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# β-Closed Topological Spaces in Terms of Grills

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## ABSTRACT

The concept of β-closedness, a kind of covering property for topological spaces, has already been studied with meticulous care from different angles and via different approaches. In this paper, we continue the said investigation in terms of a different concept viz. grills. The deliberations in the article include certain characterizations and a few necessary conditions for the β-closedness of a space, the latter conditions are also shown to be equivalent to β-closedness in a β-almost regular space. All these and the associated discussions and results are done with grills as the prime supporting tool.

**Key words:** β-open set, β-closure; β-convergence; Grill; β(θ)-convergence and β(θ)-adherence of a grill; β-closed space.

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## 1. INTRODUCTION AND PRELIMINARIES

Generalized open sets play a very important role in general topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in general topology and real analysis concerns the variously modified forms of continuity, separation axioms etc. by utilizing generalized open sets. For a subset A of a topological space (X, τ), Cl(A) and Int(A) denote the closure of A and the interior of A, respectively. A subset A of a topological space (X, τ) is called a β-open [1] (= semi-pre-open [5]) set if A ⊆ Cl(Int(Cl(A))). The complement of a β-open set is called a β-closed set. The intersection of all β-closed sets of X containing A is called the β-closure [1] of A and is denoted by βCl(A). For each x ∈ X, the family of all β-open sets of (X, τ) containing a point x is denoted by βO(X, x). The β-interior of A is the union of all β-open sets contained in A and is denoted by βInt(A). A set A is called a β-regular set [7] if it is both β-open and β-closed. The β-θ-closure [7] of a subset A, denoted by $\beta Cl_\theta(A)$, is the set of all x ∈ X such that βCl(U) ∩ A ≠ ϕ for every U ∈ βO(X, x). A subset A is called β-θ-closed [7] if A = $\beta Cl_\theta(A)$. By [7], it is proved that, for a subset A, $\beta Cl_\theta(A)$ is the intersection of all β-θ-closed sets containing A. The complement of a β-θ-closed set is called a β-θ-open set. In [3], the authors introduced the notion of β-closed spaces and investigated its fundamental properties. In this paper, we investigate some more properties of this type of closed spaces via grills.

## 2. GRILLS: β(θ)-ADHERENCE AND β(θ)-CONVERGENCE

we shall define the β(θ)-adherence and β(θ)-convergence of a grill, and develop the concept to some extent so that the results we derive here may support our subsequent deliberations.

**Definition 2.1.** [6] A grill $\mathcal{G}$ on a topological space X is defined to be a collection of nonempty subsets of X such that

(a) A ∈ $\mathcal{G}$ and A ⊆ B ⊆ X ⇒ B ∈ $\mathcal{G}$ and

(b) A, B ⊆ X and A ∪ B ∈ $\mathcal{G}$ ⇒ A ∈ $\mathcal{G}$ or B ∈ $\mathcal{G}$.

**Definition 2.2.** A grill $\mathcal{G}$ on a topological space X is said to be:

(a) β(θ)-adhere at x ∈ X if for each U ∈ βO(X, x) and each G ∈ $\mathcal{G}$, βClU ∩ G ≠ ϕ.

(b) β(θ)-converge to a point x ∈ X if for each U ∈ βO(X, x), there is some G ∈ $\mathcal{G}$, such that G ⊆ βCl(U).

**Remark 2.3.** A grill $\mathcal{G}$ is β(θ)-convergent to a point x ∈ X if and only if $\mathcal{G}$ contains the collection {βCl(U) : U ∈ βO(X, x)}.

**Definition 2.4.** A filter $\mathcal{F}$ on a topological space X is said to β(θ)-adhere at x ∈ X (β(θ)-converge to x ∈ X) if for each F ∈ $\mathcal{F}$ and each U ∈ βO(X, x), F ∩ βCl(U) ≠ ϕ (resp., to each U ∈ βO(X, x), there corresponds F ∈ $\mathcal{F}$ such that F ⊆ βCl(U)).

**Definition 2.5.** [8] If $\mathcal{G}$ is a grill (or a filter) on a topological space X, then the section of $\mathcal{G}$, denoted by sec$\mathcal{G}$, is given by,

$$\sec\mathcal{G} = \{A \subseteq X : A \cap G \neq \phi \text{ for all } G \in \mathcal{G}\}.$$

**Theorem 2.6.** [8] Let X be a topological space. Then we have

(a) For any grill (filter) $\mathcal{G}$ on X, sec$\mathcal{G}$ is a filter (resp., grill) on X.

(b) If $\mathcal{F}$ and $\mathcal{G}$ are respectively a filter and a grill on X with $\mathcal{F}$ ⊆ $\mathcal{G}$, then there is an ultrafilter $\mathcal{U}$ on X such that $\mathcal{F} \subseteq \mathcal{U} \subseteq \mathcal{G}$.

**Theorem 2.7.** If a grill $\mathcal{G}$ on a topological space X, β(θ)-adheres at some point x ∈ X, then $\mathcal{G}$ is β(θ)-converges to x.

**Proof.** Let a grill $\mathcal{G}$ on X, β(θ)-adheres at some point x ∈ X. Then for each U ∈ βO(X, x) and each G ∈ $\mathcal{G}$, βCl(U) ∩ G ≠ ϕ so that βCl(U) ∈ sec$\mathcal{G}$ for each U ∈ βO(X, x), and hence X \ βCl(U) ∉ $\mathcal{G}$. Then βCl(U) ∈ $\mathcal{G}$ (as $\mathcal{G}$ is a grill and X ∈ $\mathcal{G}$) for each U ∈ βO(X). Hence $\mathcal{G}$ must β(θ)-converge to x.

The following Example shows that a β(θ)-convergent grill need not β(θ)-adhere at any

point of the space even if the space is finite.

**Example 2.8.** Let X = {a, b, c}, τ = {X, ϕ, {b}, {c}, {b, c}, {a, c}}. Then (X, τ) is a topological space such that βO(X) = τ. Let

$$\mathcal{G} = \{\{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, X\}$$

Then the grill $\mathcal{G}$ is β(θ)-convergent but not β(θ)-adheres.

**Definition 2.9.** Let X be a topological space. Then for any x ∈ X, we adopt the following notation:

$$\mathcal{G}(\beta(\theta), x) = \{A \subseteq X : x \in \beta Cl_\theta(A)\},$$

$sec\mathcal{G}(\beta(\theta), x) = \{A \subseteq X : A \cap G \neq \phi, \text{ for all } G \in \mathcal{G}(\beta(\theta), x)\}$.

**Theorem 2.10.** A grill $\mathcal{G}$ on a topological space X, $\beta(\theta)$-adheres to a point x of X if and only if $\mathcal{G} \subseteq sec\mathcal{G}(\beta(\theta), x)$.

**Proof.** A grill $\mathcal{G}$ on a topological space X, $\beta(\theta)$-adheres to a point x of X, we have $\beta Cl(U) \cap G \neq \phi$ for all $U \in \beta O(X, x)$ and all $G \in \mathcal{G}$; hence $x \in \beta Cl_\theta(G)$ for all $G \in \mathcal{G}$. Then $G \in \mathcal{G}(\beta(\theta), x)$, for all $G \in \mathcal{G}$; hence $\mathcal{G} \subseteq \mathcal{G}(\beta(\theta), x)$. Conversely, let $\mathcal{G} \subseteq \mathcal{G}(\beta(\theta), x)$. Then for all $G \in \mathcal{G}$, $\beta Cl(U) \cap G \neq \phi$, so that for all $U \in \beta O(X, x)$ and for all $G \in \mathcal{G}$, $\beta Cl(U) \cap G \neq \phi$. Hence $\mathcal{G}$ $\beta(\theta)$-adheres at x.

**Theorem 2.11.** A grill $\mathcal{G}$ on a topological space X, $\beta(\theta)$-convergent to a point x of X if and only if $sec\mathcal{G}(\beta(\theta), x) \subseteq \mathcal{G}$.

**Proof.** Let $\mathcal{G}$ be a grill on a topological space X, $\beta(\theta)$-convergent to a point $x \in X$. Then for each $U \in \beta O(X, x)$ there exists $G \in \mathcal{G}$ such that $G \subseteq \beta Cl(U)$, and hence $\beta Cl(U) \in \mathcal{G}$ for each $U \in \beta O(X, x)$. Now, $B \in sec\mathcal{G}(\beta(\theta), x) \Rightarrow X \setminus B \notin \mathcal{G}(\beta(\theta), x) \Rightarrow x \notin \beta Cl_\theta(X \setminus B) \Rightarrow$ there exists $U \in \beta O(X, x)$ such that $\beta Cl(U) \cap (X \setminus B) = \phi \Rightarrow \beta Cl(U) \subseteq B$, where $U \in \beta O(X, x) \Rightarrow B \in \mathcal{G}$. Conversely, let if possible, $\mathcal{G}$ not to $\beta(\theta)$-converge to x. Then for some $U \in \beta O(X, x)$, $\beta Cl(U) \notin \mathcal{G}$ and hence $\beta Cl(U) \notin sec\mathcal{G}(\beta(\theta), x)$. Thus for some $A \in \mathcal{G}(\beta(\theta), x)$, $A \cap \beta Cl(U) = \phi$. But $A \in \mathcal{G}(\beta(\theta), x) \Rightarrow x \in \beta Cl_\theta(A) \Rightarrow \beta Cl(A) \cap U \neq \phi$.

## 3. β-CLOSEDNESS AND GRILLS

As proposed earlier, in this section we investigate β-closedness of a topological space in terms of grills. We begin by recalling the definition of β-closedness from.

**Definition 3.1.** [3] A nonempty subset A of a topological space X is called β-closed relative to X if for every cover $\mathcal{U}$ of A by β-open sets of X, there exists a finite subset $\mathcal{U}_0$ of $\mathcal{U}$ such that $A \subseteq \cup\{\beta Cl U : U \in \mathcal{U}_0\}$. If, in addition, A = X, then X is called a β-closed space.

**Theorem 3.2.** For a topological space X, the following statements are equivalent:

(a) X is β-closed;

(b) Every maximal filter base $\beta(\theta)$-converges to some point of X;

(c) Every filter base $\beta(\theta)$-adhere to some point of X;

(d) For every family $\{V_\alpha : \alpha \in I\}$ of β-closed sets that $\cap\{V_\alpha : \alpha \in I\} = \phi$, there exists a finite subset $I_0$ of I such that $\cap\{\beta Int(V_\alpha) : \alpha \in I_0\} = \phi$.

**Proof.** (a) ⇒ (b): Let $\mathcal{F}$ be a maximal filter base on X. Suppose that $\mathcal{F}$ does not β-converge to any point of X. Since $\mathcal{F}$ is maximal, $\mathcal{F}$ does not β-θ-accumulate at any point of X. For each $x \in X$, there exist $F_x \in \mathcal{F}$ and $V_x \in \beta O(X, x)$ such that $\beta Cl(V_x) \cap F_x = \phi$. The family $\{V_x : x \in X\}$ is a cover of X by β-open sets of X. By (a), there exists a finite number of points $x_1, x_2, ..., x_n$ of X such that $X = \cup\{\beta Cl(V_{x_i}) : i = 1, 2, ..., n\}$. Since $\mathcal{F}$ is a filter base on X, there exists $F_0 \in \mathcal{F}$ such that $F_0 \subseteq \cap\{F_{x_i} : i = 1, 2, ..., n\}$. Therefore, we obtain $F_0 = \phi$. This is a contradiction.

(b) ⇒ (c): Let $\mathcal{F}$ be any filter base on X. Then, there exists a maximal filter base $\mathcal{F}_0$ such that $\mathcal{F} \subseteq \mathcal{F}_0$. By (b), $\mathcal{F}_0$ β-θ-converges to some point $x \in X$. For every $F \in \mathcal{F}$ and every $V \in \beta O(X, x)$, there exists $F_0 \in \mathcal{F}_0$ such that $F_0 \subseteq \beta Cl(V)$; hence $\phi \neq F_0 \cap F \subseteq \beta Cl(V) \cap F$. This shows that $\mathcal{F}$ β-θ-accumulates at x.

(c) ⇒ (d): Let $\{V_\alpha : \alpha \in I\}$ be any family of β-closed subsets of X such that $\cap\{V_\alpha : \alpha \in I\} = \phi$. Let $\Gamma(I)$ denote the ideal of all finite subsets of A. Assume that $\cap\{\beta Int(V_\alpha) : \alpha \in I\} = \phi$ for every $I \in \Gamma(I)$. Then, the family $\mathcal{F} = \{\cap_{\alpha\in I} \beta Int(V_\alpha) : I \in \Gamma(I)\}$ is a filter base on X. By (c), $\mathcal{F}$ β-θ-accumulates at some point $x \in X$. Since $\{X \setminus V_\alpha : \alpha \in I\}$ is a cover of X, $x \in X \setminus V_{\alpha_0}$ for some $\alpha_0 \in I$. Therefore, we obtain $X \setminus V_{\alpha_0} \in \beta O(X, x)$, $\beta Int(V_{\alpha_0}) \in \mathcal{F}$ and $\beta Cl(X \setminus V_{\alpha_0}) \cap \beta Int(V_{\alpha_0}) = \phi$, which is a contradiction.

(d) ⇒ (a): Let $\{V_\alpha : \alpha \in I\}$ be a cover of X by β-open sets of X. Then $\{X \setminus V_\alpha : \alpha \in I\}$ is a family of β-closed subsets of X such that $\cap\{X \setminus V_\alpha : \alpha \in I\} = \phi$. By (d), there exists a finite subset $I_0$ of I such that $\cap\{\beta Int(X \setminus V_\alpha) : \alpha \in I\} = \phi$; hence $X = \cup\{\beta Cl(V_\alpha) : \alpha \in I_0\}$. This shows that X is β-closed.

**Theorem 3.3.** A topological space X is β-closed if and only if every grill on X is $\beta(\theta)$-convergent in X.

**Proof.** Let $\mathcal{G}$ be any grill on a β-closed space X. Then by Theorem 2.6(a), $sec\mathcal{G}$ is a filter on X. Let $B \in sec\mathcal{G}$, then $X \setminus B \notin \mathcal{G}$ and hence $B \in \mathcal{G}$ (as $\mathcal{G}$ is a grill). Thus $sec\mathcal{G} \subseteq \mathcal{G}$. Then by Theorem 2.6(b), there exists an ultrafilter $\mathcal{U}$ on X such that $sec\mathcal{G} \subseteq \mathcal{U} \subseteq \mathcal{G}$. Now as X is β-closed, in view of Theorem 3.2, the ultrafilter $\mathcal{U}$ is $\beta(\theta)$-convergent to some point $x \in X$. Then for each $U \in \beta O(X, x)$, there exists $F \in \mathcal{U}$ such that $F \subseteq \beta Cl(U)$. Consequently, $\beta Cl(U) \in \mathcal{U} \subseteq \mathcal{G}$, That is $\beta Cl(U) \in \mathcal{G}$, for each

$U \in \beta O(X, x)$. Hence $\mathcal{G}$ is $\beta(\theta)$-convergent to x. Conversely, let every grill on X be $\beta(\theta)$-convergent to some point of X. By virtue of Theorem 3.2 it is enough to show that every ultrafilter on X is $\beta(\theta)$-converges in X, which is immediate from the fact that an ultrafilter on X is also a grill on X.

**Theorem 3.4.** A topological space X is $\beta$-closed relative to X if and only if every grill $\mathcal{G}$ on X with $A \in \mathcal{G}$, $\beta(\theta)$-converges to a point in A.

**Proof.** Let A be $\beta$-closed relative to X and $\mathcal{G}$ a grill on X satisfying $A \in \mathcal{G}$ such that $\mathcal{G}$ does not $\beta(\theta)$-converge to any $a \in A$. Then to each $a \in A$, there corresponds some $U_a \in \beta O(X, a)$ such that $\beta Cl(U_a) \notin \mathcal{G}$. Now $\{U_a : a \in A\}$ is a cover of A by $\beta$-open sets of X. Then $A \subseteq \bigcup_{i=1}^{n} \beta Cl(U_{a_i}) = U$ (say) for some positive integer n. Since $\mathcal{G}$ is a grill, $U \notin \mathcal{G}$; hence $A \notin \mathcal{G}$, which is a contradiction. Conversely, let A be not $\beta$-closed relative to X. Then for some cover $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ of A by $\beta$-open sets of X, $\mathcal{F} = \{A \setminus \bigcup_{\alpha \in I_0} \beta Cl(U_\alpha) : I_0$ is finite subset of I$\}$ is a filterbase on X. Then the family $\mathcal{F}$ can be extended to an ultrafilter $\mathcal{F}^*$ on X. Then $\mathcal{F}^*$ is a grill on X with $A \in \mathcal{F}^*$ (as each F of $\mathcal{F}$ is a subset of A). Now for each $x \in A$, there must exists $\lambda \in I$ such that $x \in U_\lambda$, as $\mathcal{U}$ is a cover of A. Then for any $G \in \mathcal{F}^*$, $G \cap (A \setminus \beta Cl(U_\lambda)) \neq \phi$, so that $G \supseteq \beta Cl(U_\lambda)$ for all $G \in \mathcal{G}$. Hence $\mathcal{F}^*$ cannot $\beta(\theta)$-converge to any point of A. The contradiction proves the desired result.

**Theorem 3.5.** If X is any topological space such that every grill $\mathcal{G}$ on X with the property that $\bigcap_{i=1}^{n} \beta Cl_\theta(G_i) \neq \phi$ for every finite subfamily $\{G_1, G_2, \ldots, G_n\}$ of $\mathcal{G}$, $\beta(\theta)$-adheres in X, then X is a $\beta$-closed space.

**Proof.** Let $\mathcal{U}$ be an ultrafilter on X. Then $\mathcal{U}$ is a grill on X and also for each finite subcollection $\{U_1, U_2, \ldots, U_n\}$ of $\mathcal{U}$, $\bigcap_{i=1}^{n} \beta Cl_\theta(U_i) \supseteq \bigcap_{i=1}^{n} U_i \neq \phi$, so that $\mathcal{U}$ is a grill on X with the given condition. Hence by hypothesis, $\mathcal{U}$, $\beta(\theta)$-adheres. Consequently, by Theorem 3.2, X is $\beta$-closed.

**Theorem 3.6.** [7] For any $A \subseteq X$, $\beta Cl_\theta(A) = \cap\{\beta ClU : A \subseteq U \in \beta O(X)\}$.

**Definition 3.7.** A grill $\mathcal{G}$ on a topological space X is said to be:

(a) $\beta(\theta)$-linked if for any two members $A, B \in \mathcal{G}$, $\beta Cl_\theta(A) \cap \beta Cl_\theta(B) \neq \phi$,

(b) $\beta(\theta)$-conjoint if for every finite subfamily $A_1, A_2, \ldots, A_n$ of $\mathcal{G}$, $\beta Int(\bigcap_{i=1}^{n} \beta Cl_\theta(A_i)) \neq \phi$.

It is clear that every $\beta(\theta)$-conjoint grill is $\beta(\theta)$-linked. The following Example shows that the converse is need not be true in general.

**Example 3.8.** Let $X = \{a, b, c\}$, $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Then $(X, \tau)$ is a topological space such that $\beta O(X) = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Let

$$\mathcal{G} = \{\{c\}, \{b, c\}, \{a, c\}, X\}.$$

Then $\mathcal{G}$ is a grill on X. For any $A \in \mathcal{G}$, we have

$\beta Cl_\theta\{c\} = \{c\}$, $\beta Cl_\theta\{b, c\} = \{b, c\}$, $\beta Cl_\theta\{a, c\} = \{a, c\}$ and $\beta Cl_\theta X = X$. It can then be verified that the grill $\mathcal{G}$ is $\beta(\theta)$-linked but not $\beta(\theta)$-conjoint.

**Theorem 3.9.** In a $\beta$-closed space X, every $\beta(\theta)$-conjoint grill $\beta(\theta)$-adheres in X.

**Proof.** Consider any $\beta(\theta)$-conjoint grill $\mathcal{G}$ on a $\beta$-closed space X. We first note from Theorem 3.5 that for $A \subseteq X$, $\beta Cl_\theta(A)$ is $\beta$-closed (as an arbitrary intersection of $\beta$-closed sets is $\beta$-closed). Thus $\{\beta Cl_\theta(A) : A \in \mathcal{G}\}$ is a collection of $\beta$-closed sets in X such that $\beta Int(\bigcap_{i=1}^{n} \beta Cl_\theta(A_i)) \neq \phi$ for any finite subcollection $A_1, A_2, \ldots, A_n$ of $\mathcal{G}$. Then $\beta Int(\bigcap_{i=1}^{n} \beta Cl_\theta(A_i)) \neq \phi$ for any finite subcollection $A_1, A_2, \ldots, A_n$ of $\mathcal{G}$. Thus by Theorem 3.2, $\bigcap_{\alpha \in I}\{\beta Cl_\theta(A) : A \in \mathcal{G}\} \neq \phi$, That is there exists $x \in X$ such that $x \in \beta Cl_\theta(A)$ for all $A \in \mathcal{G}$. Hence $\mathcal{G} \subseteq \mathcal{G}(\beta(\theta), x)$ so that by Theorem 2.10, $\mathcal{G}$, $\beta(\theta)$-adheres at $x \in X$.

**Definition 3.10.** A subset A of a topological space X is called $\beta$-regular open if $A = \beta Int(\beta Cl(A))$. The complement a $\beta$-regular open set is called a $\beta$-regular closed set.

**Definition 3.11.** A topological space X is called $\beta$-almost regular if for each $x \in X$ and each $\beta$-regular open set V in X with $x \in V$, there is a $\beta$-regular open set U in X such that $x \in U \subseteq \beta Cl(U) \subseteq V$.

**Theorem 3.12.** In a $\beta$-almost regular $\beta$-closed space X, every grill $\mathcal{G}$ on X with the property

$\bigcap_{i=1}^{n} \beta Cl_\theta(G_i) \neq \phi$ for every finite subfamily $\{G_1, G_2, \ldots, G_n\}$ of $\mathcal{G}$, $\beta(\theta)$-adheres in X.

**Proof.** Let X be a β-almost regular β-closed space and $\mathcal{G} = \{G_\alpha : \alpha \in I\}$ a grill on X with the property that $\bigcap_{\alpha \in I_0} \beta Cl_\theta(G_\alpha) \neq \phi$ for every finite subset $I_0$ of I. We consider $\mathcal{F} = \{\bigcap_{\alpha \in I_0} \beta Cl_\theta(G_\alpha) : I_0$ is a finite subfamily of I$\}$. Then $\mathcal{F}$ is a filterbase on X. By the β-closedness of X, $\mathcal{F}$, β(θ)-adheres at some x ∈ X, that is, $x \in \beta Cl_\theta(\beta Cl_\theta(G))$ for all $G \in \mathcal{F}$, that is, $F \subseteq (\beta(\theta), x)$. Hence by Theorem 2.10, $\mathcal{F}$ β(θ)-adheres at x ∈ X.

**Corollary 3.13.** In a β-almost regular space X, the following statements are equivalent:

(a) Every grill $\mathcal{G}$ on X with the property that $\bigcap_{i=1}^{n} \beta Cl_\theta(G_i) \neq \phi$ for every finite subfamily $\{G_1, G_2, \dots, G_n\}$ of $\mathcal{G}$, β(θ)-adheres in X.

(b) X is β-closed.

(c) Every β(θ)-conjugate grill β(θ)-adheres in X.

**Theorem 3.14.** Every grill $\mathcal{G}$ on a topological space X with the property that $\bigcap\{\beta Cl_\theta(G) : G \in \mathcal{G}_0\} \neq \phi$ for every finite subsets $\mathcal{G}_0$ of $\mathcal{G}$, β(θ)-adheres in X if and only if for every family $\mathcal{F}$ of subsets of X for which the family $\{\beta Cl_\theta(F) : F \in \mathcal{F}\}$ has the finite intersection property, we have $\bigcap\{\beta Cl_\theta(F) : F \in \mathcal{G}\} \neq \phi$.

**Proof.** Let every grill on a topological space X satisfying the given condition, β(θ)-adhere in X, and suppose that $\mathcal{F}$ is a family of subsets of X such that the family $\mathcal{F}^* = \{\beta Cl_\theta(F) : F \in \mathcal{F}\}$ has the finite intersection property. Let $\mathcal{U}$ be the collection of all those families G of subsets of X for which $\mathcal{G}^* = \{\beta Cl_\theta(G) : G \in \mathcal{G}\}$ has the finite intersection property and F ⊆ G. Then $F \in \mathcal{U}$ is a partially ordered set under set inclusion in which every chain clearly has an upper bound. By Zorn's lemma, $\mathcal{F}$ is then contained in a maximal family $\mathcal{U}^* \in \mathcal{U}$. It is easy to verify that $\mathcal{U}^*$ is a grill with the stipulated property. Hence

$$\bigcap\{\beta Cl_\theta(F) : F \in \mathcal{F}\} \supseteq \bigcap\{\beta Cl_\theta(U) : U \in \mathcal{U}^*\} \neq \phi.$$

Conversely, if $\mathcal{F}$ is a grill on X with the given property, then for every finite subfamily $\mathcal{F}_0$ of $\mathcal{F}$, $\bigcap\{\beta Cl_\theta(F) : F \in \mathcal{F}\} \neq \phi$. So, by hypothesis, $\bigcap\{\beta Cl_\theta(F) : F \in \mathcal{F}\} \neq \phi$. Hence $\mathcal{F}$, β(θ)-adheres in X.

**Definition 3.15.** A topological space X is called β(θ)-linkage β-closed if every β(θ)-linked grill on X, β(θ)-adheres.

**Theorem 3.16.** Every β(θ)-linkage β-closed space is β-closed.

**Proof.** The proof is clear.

**Proposition 3.17.** [7] Let A be a subset of a topological space (X, τ). Then:

(a) If A ∈ βO(X), then $\beta Cl(A) = \beta Cl_\theta(A)$.

(b) If A is β-regular, then A is β-θ-closed

**Theorem 3.18.** In the class of β-almost regular spaces, the concept of β-closedness and β(θ)-linkage β-closedness become identical.

**Proof.** In view of Theorem 3.16, it is enough to show that a β-almost regular β-closed space is β(θ)-linkage β-closed. Let $\mathcal{G}$ be any β(θ)-linked grill on a β-almost regular β-closed space X such that $\mathcal{G}$ does not β(θ)-adhere in X. Then for each x ∈ X, there exists $G_x \in \mathcal{G}$ such that $x \in \beta Cl_\theta(G_x) = \beta Cl_\theta(\beta Cl_\theta(G_x))$. Then there exists $U_x \in \beta O(X, x)$ such that $\beta Cl(U_x) \cap \beta Cl_\theta(G_x) = \phi$, which gives $\beta Cl_\theta(U_x) \cap \beta Cl_\theta(G_x) = \phi$ by Proposition 3.17, $\beta Cl_\theta(U) = \beta Cl(U))$. Since $\beta Cl_\theta(G_x) \in \mathcal{G}$ and $\mathcal{G}$ is a β(θ)-linked grill on X, $\beta Cl_\theta(U_x) = \beta Cl(U_x) \notin \mathcal{G}$. Now, $\{U_x : x \in X\}$ is a cover of X by β-open sets of X. So by β-closedness of X, $X = \bigcup\{\beta Cl(U_{x_i}) : i = 1, 2, \dots, n\}$, for a finite subset $\{x_1, x_2, \dots, x_n\}$ of X. It is then follows that $X \notin \mathcal{G}$ (since $\beta Cl(U_{x_i}) \notin \mathcal{G}$ for i = 1, 2, ... , n), which is a contradiction. Hence $\mathcal{G}$ must β(θ)-adhere in X, proving X to be β(θ)-linkage β-closed.

**Definition 3.19.** [3] A topological space X is said to be β-compact if every cover $\mathcal{U}$ of X by β-open sets of X has a finite subcover.

**Definition 3.20.** A topological space X is β(θ)-regular if every grill on X which β(θ)-converges must β-converge (not necessarily to the same point), where β-convergence of a grill is defined in the usual way. That is a grill $\mathcal{G}$ on X is said to β-converge to x ∈ X if $\beta O(X, x) \subseteq \mathcal{G}$.

**Theorem 3.21.** A topological space X is β-compact if and only if every grill β-converges.

**Proof.** Let $\mathcal{G}$ be a grill on a β-compact space such that $\mathcal{G}$ does not β-converge to any point x ∈ X. Then for each x ∈ X, there exists $U_x \in \beta O(X, x)$ with (*) $U_x \notin \mathcal{G}$. As $\{U_x : x \in X\}$ is a cover of the β-compact space X by β-open sets, there exist finitely many points $x_1, x_2, \dots, x_n$ in X such that $X = \bigcup_{i=1}^{n} U_{x_i}$. Since $X \in \mathcal{G}$ for some i, ($1 \le i \le n$), $U_{x_i} \in \mathcal{G}$, which goes against

(*). Conversely, let every grill on X β-converge and if possible, let X be not β-compact. Then there exists a cover $\mathcal{U}$ of X by β-open sets of X having no finite subcover. Then

$$\mathcal{F} = \{X \setminus \cup\mathcal{U}_0 : \mathcal{U}_0 \text{ is a finite subcollection of } \mathcal{U}\}$$

is a filterbase on X. Then $\mathcal{F}$ is contained in an ultrafilter $\mathcal{G}$, and then $\mathcal{G}$ β-converges to some point x of X. Then for some U ∈ $\mathcal{U}$, x ∈ $\mathcal{U}$, and hence U ∈ $\mathcal{G}$. But X \ U ∈ $\mathcal{F}$ ⊆ $\mathcal{U}$. Thus U and X \ U both belong to $\mathcal{U}$, which is a filter, so giving a contradiction.

**Theorem 3.22.** A β-compact space X is β-closed, while the converse is also true if X is β(θ)-regular.

**Proof.** The proof is clear.

**Definition 3.23.** [2] A topological space (X, τ) is said to be β-regular if for any closed set F ⊆ X and any point x ∈ X \ F, there exists disjoint β-open sets U and V such that x ∈ U and F ⊆ V.

**Theorem 3.24.** [2] A topological space X is β-regular if and only if for each x ∈ X and each U ∈ βO(X, x), there exists V ∈ βO(X, x) such that βCl(V) ⊆ U.

**Theorem 3.25.** Every β-regular space is β(θ)-regular.

**Proof.** Let $\mathcal{G}$ be a grill on a β-regular X, β(θ)-converging to a point x of X. For each U ∈ βO(X, x), there exists, by β-regularity of X, a V ∈ βO(X, x) such that βCl(V) ⊆ U. By hypothesis, βCl(V) ∈ $\mathcal{G}$. Hence $\mathcal{G}$ β-converges to x, proving X to be β(θ)-regular.

**Example 3.26.** Let X = {a, b, c}, τ = {X, ϕ, {a}, {b}, {a, b}, {b, c}}. Then (X, τ) is a topological space such that βO(X) = τ. Clearly X is β-compact (X being a finite set). Hence by Theorem 3.21, every grill on X must β-converge in X. Thus X is β(θ)-regular. But it is easy to check that X is not β-regular.

**Theorem 3.27.** If a topological space X is β-closed β-regular, then X is β-compact.

**Proof.** Let X be a β-closed and β-regular space. Let $\{V_\alpha : \alpha \in I\}$ be any open cover of X. For each x ∈ X, there exists an α(x) ∈ I such that $x \in V_{\alpha(x)}$. Since X is β-regular, there exists U(x) ∈ βO(X, x) such that $U(x) \subseteq \beta Cl(U(x)) \subseteq V_{\alpha(x)}$. Then, {U(x); x ∈ X} is a β-open cover of the β-closed space X and hence there exists a finite amount of points, say, $x_1, x_2, \ldots, x_n$ such that $X = \cup_{i=1}^{n} \beta Cl(U(x_i)) = \cup_{i=1}^{n} V_{\alpha(x_i)}$. This shows that X is compact.

## 4. SETS WHICH ARE β-CLOSED RELATIVE TO A SPACE

**Theorem 4.1.** For a topological space X, the following statements are equivalent:

(a) A is β-closed relative to X;

(b) Every maximal filter base β(θ)-converges to some point of X;

(c) Every filter base β(θ)-adhere to some point of X;

(d) For every family $\{V_\alpha : \alpha \in I\}$ of β-closed sets such that $\cap\{V_\alpha : \alpha \in I\} \cap A = \phi$, there exists a finite subset $I_0$ of I such that $\cap\{\beta Int(V_\alpha) : \alpha \in I_0\} \cap A = \phi$.

**Proof.** The proof is clear.

**Theorem 4.2.** If X is a β-closed space, then every cover of X by β-θ-open set has a finite subcover.

**Proof.** Let $\{V_\alpha : \alpha \in I\}$ be any cover of X by β-θ-open subsets of X. For each x ∈ X, there exists α(x) ∈ I such that $x \in V_{\alpha(x)}$ is β-θ-open, there exists $V_x \in \beta O(X, x)$ such that $V_x \subseteq \beta Cl(V_x) \subseteq V_{\alpha(x)}$. The family $\{V_x : x \in X\}$ is a β-open cover of X. Since X is β-closed, there exists a finite number of points, say, $x_1, x_2, \ldots, x_n$ such that $X = \cup_{i=1}^{n} \beta Cl(V_{x_i})$. Therefore, we obtain that $X = \cup_{i=1}^{n} V_{x_i}$.

**Theorem 4.3.** Let A, B be subsets of a topological space X. If A is β-θ-closed and B is β-closed relative to X, then A ∩ B is β-closed relative to X.

**Proof.** Let $\{V_\alpha : \alpha \in I\}$ be any cover of A ∩ B by β-open subsets of X. Since X \ A is β-θ-open, for each x ∈ B \ A there exists $W_x \in \beta O(X, x)$ such that $\beta Cl(W_x) \subseteq X \setminus A$. The family $\{W_x : x \in B \setminus A\} \cup \{V_\alpha : \alpha \in I\}$ is a cover of B by β-open sets of X. Since B is β-closed relative to X, there exists a finite number of points, say, $x_1, x_2, \ldots, x_n$ in B \ A and a finite subset $I_0$ of I such that $B \subseteq \cup_{i=1}^{n} \beta Cl(W_{x_i}) \cup \cup_{\alpha \in I_0} \beta Cl(V_\alpha)$. Since $\beta Cl(W_{x_i}) \cap A = \phi$ for each i, we obtain that $A \cap B \subseteq \cup\{\beta Cl(V_\alpha) : \alpha \in I_0\}$. This shows that A ∩ B is β-closed relative to X.

**Corollary 4.4.** If K is $\beta$-$\theta$-closed of a $\beta$-closed space X, then K is $\beta$-closed relative to X.

**Definition 4.5.** [4] A topological space X is called $\beta$-connected if X cannot be expressed as the union of two disjoint $\beta$-open sets. Otherwise, X is $\beta$-disconnected.

**Theorem 4.6.** Let X be a $\beta$-disconnected space. Then X is $\beta$-closed if and only if every $\beta$-regular subset of X is $\beta$-closed relative to X.

**Proof.** Necessity: Every $\beta$-regular set is $\beta$-$\theta$-closed by Proposition 3.17. Since X is $\beta$-closed, the proof is completed by Corellary 4.4.

Sufficiency: Let $\{V_\alpha : \alpha \in I\}$ be any cover of X by $\beta$-open subsets of X. Since X is $\beta$-disconnected, there exists a proper $\beta$-regular subset A of X. By our hypothesis, A and X \ A are $\beta$-closed relative to X. There exist finite subsets $A_1$ and $A_2$ of A such that $A \subseteq \cup_{\alpha\in A_1} \beta Cl(V_\alpha)$, $X \setminus A \subseteq \cup_{\alpha\in A_2} \beta Cl(V_\alpha)$. Therefore, we obtain that $X = \cup\{\beta Cl(V_\alpha) : \alpha \in A_1 \cup A_2\}$.

**Theorem 4.7.** If there exists a proper $\beta$-regular subset A of a topological space X such that A and X \ A are $\beta$-closed relative to X, then X is $\beta$-closed.

**Proof.** This proof is similar to the Theorem 4.6 and hence omitted.

**Definition 4.8.** A function $f : (X, \tau) \to (Y, \sigma)$ is called $\beta$-irresolute [7] if $f^{-1}(V)$ is $\beta$-open in X for every $\beta$-open subset V of Y.

**Lemma 4.9.** [7] A function $f : (X, \tau) \to (Y, \sigma)$ is $\beta$-irresolute if and only if for each subset A of X, $f(\beta Cl(A)) \subseteq \beta Cl(f(A))$.

**Theorem 4.10.** If a function $f : (X, \tau) \to (Y, \sigma)$ is $\beta$-irresolute surjection and K is $\beta$-closed relative to X, then f(K) is $\beta$-closed relative to Y.

**Proof.** Let $\{V_\alpha : \alpha \in I\}$ be any cover of f(K) by $\beta$-open subsets of Y. Since f is $\beta$-irresolute, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is a cover of K by $\beta$-open subsets of X, where K is $\beta$-closed relative to X. Therefore, there exists a finite subset $I_0$ of I such that $K \subseteq \cup_{\alpha\in I_0} \beta Cl(f^{-1}(V_\alpha))$. Since f is $\beta$-irresolute surjective, by Lemma 4.9, we have

$$f(K) \subseteq \cup_{\alpha\in I_0} f(\beta Cl(f^{-1}(V_\alpha))) \subseteq \cup_{\alpha\in I_0} \beta Cl(V_\alpha).$$

**Corollary 4.11.** If a function $f : (X, \tau) \to (Y, \sigma)$ is $\beta$-irresolute surjection and X is $\beta$-closed, then Y is $\beta$-closed.

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