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SPECIAL SELFGENERATOR MODULES

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ABSTRACT

Let R be a commutative ring with 1, and let M be a unitary (left) R-module. M is called a selfgenerator module or weak multiplication module if for each cyclic submodule Ra of M (equivalently for each submodule N of M) there exists a family (f_i) of endomorphisms of M, such that $Ra=\sum_i f_i(M)$ (equivalently $N=\sum_i f_i(M)$. In this paper we introduce classes of modules properly contained in selfgenerator modules called special selfgenerator modules, and we study some of the properties of these modules.

INTRODUCTION

Let R be a commutative ring with 1, and let M be a unitary (left) R-module. M is called a selfgenerator module [14] or weak multiplication [3], if for each cyclic submodule Ra of M (equivalently for each submodule N of M), there exists a family (f_i) of endomorphisms of M, such that $Ra=\sum_i f_i(M)$ (equivalently $N=\sum_i f_i(M)$. It is easy to see that every multiplication module is selfgenerator and the converse is true if End is commutative [3]. In this paper we introduce classes of modules properly contained selfgenerator modules, and we study some of their properties.

we Consider in particular the relation of these modules with multiplication modules.

1. GENERAL REMARKS AND EXAMPLES

In this section we introduce the concepts of special selfgenerator modules, and we give some general results with some examples.

1.1. Definition

For each natural number n, we call the R-module M a selfgenerator module of type S_n (briefly of type S_n) if for each n-generated submodule N of M, there exists an epimorphism f: M---> N. And we say that M is of type $S\infty$, if it is of type Sn for each $n \in N$. Finally, we say that M is of type S if each submodule N of M is an epimorphic image of M.

It is clear that each module of type S is of type $S\infty$ and each module of type S_n is of type S_m for sach $m \leq n$. Moreover, each module of type S_1 is a selfgenerator.

Since in a semisimple module every submodule is a direct summand, then it is clear that every semisimple module is of type S.

Recall that an R-module M is said to be a C.P. module if each cyclic submodule is projective and M is an FGP module if every f.g submodule is projective, and M is a hereditary module of every submodule of M is projective.

Let us also recall that M is called Z regular if each cyclic submodule (equivalently, every f.g submodule) is projective and direct summand, [13].

1.2. Proposition

Let M be an R-module. The following statements are equivalent.

1- M is Z regular.
2- M is of the $S\infty$ type and is an FGP module.
3- M is of the S_1 type and is a C.P. module.

Proof:

(1)------>(2). Let N be a f.g submodule of M, then N is a direct summand of M and is projective [13]. It is clear that N is an epimorphic image of M.

(2)------> (3). Clear.

(3)------> (1).Let N be any cylic submodule of M, there exists an epimorphism f:M ------> N. But N is projective, thus the sequence splits [5], and thus N is a direct summand, hence M is Z regular.

Sinceevery projective module over a regular ring is Z regular, we have.

1.3. Corollary:

Every projective module over a regular ring R is of type S.

1.4. Definitions:

A submodule N of the R-module M is said to be pure in M if N ∩ IM = IN for each I deal I of R, and the module M is said to be F regular if each submodule N of M is pure [2]. And the submodule N of M is called strongly pure if for each a ∈ N(equivalently for each finite submodule N` of N), there exists an epimorphism f:M ------>N such that f (a)=a (equivalently f (n)=n ∀ n ∈ N [7]. The module M is called strongly F regular if every submodule N of M is strongly pure [7]. It is clear that every strongly F regular module is F regular, and it is Known that a Z regular module is F regular and the converse is true if the module is projective [7].

The proofs of the following are simple and hence are omitted.

1.5. Proposition

Let M be an R-module. If M is strongly F-regular, then M is of the type S∞, and the converse is true if M is a C.P. module.

1.6. Proposition

let M be a C.P. module. The following statements are equivalent.

1. M is of type S1.
2. M is of type S∞.
3. M is Z regular.
4. M is strongly F regular.

2. SOME PROPERTIES OF MODULES OF TYPE SN, S∞ AND S:

In this section we make some remarks about the classes of modules of the type Sn, S∞ and S. We also give some general results. We start by the following.

2.1. Proposition:

let R be any ring. Then

1- R satisfies the S1 property as an R-module.

2- R satisfies the S∞ property as an R-module iff R is a Bezout ring (i.e. every f.g ideal is principal).

3- R satisfies the S property as an R-module iff R is a principal ideal ring.

Proof:

1- Let Ra be a principal ideal in R , then the map f :R-----> Rα defined by f (x)=xa is an epimorphism.

2- let I be a fig I deal in R, then there exists an epimorphism. F:R ------> I. But R is a cyclic R-module, hence I is principal. The converse is clear.

3-The proof is semilar to (2).

Recall that a ring R is called a valuation ring if for any two ideals A, B in R, A ≤ B or B ≤ A [4].

2.2 Theorem

let M be a module over the valuation ring R. If M is of type S_1, then it is of type $S\infty$.

Proof:

For any two elements $x, y \in M$, let $A = \text{ann}(x)$, $B = \text{ann}(y)$ then either $A \subseteq B$ or $B \subseteq A$, assume $A \subseteq B$. Define a map $f: Rx \dashrightarrow Ry$ by putting $f(rx) = ry$, then it is easily seen that f is well defined R-epimorphism. Now, let N be a finitely generated submodule of M, generated by $(x_0, x_1, x_2, \ldots, x_n)$. Let $A_i = \text{ann}(xi)$ $0 \le I \le n$. Since R is a valuation ring, then the ideals (Ai) are linearly ordered by inclusion. We may assume that $A_o \le Ai$, $0 \le i \le n$. thus there exists an epimorphism $fi : Rx_o \dashrightarrow Rxi$. On the other hand, since M is of the S_1 type, there exists an epimorphism $f_o: M \dashrightarrow Rx_o$ and thus there m E M with $f_o(m) = x_o$. Hence fio $f_o: M \dashrightarrow Rxi$ is an epimorphism, $1 \le i \le n$. Define a map $F : M \oplus \sum_{i=0}^{n} Rxi$ by $F(x) = (f_o(x), f_1 o F_o(x), \ldots, f_n o f_o(x))$ $\forall x$ E M and define $H: \oplus \sum_{i=0}^{n} Rxi \dashrightarrow N$ by $H(a_o x_o, a_1 x_1, \ldots m\, a_n x_n) = \sum_{i=0}^{n} ai\, x\, i$, $ai \in R$. It is easily checked that each of F and H is a well defined R-epimorphism, (note that $F(m) = (x_o, x_1, \ldots, x_n)$ and hence $H o F: M \dashrightarrow N$ is an R-epimorphism.

Recall that an R-module M over an integral domain R is called tortion free if $rm = 0$, $r \in R$, $m \in M$, $m \in M$ then $r = 0$ or $m = 0$. Using a proof similar to the proof of the last theorem, we get the following.

2.3. Theorem

let M be an R-module over the integral domain R If M is of type S1, Then it is of type $S\infty$.

2.4. Theorem:

Let M be an R-module. Assume that M contains an element x_o with ann (x_o) = ann (M). If there exists an epimorphism $f : M \dashrightarrow Rx_o$, then M is of type $S\infty$.

Proof:

For each $x \in M$, define a map $f_x: Rx_o \dashrightarrow Rx$ by $f_x(rx_o) = rx$. Then f is well defined, in fact, if $rx_o = \emptyset$, then $r \in$ ann (x_o)= ann (M), hence $rx = \emptyset$.

By the argument used in the proof of Theorem (2.2), we get M is of type S_1 and by the same theorem, M is of type $S\infty$.

As an immediate consequence, we have:

2.5. Corollary:

If M is an R-module which contains a free direct summand then M is of type $S\infty$.

The following result shows that the class of modules of type $S\infty$ is "quite large".

2.6. Corollary:

Let M be an R-module. Then $R \oplus M$ is of type $S\infty$.

The following theorem gives plenty of examples of Z regular modules, see [9].

2.7. Theorem:

Let M be an R-module, and R is a P.P. ring If M is a projective module or M is a C.P. module, then $R \oplus M$ is a Z-regular R-module.

Proof:

Assume M is projective, thus $R \oplus M$ is a projective R-module. Since R is a P.P. ring, then by [8], $R \oplus M$ is a C.P. module. On the other hand, by corollary (2.6) $R \oplus M$ is of type $S\infty$. It follows from

(1.2) that M is Z regular module. If M a C.P. module, then by [8] R ⊕ M is a C.P. module. Moreover, by corollary (2.6), R ⊕ M if of type S, hence by (1.2) R ⊕ M is a Z regular module.

2.8. Note:

The known examples of Z regular modules which are not projective are rare. However, theorem (2.7) gives plenty of such examples, just take any C.P. R-module which is not projective, then R ⊕ M is a z regular R-module that is not projective. For example Z ⊕ Q is Z regular but not projective. Let us observe that Q as a Z module is easily seen, not of the type S_1.

3. MULTIPLICATION MODULES:

Recall that an r-module M is called a multiplication module if for each submodule N of M , there exists an ideal I of R such that N= IM [1]. In this section we study multiplication module that are of type S_n, S∞ and S.

We start by the following.

3.1 Theorem:

let R be any ring. Then R is a P.I.R. iff every multiplication R-module is of type S.

Proof:

Let R be a P.I.R., and let M be a multiplication R-module. Let N be a submodule of M, then there exists an ideal I of R such that N =IM. Thus there exists a∈R such that I=Ra and N = aM. Define fα:M ------>N by f (m) =am ∀ m ∈M, then fα is an R-epimorphism. Conversely, R is a multiplication R-module, hence by (2.1) R is a P.I.R.

3.2 Theorem:

let R be any ring. Then R is a Bezout ring iff every multiplication R-module is of type $S\infty$.

Proof:

Let R be a Bezout ring and let M be a multiplication R-module. Let N be a f.g submodule of M, then there exists an ideal I of R such that N= IM. It is easily seen that one may choose I to be finitely generated [11]. Thus there exists a E R such that I = Ra and N= aM.As in the last proof, there exists an epimorphism $f\alpha$: M-----> N.

The converse follows from 3.1.

3.3 Proposition:

Let M be a multiplication R-module. If M is of the S_2 type, then every f.g submodule of M is a multiplication module.

Proof:

For any 2-generated submodule N of M, there exists an epimorphism f: M ------> N, and hence N is a multiplication module. Thus N is locally cyclic [1]. If H is generated by (a_1, a_2, a_3) in M, then $H=Ra_1 + Ra_2 + Ra_3$. Hence, locally H is 2-generated. I.e for each prime ideal P of R, there exists a 2-generated submodule N of M such that $N_p = H_p$. But N is a multiplication module, hence by [5], $N_p = H_p$ is cyclic and this implies H is a multiplication module. Now an induction argument finishes the proof.

Next we consider regular multiplication modules.

3.4 Proposition:

Let M be a faithful multiplication R-module. If R is a P.P. ring and M is of S_1 type, then M is Z regular and hence is of the $S\infty$ type.

Proof:

It follows from [8] that M is a C.P. module and the result followos from (1.2).

Note that the above result is false without the assumption on R. In fact, every R is of type S_1 as an R-module, but it is of type $S\infty$ only if it is a Bezout ring, see (2.1).

Recall that if M is a multiplication R-module, then End (M) is commutative, in fact, for each $f \in$ End (M) and $m \in M$, $\exists\ r \in R$, r may depend on m such that f (m) = rm [6]. It was shown in [6] that if f (M) is f.g, then $\exists\ r \in R$ such that f (m)= r m $\forall m \in M$ Now we have:

3.5 Theorem

Let M be a multiplication R-module. The following statements are equivalent.

1. End (M) is a regular ring (in the sense of Von Neumann) and M is a C.P. module.
2. M is of the S_1 type and M is a C.P. module.
3. M is of the $S\infty$ type and M is an FGP. Module.

Proof:

The equivalence of (2) and (3) was proved in (1.2). Moreover, (2) implies by (1.2) that M is Z regular. Hence by [7]., M is strongly F regular. Thus End (M) is regular by [7]. To show (1) implies (2), by [7], M is strongly F regular, hence it is of type S1 and this implies (2).

The following result gives a partial answer to the question raised in [6].

3.6. Theorem

let M be a faithful multiplication R-module over the P.P. ring R. Then the following statements are equivalent.

1- End (M) is regular.

2- M is of the $S\infty$ type.

3- M is of the S_1 type.

Proof:

Assume (1), then by [7], M is strongly F regular, hence it is of type $S\infty$.

(2) ------> (3) trivial.

Assume (3), Since R is a P.P. ring, then by (3.4) M is a Z regular module, and thus End (M) is a regular ring [7].

We end this section by the following.

3.7. Proposition

Let M be a multiplication R-module, and N a submodule of M. If M is of type $S\infty$ (type S), then so is M/N.

Proof:

Let K / N be a f.g submodule of M/N, where K is a f.g submodule of M. Since M is of type $S\infty$, there exists an epimorphism f:M------>K. By [6], N is an invariant submodule of M, i.e $f(N) \subseteq N$, thus f induces an epimorphism f:M / N------> K / N.

4. Further results:

We start this section by the following.

4.1. Proposition

let M be an R-module of type S (type $S\infty$) then Mp is of type S (type $S\infty$) for each prime ideal P of R.

Proof:

Let P be a prime ideal of R and let A be an R_p submodule of M_p. Thus there exists a submodule N of M with N_p= A (N is f.g if A is f.g). There exists an R epimorphism f:M ------> N. f induces an R_p -homomorphism f_p: M_p------>Np defined by f_p (m/t)= f (m)/ t. It is easily checked that f_p is an epimorphism.

4.2 Note:

The converse of (4.1) is false In fact, let R be a regular ring which is not semi-simple. It is know that R_p is a field for each prime ideal p of R, thus R_p is an R module of type S. However, R is not of type S, it is of type S∞.

4.3. Proposition:

Let M = $M_1 \oplus M_2$ where M_1, M_2 are modules. If each of M_1 and M_2 is of type S (type S), then M is of type Sn (type S_n) provided ann (M_1) + ann (M_2) = R.

Proof:

Let N be a submodule of M then by [10], N= $N_1 \oplus N_2$ where N_1 is a submodule of M, i=1,2. (It is clear that if N is n-generated, then N_1 , N_2 are n-generated). There exist epimorphisms fi, i=1,2, f: M --- N. Then it is clear that $f_1 \oplus f_2$ is an epimorphism from $M_1 \oplus M_2$ ------> $N_1 \oplus N_2$.

Note that if n=1 then the condition on annihilators is not needed.

4.4. Proposition:

Let M = $M_1 \oplus M_2$ where M_1 and M_2 are R-modules and M is of type S (type S∞). If Hom (M_1, M_2) = Ø, then M_1 is of type S (type S∞).

Proof:

Let N_1 be a submodule of M_1, hence N_1 is a submodule of M (f.g) if M is of type $S\infty$). Then there exists an epimorphism f:M ------> N. Let $f_1 = f/M$. Since Hom $(M_1, M_2) = \emptyset$, then $f(M_1) \subseteq M$. It is clear that $f(M_1) = N_1$ and hence M_1 is of type S (of type $S\infty$).

4.5. Note:

It was shown is (2.8) that $Z \oplus Q$ is a Z module of type $S\infty$. Observe that Hom (Z, Q) $\neq \emptyset$. Note also that Z is of type $S\infty$ and Hom $(Q, Z) = \emptyset$.

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