒ (3), and if U is weakly Pr-supplemented then (3) ⇒ (1).

Section 3 is devoted to dualize the concept of semi-extending module, we call it cosemi-extending modules, we give conditional characterization for cosemi-extending module, see Theorem 3.4. In addition, we show in Proposition 3.9, that every free multiplication R -module is cosemi-extending if and only if every projective R-module cosemi-extending. Moreover, we discuss the direct sum of cosemi-extending modules for example we prove that if $U = X \oplus Y$ is a duo module with $ann_R X + ann_R Y = R$, where X and Y be R-modules, then X and Y are cosemi-extending module if and only if U is cosemi-extending module, see Theorem 3.10.

Section 4; discuss the relationships between cosemi-extending and some other related concepts such as cosemi-uniform, Pr-hollow, semisimple and Pr-lifting modules, see Propositions 4.2, 4.3, and Theorem 4.8. Moreover, we discuss in 4.9 and 4.10 the relationships of a cosemi-extending module with certain kind of rings.

2. CSt-closed submodules

In this section we introduce the following concept which is a dualization of St-closed submodules.

Definition 2.1. *A submodule V of U is said to be CSt-closed (simply $V \leq_{CSt} U$), if V has no proper cosemi-essential extensions inside U. That is if $V/A \ll_P U/A$, then $V = A$ for all submodules A of U contained in V.*

Remarks and examples 2.2.

i. Every R-module is CSt-closed submodule of itself.

ii. $(\bar{0})$ may not be CSt-closed submodule of a non-zero module, for example: consider the Z-module Z_4, $(\bar{0}) \nleq_{CSt} Z_4$ since $(\bar{0}) \leq_{cosm} (\bar{2}) \leq Z_4$ ([5], Ex.(2.3)(1)).

iii. Consider the Z-module of rational number Q, since Z is P-small submodule of Q, then $Z/A \ll_P Q/A$, for every submodule A with $A \leq Z \leq Q$ ([13]). Thus $Z \nleq_{CSt} Q$.

iv. Every CSt-closed submodule is coclosed. To show that; let $V \leq_{CSt} U$, then V has no proper cosemi-essential submodule in U, and by the direct implication between coessential and cosemi-essential submodules in [5]; V has no proper coessential submodule inside U, that is V is coclosed in U.

v. Consider the Z_{P^∞} as Z-module. The only proper CSt-closed submodule in Z_{P^∞} is zero. In fact, Z_{P^∞} is an almost finitely generated module, where "an R-module U is called almost finitely generated if U is not finitely generated and every proper submodule of U is finitely generated" ([13]). Note that in an almost finitely generated module; every P-small is small ([13]), so if V is a submodule of Z_{P^∞} with $V \leq_{CSt} Z_{P^\infty}$, then clearly V is coclosed submodule of Z_{P^∞}. But, the only coclosed submodule in Z_{P^∞} is zero, therefore $V = 0$.

vi. For the Z-module Z_6; $(\bar{2}) \leq_{CSt} Z_6$, since $(\bar{2})$ has only two submodules $(\bar{0})$ and $(\bar{2})$, and clearly $(\bar{0}) \leq_{cosm} (\bar{2})$ in Z_6. In fact, there is a prime submodule $(\bar{3})$ in Z_6 such that that $(\bar{2})/(\bar{0}) + (\bar{3})/(\bar{0}) = Z_6/(\bar{0})$. Thus the only submodule contained $(\bar{2})$ such that $(\bar{2})/W \leq_P Z_6/W$ is $(\bar{2})$ itself, hence $(\bar{2}) \leq_{CSt} Z_6$. Note that, every submodule of Z_6 is CSt-closed.

vii. For the Z-module Z_{12}, since the only submodule W contained in $(\bar{3})$ such that $(\bar{3})/W \ll_P Z_{12}/W$ is $(\bar{3})$ itself, thus $(\bar{3}) \leq_{CSt} Z_{12}$.

viii. If W is cosemi-essential submodule of V in U, and V is CSt-closed in U, then $V = U$.

Proof. Since W is cosemi-essential submodule of V in U then $V/W \ll_P U/W$, but V is CSt-closed, therefore $V = W$. □

Remark 2.3. A direct summand of an R-module U may not be CSt-closed submodule, for example $(\bar{0})$ is a direct summand of Z_4, but $(\bar{0}) \nleq_{CSt} Z_4$ as was showed in Example 2.2(ii). □

"An R-module U is called multiplication, if every submodule V of U can be written in the form $V = IU$ for some ideal I of R" ([9]).

Proposition 2.4. *If a module U is multiplication (finitely generated) then every non-zero direct summand of U is CSt-closed.*

Proof. Let $U = V \oplus W$ where both of V and W be submodules of U, with $V \neq (0)$. We have to show that $V \leq_{CSt} U$. Assume that X is a submodule of V with $V/X \ll_P U/X$. Now, $U/X = V/X + (W + X)/X$. On the other hand, U is multiplication module (finitely generated), therefore $V/X \ll U/X$ ([13]). This implies that $U/X = (W + X)/X$. To complete the proof, we must show that $V = X$. Let $a \in V$, then $a + X \in U/X = (V + X)/X$. So $a + X = b + X$ for some $b \in W$. This implies that $a - b \in X$, hence $a - b = c$ for some $c \in X$. Therefore, $a - c = b \in W \cap V = 0$, thus $a = c$, hence $a \in X$. So $V = X$, that is $V \leq_{CSt} U$. □

Corollary 2.5. *If U is a multiplication (finitely generated) module, then the concept of CSt-closed submodules coincide with the coclosed submodules.*

Proof. The proof follows by the equivalence between "small" and "P-small" submodules in the class of multiplication (finitely generated) modules ([13] Prop.(1.4)). □

Proposition 2.6. *Let U be an R-module, and X, Y be submodules of U such that $X \leq Y \leq U$, then If X is CSt-closed submodule in U, then X is CSt-closed submodule in Y.*

Proof. Suppose that V is a submodule of U with $X \leq_{cosm} V \leq Y$. So that $X \leq_{cosm} V \leq U$. Since X is CSt-closed submodule of U, then $X = V$. □

Corollary 2.7. *For any submodules X and Y of an R-module U, the following are hold:*

1. *If $X \cap Y \leq_{CSt} U$, then $X \cap Y \leq_{CSt} X$ and $X \cap Y \leq_{CSt} Y$.*
2. *If $X \leq_{CSt} U$ and $Y \leq_{CSt} U$, then $X \leq_{CSt} X + Y$ (also $Y \leq_{CSt} X + Y$). The proof of (1) and (2) follows directly by Proposition 2.6.*
3. *If V is a CSt-closed submodule of U, and W is a submodule of U such that $V \cong W$, then it is not necessary that W is CSt-closed in U. For example, the Z-module Z is CSt-closed in itself, and $Z \cong 2Z$, but 2Z is not CSt-closed submodule in Z, since $4Z$ is a cosemi-essential submodule of $2Z$ in Z.*

Remark 2.8. A submodule of CSt-closed need not be St-closed. That is if Y is CSt-closed submodule of U and $X \leq Y$, then X may not be CSt-closed submodule of U (or Y). In fact for the Z-module Z, if $U = Y = Z$ and $X = 4Z$, then $Z \leq_{CSt} Z$ but $4Z \leq_{CSt} Z$, since $2Z/4Z \ll_P Z/4Z$.

Proposition 2.9. *Let $U = U_1 \oplus U_2$, where U_1 and U_2 be R-modules and V be CSt-closed in U_1, then V is CSt-closed in U.*

Proof. Suppose that V is not CSt-closed in U, so there exists a proper submodule L of V such that $V/L \ll_P U/L$. Let $f : U/L \to U_1/L$ be a projection epimorphism defined by $f(u_1 + u_2 + L) = u_1 + L$ where $u_1 \in U_1$ and $u_2 \in U_2$. Since $V/L \ll_P U/L$, then $f(V/L) = V/L \ll_P U_1/L$ ([13], Prop.(1.3)). But V is CSt-closed in U_1, therefore $V = L$ which is a contradiction, since L is a proper submodule of V. Thus, V is CSt-closed of U. □

Proposition 2.10. *Every submodule of semisimple module is CSt-closed.*

Proof. Since a semisimple module has no cosemi-essential ([5], Rem.(2.3)(9)), then we are done. □

"Recall that an R-module U is called Pr-hollow if each prime submodule of U is small" ([3]).

Remark 2.11. The only proper CSt-closed submodule of Pr-hollow module is a zero submodule.

Proof. Let U be a Pr-hollow module, and V be a proper CSt-closed in U. Since U is Pr-hollow, then $V \ll_P U$, hence $V/(0) \ll_P Q/(0)$. But V is CSt-closed, thus $V = (0)$. □

We need to give the following Lemmas.

Lemma 2.12. *Let U be a finitely generated faithful and multiplication R-module, and let V be a submodule of U. Then $V \ll_P U$ if and only if $[V : U] \ll_P R$.*

Proof. Assume that $V \ll_P U$, and I be an ideal of R with $I + [I : U] = R$, then $(I + [V : U])U = IU + [V : U]U = IU + V = U$. Since U is finitely generated, then $V \ll_P U$ implies to $V \ll U$ ([13]), so we deduce that $IU = U = RU$. Since U is finitely generated multiplication and faithful, then $I = R$ ([9]), hence $[V : U] \ll_P R$. Conversely, Suppose that $[V : U] \ll_P R$. Let W be a submodule of U with $V + W = U$. Since U is multiplication, then $[V : U]U + [W : U]U = U$, hence $([V : U] + [W : U])U = RU$. Since U is finitely generated and multiplication, so $[V : U] + [W : U] = R$ ([9]). On the other hand, since $[V : U] \ll_P R$, and R is finitely generated R-module, then $[V : U] \ll R$ ([13]). Thus $[W : U] = R$. Now, $W = [W : U]U = RU = U$, implies to $V \ll U$. But U is a finitely generated module thus $V \ll_P U$ ([13], Prop.(1.4). □

Lemma 2.13. *Let U be a finitely generated and multiplication R-module, and $W \leq V \leq U$, then $V/W \ll_P U/W$ if and only if $[V/W : U/W] \ll_P R/([W : U])$, where $[V/W : U/W$ considered as an ideal of $R/([W : U])$.*

Proof. We can easily show that U/W is finitely generated faithful and multiplication $R/[W : U]$ module. So U/W satisfies the conditions of Lemma 2.11, hence the result follows. □

Proposition 2.14. *Let U be a finitely generated faithful and multiplication R-module, and let V be a submodule of U. Then $V \leq_{CSt} U$ if and only if $[V : U] \leq_{CSt} R$.*

Proof. Assume that $V \leq_{St} U$, and let $I[V:U]R$ with $([V:U])/I \ll_P R/I$. Since $[IU:U]U = I$, then $([V:U])/([IU:U]) \ll_P R/([IU:U])$. We can easily show that:

$$([V:U])/([IU:U]) = [([V:U]U)/([IU:U]U) : U/([IU:U]U)].$$

Thus:

$$[([V:U]U)/([IU:U]U) : U/([IU:U]U)] \ll_P R/([IU:U]).$$

Hence:

$$V = [V:U]U = [IU:U]U = IU.$$

But $V \leq_{CSt} U$, therefore $[V:U] = I$. Conversely, let W be a submodule of V such that $V/W \ll_P U/W$. By Lemma 2.12:

$$[V/W : U/W] \ll_P R/([W:U]).$$

It is clear that:

$$[V/W : U/W] = ([V:U])/([W:U]).$$

This implies that:

$$([V:U])/([W:U]) \ll_P R/([W:U]).$$

Since $[V:U] \leq_P R$, then $[V:U] = [W:U]$. Thus $V = W$, that is $V \leq_{CSt} U$.□

Now, we can give the following result.

Proposition 2.15. *If V is CSt-closed submodule of an R-module U, then V/L is CSt-closed in U/L for any submodule L of V.*

Proof. Let L be a submodule of U, and W be a submodule of V containing L such that $\frac{V/L}{W/L} \ll_P \frac{U/L}{W/L}$. So that $V/W \ll_P U/W$. Since V is CSt-closed of U, then $V = W$, hence $V/L = W/L$, and we are done.

It is known that if $X \leq V \leq U$ with $X \ll_P U$ then X may not be P-small submodule of V ([13]), so we have the following.

Proposition 2.16. *Let V be a CSt-closed submodule of an R-module U. For each submodule X of V with $X \leq V \leq U$; if $X \ll_P U$ then $X \ll_P V$.*

Proof. Let X be a submodule of V with $X \ll_P U$. Suppose that $V = X + Y$ for some prime submodule Y of V. We claim that $V/Y \ll_P U/Y$. To prove this; assume that $V/Y + W/Y = U/Y$ where W/Y is a prime submodule of U/Y. This implies that W is prime submodule of U ([15]). On the other hand, $V + W = U$, hence $U = X + Y + W = X + W$. So $U = X + W$ which is a contradiction since $X \ll_P U$, thus $U \neq X+W$, hence $U/Y \neq X/Y+W/Y$, thus $V/Y \ll_P U/Y$. Now, we have $V \leq_{CSt} U$ and $V/Y \ll_P U/Y$, then $V = Y$, which is a contradiction since Y is proper, therefore $V \neq X + Y$, that is $X \ll_P V$. □

From Proposition 2.16, we conclude the following.

Corollary 2.17. *Let X and V be submodules of an R-module U such that $X \leq V \leq U$. If $X \leq_{CSt} V$ and $V \leq_{CSt} U$ then $X \leq_{CSt} U$.*

Proof. Let $W \leq X$ with $X/W \ll_P U/W$. Since $V \leq_{CSt} U$, then by Proposition 2.14, $V/W \leq_{CSt} U/W$, and by Proposition 2.15, $X/W \ll_P V/W$. But $X \leq_{CSt} V$, therefore $X = W$, that is $X \leq_{CSt} U$. □

"Recall that a submodule V of an R-module is called supplement (weakly supplement) of W in U, if V is minimal with the property $U = W + V$ equialently, $U = W + V$ and $W \cap V \ll V$ (resp. $W \cap V \ll U$)" ([14]). "A submodule V of U is called a supplement submodule of U if V is a supplement of some submodule of U, and an R-module U is called supplemented (weakly supplemented) if every submodule of U has a supplement (resp. weakly supplement) in U" ([14]). Now we need to give generalizations for these classes.

Definition 2.18. *A submodule V of an R-module U is said to be Pr-supplement (weakly Pr-supplement) of W in U if $U = W + V$ and $W \cap V \ll_P V$ (resp. $W \cap V \ll_P U$). A submodule V of U is said to be Pr-supplement (weakly Pr-supplement) submodule of U if V is a Pr-supplement (weakly Pr-supplement) of some submodule of U. An R-module U is called Pr-supplemented (weakly Pr-supplemented) if every submodule of U has a Pr-supplement (weakly Pr-supplement) in U.*

It is clear that every supplement is Pr-supplement submodule.

Examples 2.19.

i. Z_4 is a Pr-supplemented module, since every submodule of Z_4 has a Pr-supplement.

ii. Z_{P^∞} is Pr-supplemented, since Z_{P^∞} an Pr-supplement of every proper submodule of the Z-module Z_{P^∞}.

iii. Z is not Pr-supplemented, since $2Z$ has no Pr-supplement submodule in Z.

"An R-module U is called Noetherian if every submodule of U is finitely generated" ([10], P.7).

Proposition 2.20. *If U be a Noetherian module, then every Pr-supplement submodule of U is CSt-closed submodule of U.*

Proof. Let V be a Pr-supplement of a submodule L in U, then $U = V + L$ and $V \cap L \ll_P V$. Let $W \leq V \leq U$ such that $V/W \ll_P U/W$. Since U is Noetherian, then $V/W \ll U/W$ ([13], Prop.(1.7)), We can write U/W as follows:

$$U/W = (V + L)/W = V/W + (L + W)/W.$$

This implies that:

$$U/W = (L + W)/W$$

hence $U = L + W$, and by minimality of V, we conclude that $W = V$, that is $V \leq_{CSt} U$. □

Remark 2.21. If we replace the condition "Noetherian" in Proposition 2.20 by "finitely generated" or "multiplication", then we deduce the same result, since Hadi and Ibrahiem in ([13], Prop.(1.4)) showed that under these conditions the concepts small and P-small submodules are equivalent.

From Propositions 2.16 and 2.20, we have the following

Theorem 2.22. *Let U be a Noetherian (or multiplication) module, and $V \leq U$. Consider the following statements:*

1. *V is a Pr-supplement submodule of U.*
2. *V is CSt-closed submodule of U.*
3. *Condition*: "For each submodule X of V; if $X \ll_P U$ then $X \ll_P V$".*

Then (1) ⇒ (2) ⇒ (3), *and if U is weakly Pr-supplemented then* (3) ⇒ (1).

Proof. (1) ⇒ (2) It is just Proposition 2.20.

(2) ⇒ (3) Proposition 2.16.

(3) ⇒ (1) Suppose the condition*, since U is weakly Pr-supplemented module, then there exists a submodule L of U such that $U = V + L$ and $V \cap L \ll_P U$. But $V \cap L \subseteq V$, so by assumption $V \cap L \ll_P V$, hence V is Pr-supplement of U. □

We need the following lemma.

Lemma 2.23 ([5], Prop. (2.4)). *"For a chained of submodules $A \leq B \leq C \leq U$ of an R-module U; if $A \leq_{cosm} B$ in U and $B \leq_{ce} C$ in U then $A \leq_{cosm} B$ in U".*

Proposition 2.24. *Let U be a multiplication (finitely generated) module. For a chain of a submodules $A \leq B \leq C \leq U$, if $A \leq_{cosm} B$ in U and $B \leq_{cosm} C$ in U, then $A \leq_{cosm} C$ in U.*

Proof. The expression $B \leq_{cosm} C$ in U, means $C/B \ll_P U/B$. Since U is multiplication (finitely generated) then $C/B \ll U/B$ ([13], Prop. (1.4)), hence $B \leq_{ce} C$ in U. Now, we have $A \leq_{cosm} B$ in U and $B \leq_{ce} C$ in U, by Lemma 2.23 we get $A \leq_{cosm} C$ in U. □

Proposition 2.25. *Let U be a multiplication (finitely generated) R-module, then for every non-zero submodule V of U, there exists a CSt-closed submodule K of U with $V \leq_{cosm} K$ in U.*

Proof. Consider the set: $F = \{A|A$ is a submodule of U such that $V \leq_{cosm} A\}$ Note that $F \neq \Phi$, so by Zorns Lemma, F has a maximal element say K. In order to prove that K is an CSt-closed submodule in U; assume that there exists a submodule X of U such that $K \leq_{cosm} X \leq U$. Since $V \leq_{cosm} K$ and $K \leq_{cosm} X$, and U is multiplication so by Proposition 2.24, $V \leq_{cosm} X$. But this contradicts the maximality of K, thus $K = X$. Therefore $K \leq_{CSt} U$ with $V \leq_{cosm} K$ in U. □

Proposition 2.26. *Let $U = X \oplus Y$ where X and Y be two R-modules and $ann_R X + ann_R Y = R$. Assume that $C = A \oplus B$ where $A \leq X$ and $B \leq Y$. If $C \leq_{CSt} U$ then $A \leq_{CSt} U$ and $B \leq_{CSt} Y$.*

Proof. Suppose that $(A/S) \ll_P (X/S)$ and $(B/W) \ll_P (Y/W)$, where $S \leq A$ and $W \leq B$. Since $ann_R X + ann_R Y = R$, then by [13]:

$$(A/S) \oplus (B/W) \ll_P (X/W) \oplus (Y/W).$$

Hence:

$$(A \oplus B)/(S \oplus W)) \ll_P (X \oplus Y)/(S \oplus W)$$

and so that:

$$(C/S \oplus W) \ll_P (U/S \oplus W).$$

But $C \leq_{CSt} U$, therefore $C = S \oplus W$. This implies that $(A \oplus B) = (S \oplus W)$, but $S \leq A$ and $W \leq B$, therefore $A = S$ and $B = W$. Thus $A \leq_{CSt} X$ and $B \leq_{CSt} Y$. □

Proposition 2.27. *Let U be a multiplication module with $U = X \oplus Y$ where X and Y be two R-modules and $ann_R X + ann_R Y = R$. Then a submodule $C \leq_{CSt} U$ if and only if there exist CSt-closed submodules A, B of X and Y respectively such that $C = A \oplus B$.*

Proof. Assume that $C \leq_{CSt} U$. since $ann_R X + ann_R Y = R$, then by ([1], Prop.(4.2)) there exist submodules A, B of X and Y respectively such that $C = A \oplus B$. By Proposition 2.26 both of A and B are CSt-closed submodules X and Y respectively. Conversely, in order to prove that $C \leq_{CSt} U$; suppose that $C/W \ll_P U/W$, where $W \leq C \leq U$. Since $ann_R X + ann_R Y = R$, so by the same proof of ([1], Prop(4.2)), there exist $W_1 \leq X$ and $W_2 \leq Y$ such that $W = W_1 \oplus W_2$. Now,

$$C/W = (A \oplus B)/(W_1 + W_2) \ll_P (X \oplus Y)/(W_1 + W_2)$$

implies to:

$$A/W_1 \oplus B/W_2 \ll_P X/W_1 \oplus Y/W_2.$$

But U is multiplication, thus by [13]:

$$A/W_1 \oplus B/W_2 \ll X/W_1 \oplus Y/W_2.$$

Hence:

$$A/W_1 \ll X/W_1$$

and $B/W_2 \ll Y/W_2$ ([10], P.20) Again, U is multiplication, so that:

$$A/W_1 \ll_P X/W_1$$

and

$$B/W_2 \ll_P Y/W_2.$$

Since A and B are CSt-closed submodules of X and Y respectively, thus $A = W_1$ and $B = W_2$, and hence $C = A \oplus B$, thus $C = W_1 \oplus W_2 = W$, hence $C \leq_{CSt} U$.□

3. Cosemi-extending module

In this section, we dualize the concept of semi-extending modules which is appeared in [7]. We start by the following definition.

Definition 3.1. *An R-module U is called cosemi-extending if every CSt-closed submodule of U is direct summand.*

Remarks and examples 3.2.

i. It is clear that every coextending module is cosemi-extending, where "an R-module U is called coextending (or CCS-module), if every coclosed submodule of U is a direct summand of U" ([12]). This is follows from the direct implication between CSt-closed and coclosed submodules.

ii. Z is cosemi-extending Z-module. In fact, the only CSt-closed submodule of Z is (0) which is a direct summand of Z.

iii. Hollow modules is cosemi-extending, since it is coextending ([12]). In particular Z_{P^∞} is cosemi-extending module.

iv. $Z_2 \oplus Z_4$ is a cosemi-extending Z-module, since it is coextending ([12]).

v. Every simple module is cosemi-uniform.

Proposition 3.3. *Let U be an R-module, if every submodule of U is cosemi-essential in a direct summand of U, then U is a cosemi-extending module.*

Proof. Suppose that V is a CSt-closed submodule of U. By hypothesis, $V \leq_{cosm} K$ in U, where K is a direct summand of U. But V is CSt-closed in U, thus $V = K$, that is V is a direct summand of U. □

By using Proposition 2.25, the following gives a partial characterization of cosemi-extending module.

Theorem 3.4. *A multiplication (finitely generated) module U is cosemi-extending if and only if every submodule is cosemi-essential in a direct summand of U.*

Proof. Assume that U is a cosemi-extending module, and let V be a submodule of U. In case $V = (0)$, then clearly V is a cosemi-essential submodule in a direct summand of U. Otherwise; since U is multiplication (finitely generated) so by Proposition 2.25, there exists an St-closed submodule K in U such that $V \leq_{cosm} K$ in U. By hypotheses, K is a direct summand of U, therefore $V \leq_{cosm} K$ in U. The converse is just Proposition 3.3. □

Proposition 3.5. *Let U be a multiplication (finitely generated) and cosemi-extending module. For every submodules X and Y of U. If $X \cap Y \leq_{CSt} U$, then $X \cap Y$ is a direct summand of X and Y.*

Proof. We have to show that $X \cap Y$ is a direct summand of X. Since U is a multiplication (finitely generated) and cosemi-extending module, then by Theorem 3.4, $X \cap Y$ is cosemi-essential in a direct summand of U. But $X \cap Y \leq X \leq U$, so clearly $X \cap Y$ is a direct summand of X. □

Proposition 3.6. *A direct summand of multiplication cosemi-extending module is a cosemi-extending module.*

Proof. Let U be an R-module, and V be a direct summand of U. Assume that K is a CSt-closed submodule of V. Since U is multiplication and V is a direct summand of U, then by Proposition 2.4, V is a CSt-closed submodule of U, and by Corollary 2.16, K is a CSt-closed submodule of U. But U is cosemi-extending, then $U = L \oplus K$ for some submodule L of U. Now, $V = U \cap V = (K \oplus L) \cap V = K \oplus (L \cap V)$ by Modular Law. Thus K is a direct summand of V, that is, i.e V is a cosemi-extending module. □

As a consequence of Proposition 3.6 we have the following. Before that, an R-module U is said to be projective if every short exact sequence of the form:

$$(0) \to W {\to} V {\to} U \to (0)$$

splits ([16], P.23).

Corollary 3.7. *Let $f : U_1 \to U_2$ be an epimorphism from an R-module U_1 to a projective R-module U_2. If U_1 is a multiplication and cosemi-extending module, then U_2 is cosemi- extending.*

Proof. Consider the following short exact sequence:

$$(0) \to kerf \xrightarrow{i} U_1 \xrightarrow{f} U_2 \to (0),$$

where i is the inclusion homomorphism. Since U_2 is a projective, then the sequence splits. This implies that $U_1 \cong kerf \oplus U_2$, so U_2 is isomorphic to a direct summand of U_1. Since U_1 is multiplication and cosemi-extending, so by proposition 3.6, U_2 is cosemi-extending. □

Corollary 3.8. *Let U be a multiplication and cosemi-extending R-module, and V is a CSt-closed submodule of U, then U/V is cosemi-extending.*

Proof. Since V is a CSt-closed submodule of U, and U is cosemi-extending, then V is a direct summand of U, so $U = V \oplus L$ for some submodule W of U. This implies that $U/V \cong L$. But L is a direct summand of U, so by Proposition 3.6, U/V is cosemi-extending. □

Recall that an R-module U is called free if it has a basis ([16], P.21).

Proposition 3.9. *Let U be a multiplication R-module, then every free R-module is cosemi-extending if and only if every projective R-module is cosemi-extending.*

Proof. For the necessity; let U be a projective R-module, then U is an epimorphic image of a free R-module say F ([16], P.23). By the hypothesis, F is a cosemi-extending module. But U is multiplication, then by Corollary 3.7, U is cosemi-extending. The converse is straightforward. □

"Recall that an R-module U is called duo, if every submodule of U is a fully invariant" ([2]). The following theorem deals with the direct sum of two cosemi-extending modules.

Theorem 3.10. *Let $U = X \oplus Y$ be a duo module, where X and Y are R-modules. Assume that $ann_R X + ann_R Y = R$. If X and Y are cosemi-extending module then U is cosemi-extending. the converse is true when U is multiplication*

Proof. Assume that X and Y are cosemi-extending, and let $V \leq_{CSt} U$. Since U is a duo module, then V is fully invariant, hence $V = (V \cap X) \oplus (V \cap Y)$ ([2]). By Proposition 2.8(1), $V \cap X \leq_{CSt} X$ and $V \cap Y \leq_{CSt} Y$ respectively. But X and Y are cosemi-extending, so $V \cap X \oplus S = X$ and $V \cap Y \oplus T = Y$ for some $S \leq X$ and $T \leq Y$. This implies that:

$$X \oplus Y = (V \cap X \oplus S) \oplus (V \cap Y \oplus T),$$

$$U = [(V \cap X) \oplus (V \cap Y)] \oplus (S \oplus T).$$

If we put $W = S \oplus T$, then $U = V \oplus W$, thus V is a direct summand of U, hence U is cosemi-extending. For the converse; since U is multiplication, then the result follows by Proposition 3.6. □

Since every multiplication module is duo, then form Theorem 3.10 we deduce the following.

Corollary 3.11. *Let U be a multiplication module such that $U = X \oplus Y$, where X and Y be R-modules. If $ann_R X + ann_R Y = R$, then U is a cosemi-extending module if and only if X and Y are cosemi-extending.*

4. Cosemi-extending module related concepts

This section deals with the relationships between cosemi-extending module and other related concepts such as semisimple, Pr-hollow module, cosemi-uniform, Pr-lifting and St-semisimple modules.

Remark 4.1. Every semisimple module is cosemi-extending. This follows by every submodule of semisimple module is CSt-closed. The converse is not true in general, for example: Z_{12} is cosemi-extending Z-module, but it is not semisimple.

"An R-module U is said to be Pr-hollow if every prime submodule of U is small submodule" ([3]).

Proposition 4.2. *Every Pr-hollow module is cosemi-extending module.*

Proof. Assume that U is a Pr-hollow module. By Remark 2.10; the only CSt-closed submodule of Pr-hollow module is zero, hence the result follows. □

The converse of Proposition 4.2 is not true in general, for example; Z is a cosemi-extending Z-module, but not Pr-hollow module ([3], (1.2)(2)).

"Recall that a non-zero R-module U is called cosemi-uniform, if every proper submodule V of U is either zero or there exists a proper submodule S of V such that $V/S \ll_P U/S$ " ([5]).

Proposition 4.3. *Every cosemi-uniform module is cosemi-extending module.*

Proof. Let U be a cosemi-extending module, and V be a submodule of U. If $V = (0)$ then either V is CSt-closed or not, in each case V is a direct summand of U. If $V \neq (0)$, since U is a cosemi-uniform module so there exists a proper W of V such that $V/W \ll_P U/W$, therefore, V is not CSt-closed in U. So U has a non-zero CSt-closed submodule, thus U is cosemi-extending.

The converse of Proposition 4.3 is not true in general, for example the Z-module Z_{10} is a cosemi-extending module because it is a semisimple module, but not a cosemi-uniform module, see ([5], Rem (3.2)(3)).

"A non-zero module U is called couniform, if every proper submodule V of U is either zero or there exists a proper submodule W of V such that $V/W \ll U/W$. That is for each proper submodule V of U, either $N = (0)$ or there exists a proper submodule W of V such that $W \leq_{ce} V in U$" ([11]).

Since every couniform module is cosemi-uniform, then we have the following.

Corollary 4.4. *Every couniform module is cosemi-extending.*

The converse of Corollary 4.4 is not true in general. In fact the Z-module Z_6 s not couniform module ([11], Rem.(1.2)(2)), while Z_6 is cosemi-extending because it is semisimple.

"Recall that a module U is called lifting, if for every submodule V of U there exists a direct summand W of U such that $W \leq_{ce} V$ in U" ([14]). This motivated us to define the following.

Definition 4.5. *An R-module U is called Pr-lifting, if for every submodule V of U there exists a direct summand W of U such that $W \leq_{cosm} V$ in U.*

This concept is clearly a proper subclass of lifting module, and we can prove the following.

Proposition 4.6. *If a module U is a Pr-lifting module, then U is cosemi-extending.*

Proof. Let V be a CSt-closed submodule of U. Since U is a Pr-lifting module, so there exists a direct summand W of U, such that $W \leq_{cosm} V$ in U, that is $V/W \ll_P U/W$. But $V \leq_{CSt} U$, then $V = W$. That is U is a cosemi-extending module. □

The converse of Proposition 4.6 is not true in general, for example: Z as Z-module is a cosemi-extending module, but it is not Pr-lifting.

"An R-module U is called St-semisimple if every submodule of U is St-closed" ([4]). As a dual of this concept, we introduce the following.

Definition 4.7. *An R-module U is called CSt-semisimple, if every submodule of U is CSt-closed.*

The following theorem gives some useful relationships of a cosemi-essential module with some related concepts.

Theorem 4.8. *If U is CSt-semisimple, then the following statements are equivalent.*

1. *U is a Pr-lifting module.*
2. *U is a cosemi-extending module.*
3. *U is a semisimple module.*

Proof. (1) $\Rightarrow$ (2) It is just Proposition 4.6.

(2) $\Rightarrow$ (3) Let V be a submodule of U, since U is CSt-semisimple, then $V \leq_{CSt} U$. But U is cosemi-extending, therefore V is a direct summand of U.

(3) $\Rightarrow$ (1) Let V be a submodule of U, by (3), V is a direct summand of U. On the other hand, $V \leq_{cosm} V$ in U ([5], Rem (2.3)(6)). So U satisfies the definition of Pr-lifting, and we are done. □

Hadi and Ibrahiem in [13] defined P-Rad(U) as a summation of all P-small submodules of U, so we have the following.

Theorem 4.9. *Let R be a ring such that $P - Rad(R/A) = 0$, for each ideal A of R. Then every R-module is a cosemi-extending module if and only if R is a semisimple ring.*

Proof. Assume that every R-module is cosemi-extending, then R is a cosemi-extending R-module. So if I is a CSt-closed ideal of R, then I is a direct summand of R. Assume that I is not CSt-closed ideal in R, then there exists a proper ideal A of I such that $I/A \ll_P R/A$. This implies that $I/A \leq P - Rad(R/A) = 0$, thus $I/A = 0$, hence $A = I$ which a contradiction, therefore $I \leq_{CSt} R$. Since R is cosemi-extending R-module, then I is a direct summand of R. i.e R is a semisimple ring. Conversely; since R is a semisimple ring, then every R-module U is semisimple, hence every submodule of U is direct summand. In particular; every CSt-closed submodule of U is direct summand. □

If the condition "semisimple ring" in Theorem 4.9 is replaced by "St-semisimple", then we need to add another condition as the following theorem shows.

Theorem 4.10. *Let R be a ring such that $P - Rad(R/A) = 0$, for each ideal A of R. Then every finitely generated R-module is a cosemi-extending module if and only if R is an St-semisimple ring.*

Proof. ⇒) It is as the same proof of Theorem 4.9.

⇐) Since R is finitely generated, then the concept of a CSt-closed submodule coincide with coclosed submodule, see Corollary 2.5, and according that, there is no difference between St-semisimple and semisimple modules, thus by Theorem 4.9, every R-module (hence every finitely generated) is cosemi-extending. □

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