Fibrewise Near Compact and Locally Near Compact Spaces

M. E. Abd El-monsef
Department of Mathematics, Faculty of Science, Tanta University, Tanta-Egypt
PO box 31527, Tanta-Egypt, Email address: me_monsef@yahoo.com
A. E. Radwan
Department of Mathematics, Faculty of Science, Ain Shams University, Cairo-Egypt
PO box 11566, Cairo-Egypt, Email address: zezoradwen@yahoo.com
Y. Y. Yousif (Corresponding author)
Department of Mathematics, Faculty of Education for Pure Science (Ibn Al-Haitham),
Baghdad University, Iraq-Baghdad
PO box 4150, Iraq-Baghdad, Email address: yoyayousif@yahoo.com

Abstract

In this paper we define and study new concepts of fibrewise topological spaces over B namely, fibrewise near compact and fibrewise locally near compact spaces, which are generalizations of well-known concepts near compact and locally near compact topological spaces. Moreover, we study relationships between fibrewise near compact (resp., fibrewise locally near compact) spaces and some fibrewise near separation axioms.

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1. Introduction and Preliminaries

To begin with we work in the category of fibrewise sets over a given set, called the base set. If the base set is denoted by B then a fibrewise set over B consists of a set X together with a function $p : X \to B$, called the projection. For each point b of B the fibre over b is the subset $X_b = p^{-1}(b)$ of X; fibres may be empty since we do not require p to be surjective, also for each subset B* of B we regard $X_{B*} = p^{-1}(B*)$ as a fibrewise set over B* with the projection determined by p. In fibrewise topology the term neighbourhood (nbd) is used in precisely the same sense as it is in ordinary topology. For a subset A of a topological space X, the closure (resp., interior) of A is denoted by Cl(A) (resp., Int(A)). For other notions or notations which are not defined here we follow closely James [11], Engelking [10], and Bourbaki [7].

Definition 1.1. [11] Let X and Y are fibrewise sets over B, with projections $p_X : X \to B$ and $p_Y : Y \to B$, respectively, a function $\varphi : X \to Y$ is said to be fibrewise if $p_Y \varnothing \varphi = p_X$, in other words if $\varphi(X_b) \subset Y_b$ for each point b of B.

Note that a fibrewise function $\varphi : X \to Y$ over B determines, by restriction, a fibrewise function $\varphi_{B*} : X_{B*} \to Y_{B*}$ over B* for each subset B* of B.

Definition 1.2. [11] Suppose that B is a topological space, the fibrewise topology on a fibrewise set X over B, mean any topology on X for which the projection p is continuous.

Remark 1.3. [11]

(a) The coarsest such topology is the topology induced by p, in which the open sets of X are precisely the inverse images of the open sets of B; this is called the fibrewise indiscrete topology.
(b) The fibrewise topological space over B is defined to be a fibrewise set over B with a fibrewise topology.
(c) We regard the topological product B×T, for any topological space T, as a fibrewise topological spaces over B, using the first projection, and similarly for any subspace of B×T.
(d) The equivalences in the category of fibrewise topological spaces are called fibrewise topological equivalences.

Definition 1.4. [11] The fibrewise function $\varphi : X \to Y$, where X and Y are fibrewise topological spaces over B is called:

(a) Continuous if for each point $x \in X_b$, where $b \in B$, the inverse image of each open set of $\varphi(x)$ is an open set of x.
(b) Open if for each point $x \in X_b$, where $b \in B$, the direct image of each open set of x is an open set of $\varphi(x)$.

Definition 1.5. [11] The fibrewise topological space X over B is called fibrewise closed if the projection p is closed function.

Definition 1.6. [10] The function $\varphi : X \to Y$ is called proper function if it is continuous, closed, and for each $y \in Y$, $\varphi^{-1}(y)$ is compact set.

Definition 1.7. A subset A of a topological space (X, τ) is called:

(a) Pre-open [15] (briefly P-open) if A ♥ Int(Cl(A)),

(b) Semi-open [12] (briefly S-open) if A ♥ Cl(Int(A)),
(c) γ-open [9] (= b-open [4]) (briefly γ-open) if A ♥ Cl(Int(A)) Ɽ Int(Cl(A)),
(d) α-open [18] (briefly α-open) if A ♥ Int(Cl(Int(A))),
(e) β-open [1](=semi-pre-open set [5]) (briefly β-open) if A ♥ Cl(Int(Cl(A))).

The complement of a P-open (resp., S-open, γ-open, α-open, β-open) is called P-closed (resp., S-closed, γ-closed, α-closed, β-closed). The family of all P-open (resp., S-open, γ-open, α-open, β-open) are larger than τ and closed under forming arbitrary union, we will called this family near topology (briefly j-topology), where j∈{P, S, γ, α, β}.

Definition 1.8. A function φ : X → Y is said to be P-continuous [15] (resp., S-continuous [12], γ-continuous [9], α-continuous [17], β-continuous [1]) if the inverse image of each open set in Y is P-open (resp., S-open, γ-open, α-open, β-open) in X.

Definition 1.9. A function φ : X → Y is said to be P-open [15] (resp., S-open [12], γ-open [9], α-open [17], β-open [1]) if the image of each open set in X is P-open (resp., S-open, γ-open, α-open, β-open) in Y.

Definition 1.10. A topological space X is called P-compact [16] (resp., S-compact [8], γ-compact [9], α-compact [6, 13], β-compact [2]) space if each P-open (resp., S-open, γ-open, α-open, β-open) cover of X has a finite subcover.

Definition 1.11. A topological space X is called locally P-compact [16] (resp., locally S-compact [8], locally γ-compact [6], locally α-compact [6, 13], locally β-compact [2]) spaces if for every point x in X, there exists an open nbd U of x such that the closure of U in X is P-compact (resp., S-compact, γ-compact, α-compact, β-compact) space.

Definition 1.12. [10] For every topological space X* and any subspace X of X*, the function $i_X : X \to X^*$ define by $i_X(x) = x$ is called embedding of the subspace X in the space X*. Observe that i_X is continuous, since $i_X^{-1}(U) = X \cap U$, where U is open set in X*. The embedding i_X is closed (resp., open) if and only if the subspace X is closed (resp., open).

2. Fibrewise Near Compact and Locally Near Compact Spaces

In this section, we introduce the following new concepts.

Definition 2.1. The function φ : X → Y is called near proper (briefly j-proper) function if it is continuous, closed, and for each y ∈ Y, $\varphi^{-1}(y)$ is j-compact set, where j∈{S, P, γ, α, β}.

For example, let (□, τ) where τ is the topology with basis whose members are of the form (a, b) and (a, b) − N, N = {1/n ; n ∈ $□^+$}. Define f : (□, τ) → (□, τ) by f(x) = x, then f is j-proper function, where j∈{S, P,γ,α,β}.

If φ : X → Y is fibrewise and j-proper function, then φ is said to be fibrewise j-proper function, where j∈{S, P, γ, α, β}.

Definition 2.2. The fibrewise topological space X over B is called fibrewise near compact (briefly j-compact) if the projection p is j-proper, where j∈{S, P, γ, α, β}.

For example the topological product B×T is fibrewise j-compact over B, for all j-compact space T. For another example, the subset {(b, x) ∈ $□\times□^n$: ||x|| ≤ b} of $□\times□^n$ is fibrewise j-compact over □, where j∈{S, P, γ, α, β}.

Definition 2.3. [3] A function φ : X → Y is called j-biclosed function, where X and Y are topological spaces, if it is maps j-closed set onto j-closed set, where j∈{S, P, γ, α, β}.

Proposition 2.4. [3] Let X be a fibrewise topological space over B. Then

(a) X is fibrewise closed iff for each fibre X_b of X and each open set O of X_b in X, there exist an open nbd W of b such that $X_W \subset O$.
(b) X is fibrewise j-biclosed iff for each fibre X_b of X and each j-open set O of X_b in X, there exists a j-open set W of b such that $X_W \subset O$, where j∈{S, P, γ, α, β}.

Useful characterizations of fibrewise j-compact spaces are given by the following propositions, where j∈{S, P, γ, α, β}.

Proposition 2.5. The fibrewise topological space X over B is fibrewise j-compact iff X is fibrewise closed and every fibre of X is j-compact, where j∈{S, P, γ, α, β}.

Proof. (⇒) Let X be a fibrewise j-compact space, then the projection p : X → B is j-proper function i.e., p is closed and for each b ∈ B, X_b is j-compact. Hence X is fibrewise closed and every fibre of X is j-compact, where j∈{S, P, γ, α, β}.

(⇐) Let X be fibrewise closed and every fibre of X is j-compact, then the projection p : X → B is closed and it is clear that p is continuous, also for each b ∈ B, X_b is j-compact. Hence X is fibrewise j-compact, where j∈{S, P, γ, α, β}.

Proposition 2.6. Let X be fibrewise topological space over B. Then X is fibrewise j-compact iff for each fibre X_b of X and each covering Γ of X_b by open sets of X there exists a nbd W of b such that a finite subfamily of Γ covers X_W, where $j\in\{S, P, \gamma, \alpha, \beta\}$.

Proof. (⇒) Let X be fibrewise j-compact space, then the projection $p : X \to B$ is j-proper function, so that X_b is j-compact for each $b \in B$. Let Γ be a covering of X_b by open sets of X for each $b \in B$ and let $X_W = \cup X_b$ for each $b \in W$. Since X_b is j-compact for each $b \in W \subset B$ and the union of j-compact sets is j-compact, we have X_W is j-compact. Thus, there exists a nbd W of b such that a finite subfamily of Γ covers X_W, where $j\in\{S, P, \gamma, \alpha, \beta\}$.

(⇐) Let X be fibrewise topological space over B, then the projection $p : X \to B$ exist. To show that p is j-proper. Now, it is clear that p is continuous and for each $b \in B$, X_b is j-compact by take $X_b = X_W$. By Proposition (2.4), we have p is closed. Thus p is j-proper and X is fibrewise j-compact, where $j\in\{S, P, \gamma, \alpha, \beta\}$.

These are special cases of well-known results of Theorems (3.7.2), (3.7.9), and Proposition (3.7.8) in [10], as in Propositions (2.7)-(2.9) below.

Proposition 2.7. Let $\varphi : X \to Y$ be a j-proper, j-biclosed fibrewise function, where X and Y are fibrewise topological spaces over B. If Y is fibrewise j-compact then so is X, where $j\in\{S, P, \gamma, \alpha, \beta\}$.

Proof. Suppose that $\varphi : X \to Y$ is j-proper, j-biclosed fibrewise function and Y is fibrewise j-compact space i.e., the projection $p_Y : Y \to B$ is j-proper. To show that X is fibrewise j-compact space i.e., the projection $p_X : X \to B$ is j-proper. Now, clear that p_X is continuous. let F be a closed subset of X_b, where $b \in B$. Since φ is closed, then φ(F) is closed subset of Y_b. Since p_Y is closed, then $p_Y(\varphi(F))$ is closed in B. But $p_Y(\varphi(F)) = (p_Y \circ \varphi)(F) = p_X(F)$ is closed in B so that p_X is closed. Let $b \in B$, since p_Y is j-proper, then Y_b is j-compact. Now let $\{U_i; i \in \Lambda\}$ be a family of j-open sets of X such that $X_b \subset \cup_{i\in\Lambda}U_i$. If $y \in Y_b$, then there exist a finite subset M(y) of Λ such that $\varphi^{-1}(y) \subset \cup_{i\in M(y)}U_i$. Since φ is j-biclosed function, so by Proposition (2.4.b) there exist a j-open set V_y of Y such that $y \in V_y$ and $\varphi^{-1}(V_y) \subset \cup_{i\in M(y)}U_i$. Since Y_b is j-compact, there exist a finite subset C of Y_b such that $Y_b \subset \cup_{y\in C}V_y$. Hence $\varphi^{-1}(Y_b) \subset \cup_{y\in C}\varphi^{-1}(V_y) \subset \cup_{y\in C}\cup_{i\in M(y)}U_i$. Thus if $M = \cup_{y\in C}M(y)$, then M is a finite subset of Λ and $\varphi^{-1}(Y_b) \subset \cup_{i\in M}U_i$. Thus $\varphi^{-1}(Y_b) = \varphi^{-1}(p_Y^{-1}(b)) = (p_Y \circ \varphi)^{-1}(b) = p_X^{-1}(b) = X_b$ and $X_b \subset \cup_{i\in M}U_i$ so that X_b is j-compact. Thus p_X is j-proper and X is fibrewise j-compact, where $j\in\{S, P, \gamma, \alpha, \beta\}$.

The class of fibrewise j-compact spaces is multiplicative, where $j\in\{S, P, \gamma, \alpha, \beta\}$, in the following sense.

Proposition 2.8. Let $\{X_r\}$ be a family of fibrewise j-compact spaces over B. Then the fibrewise topological product $X = \prod_B X_r$ is fibrewise j-compact, where $j\in\{S, P, \gamma, \alpha, \beta\}$.

Proof. Without loss of generality, for finite products a simple argument can be used. Thus, let X and Y be fibrewise topological spaces over B. If X is fibrewise j-compact then the projection $p\times id_Y : X\times_B Y \to B\times_B Y \equiv Y$ is j-proper. If Y is also fibrewise j-compact then so is $X\times_B Y$, by Proposition (2.7).

A similar result holds for finite coproducts.

Proposition 2.9. Let X be fibrewise topological space over B. Suppose that X_i is fibrewise j-compact for each member X_i of a finite covering of X. Then X is fibrewise j-compact, where $j\in\{S, P, \gamma, \alpha, \beta\}$.

Proof. Let X be fibrewise topological space over B, then the projection $p : X \to B$ exist. To show that p is j-proper. Now, it is clear that p is continuous. Since X_i is fibrewise j-compact, then the projection $p_i : X_i \to B$ is closed and for each $b \in B$, $(X_i)_b$ is j-compact for each member X_i of a finite covering of X. Let F be a closed subset of X, then $p(F) = \cup\, p_i(X_i \cap F)$ which is a finite union of closed sets and hence p is closed. Let $b \in B$, then $X_b = \cup\, (X_i)_b$ which is a finite union of j-compact sets and hence X_b is j-compact. Thus, p is j-proper and X is fibrewise j-compact, where $j\in\{S, P, \gamma, \alpha, \beta\}$.

Definition 2.10. [3] A fibrewise function $\varphi : X \to Y$, where X and Y are fibrewise topological spaces over B is called j-irresolute if for each point $x \in X_b$, where $b \in B$, the inverse image of each j-open set of φ(x) is a j-open set of x, where $j\in\{S, P, \gamma, \alpha, \beta\}$.

Proposition 2.11. Let $\varphi : X \to Y$ be a continuous, j-irresolute fibrewise surjection, where X and Y are fibrewise topological spaces over B. If X is fibrewise j-compact then so is Y, where $j\in\{S, P, \gamma, \alpha, \beta\}$.

Proof. Suppose that $\varphi : X \to Y$ is continuous, j-irresolute fibrewise surjection and X is fibrewise j-compact i.e., the projection $p_X : X \to B$ is j-proper. To show that Y is fibrewise j-compact i.e., the projection $p_Y : Y \to B$ is j-proper. Now, it is clear that p_Y is continuous. Let F be a closed subset of Y_b, where $b \in B$. Since φ is continuous fibrewise, then $\varphi^{-1}(F)$ is closed subset of X_b. Since p_X is closed, then $p_X(\varphi^{-1}(F))$ is closed in B. But $p_X(\varphi^{-1}(F)) = (p_X \circ \varphi^{-1})(F) = p_Y(F)$ is closed in B, hence p_Y is closed. For any point $b \in B$, we have $Y_b = \varphi(X_b)$ is j-compact because X_b is j-compact and the image of a j-compact subset under j-irresolute function is j-compact. Thus, p_Y is j-proper and Y is fibrewise j-compact, where $j\in\{S, P, \gamma, \alpha, \beta\}$.

Proposition 2.12. Let X be fibrewise j-compact space over B. Then X_{B^*} is fibrewise j-compact space over B* for each subspace B* of B, where $j\in\{S, P, \gamma, \alpha, \beta\}$.

Proof. Suppose that X is fibrewise j-compact i.e., the projection $p : X \to B$ is j-proper. To show that X_{B^*} is fibrewise j-compact space over B* i.e., the projection $p_{B^*} : X_{B^*} \to B^*$ is j-proper. Now, it is clear that p_{B^*} is continuous. Let F be a closed subset of X, then $F \cap X_{B^*}$ is closed in subspace X_{B^*} and $p_{B^*}(F \cap X_{B^*}) = p(F \cap X_{B^*}) = p(F) \cap B^*$ which is closed set in B*, hence p_{B^*} is closed. Let $b \in B^*$, then $(X_{B^*})_b = X_b \cap X_{B^*}$ which is j-compact set in X_{B^*}. Thus, p_{B^*} is j-proper and X_{B^*} is fibrewise j-compact over B*, where $j \in \{S, P, \gamma, \alpha, \beta\}$.

Proposition 2.13. Let X be fibrewise topological space over B. Suppose that X_{B_i} is fibrewise j-compact over B_i for each member B_i of a j-open covering of B. Then X is fibrewise j-compact over B, where $j \in \{S, P, \gamma, \alpha, \beta\}$.

Proof. Suppose that X is fibrewise topological space over B, then the projection $p : X \to B$ exist. To show that p is j-proper. Now, it is clear that p is continuous. Since X_{B_i} is fibrewise j-compact over B_i, then the projection $p_{B_i} : X_{B_i} \to B_i$ is j-proper for each member B_i of a j-open covering of B. Let F be a closed subset of X, then we have $p(F) = \cup\, p_{B_i}(X_{B_i} \cap F)$ which is a union of closed sets and hence p is closed. Let $b \in B$ then $X_b = \cup\, (X_{B_i})_b$ for every $b = \{b_i\} \in \cup\, B_i$. Since $(X_{B_i})_b$ is j-compact in X_{B_i} and the union of j-compact sets is j-compact, we have X_b is j-compact. Thus, p is j-proper and X is fibrewise j-compact over B, where $j \in \{S, P, \gamma, \alpha, \beta\}$.

In fact the last result is also holds for locally finite j-closed coverings, instead of j-open coverings.

Proposition 2.14. Let $\varphi : X \to Y$ be a fibrewise function, where X and Y are fibrewise topological spaces over B. If X is fibrewise j-compact and $id_X \times \varphi : X \times_B X \to X \times_B Y$ is j-proper and j-biclosed then φ is j-proper, where $j \in \{S, P, \gamma, \alpha, \beta\}$.

Proof. Consider the commutative figure shown below

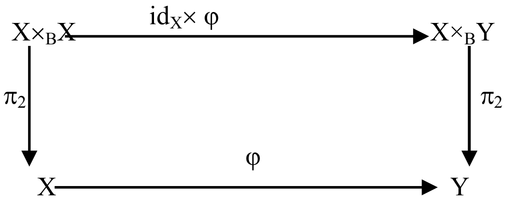

Figure 2.1.

If X is fibrewise j-compact then π_2 is j-proper. If $id_X \times \varphi$ is also j-proper and j-biclosed then $\pi_2 \circ (id_X \times \varphi) = \varphi \circ \pi_2$ is j-proper, and so φ itself is j-proper, where $j \in \{S, P, \gamma, \alpha, \beta\}$.

The second new concept in this paper is given by the following:

Definition 2.15. The fibrewise topological space X over B is called fibrewise locally near compact (briefly locally j-compact) if for each point x of X_b, where $b \in B$, there exists a nbd W of b and an open set $U \subset X_W$ of x such that the closure of U in X_W (i.e., $X_W \cap Cl(U)$) is fibrewise j-compact over W, where $j \in \{S, P, \gamma, \alpha, \beta\}$.

Remark 2.16. Fibrewise j-compact spaces are necessarily fibrewise locally j-compact by taken W = B and X_W = X. But the conversely is not true for example, let (X, τ_{dis}) where X is infinite set and τ_{dis} is discrete topology, then X is fibrewise locally j-compact over □, since for each $x \in X_b$, where b ■ □, there exists a nbd W of b and an open $\{x\} \subset X_W$ of x such that $Cl\{x\} = \{x\}$ in X_W is fibrewise j-compact over W. But X is not fibrewise j-compact space over □. Also the product space B×T is fibrewise locally j-compact over B, for all locally j-compact space T, where $j \in \{S, P, \gamma, \alpha, \beta\}$.

Closed subspaces of fibrewise locally j-compact spaces are fibrewise locally j-compact spaces, where $j \in \{S, P, \gamma, \alpha, \beta\}$. In fact we have

Proposition 2.17. Let $\varphi : X \to X^*$ be a closed fibrewise embedding, where X and X* are fibrewise topological spaces over B. If X* is fibrewise locally j-compact then so is X, where $j \in \{S, P, \gamma, \alpha, \beta\}$.

Proof. Let $x \in X_b$, where $b \in B$. Since X* is fibrewise locally j-compact there exists a nbd W of b and an open $V \subset X^*_W$ of φ(x) such that the closure $X^*_W \cap Cl(V)$ of V in X^*_W is fibrewise j-compact over W. Then $\varphi^{-1}(V) \subset X_W$ is an open set of x such that the closure $X_W \cap Cl(\varphi^{-1}(V)) = \varphi^{-1}(X^*_W \cap Cl(V))$ of $\varphi^{-1}(V)$ in X_W is fibrewise j-compact over W. Thus, X is fibrewise locally j-compact, where $j \in \{S, P, \gamma, \alpha, \beta\}$.

The class of fibrewise locally j-compact spaces is finitely multiplicative, where $j \in \{S, P, \gamma, \alpha, \beta\}$, in the following sense.

Proposition 2.18. Let $\{X_r\}$ be a finite family of fibrewise locally j-compact spaces over B. Then the fibrewise topological product $X = □_B X_r$ is fibrewise locally j-compact, where $j \in \{S, P, \gamma, \alpha, \beta\}$.

Proof. The proof is similar to that of Proposition (2.8).

3. Fibrewise Near Compact (resp., Locally Near Compact) Spaces and Some Fibrewise Near Separation Axioms

Now we give a series of results in which give relationships between fibrewise near compactness (or fibrewise locally near compactness in some cases) and some fibrewise near separation axioms which are discussed in [11, 14].

Definition 3.1. [11] The fibrewise topological space X over B is called fibrewise Hausdorff if whenever $x_1, x_2 \in X_b$, where $b \in B$ and $x_1 \neq x_2$, there exist disjoint open sets U_1, U_2 of x_1, x_2 in X.

Definition 3.2. [14] The fibrewise topological space X over B is called fibrewise near regular (briefly j-regular) if for each point $x \in X_b$, where $b \in B$, and for each j-open set V of x in X, there exists a nbd W of b in B and an open set U of x in X_W such that the closure of U in X_W is contained in V (i.e., $X_W \cap Cl(U) \subset V$), where $j \in \{S, P, \gamma, \alpha, \beta\}$.

Definition 3.3. [14] The fibrewise topological space X over B is called fibrewise near normal (briefly j-normal) if for each point b of B and each pair H, K of disjoint closed sets of X, there exists a nbd W of b and a pair of disjoint j-open sets U, V of $X_W \cap H$, $X_W \cap K$ in X_W, where $j \in \{S, P, \gamma, \alpha, \beta\}$.

Proposition 3.4. Let X be fibrewise locally j-compact and fibrewise j-regular over B. Then for each point x of X_b, where $b \in B$, and each j-open set V of x in X, there exists an open set U of x in X_W such that the closure $X_W \cap Cl(U)$ of U in X_W is fibrewise j-compact over W and contained in V, where $j \in \{S, P, \gamma, \alpha, \beta\}$.

Proof. Since X is fibrewise locally j-compact there exists a nbd W* of b in B and an open set U* of x in X_{W^*} such that the closure $X_{W^*} \cap Cl(U^*)$ of U* in X_{W^*} is fibrewise j-compact over W*. Since X is fibrewise j-regular there exists a nbd $W \subset W^*$ of b and an open set U of x in X_W such that the closure $X_W \cap Cl(U)$ of U in X_W is contained in $X_W \cap U^* \cap V$. Now $X_W \cap Cl(U^*)$ is fibrewise j-compact over W, since $X_{W^*} \cap Cl(U^*)$ is fibrewise j-compact over W*, and $X_W \cap Cl(U)$ is closed in $X_W \cap Cl(U^*)$. Hence $X_W \cap Cl(U)$ is fibrewise j-compact over W and contained in V, where $j \in \{S, P, \gamma, \alpha, \beta\}$, as required.

Proposition 3.5. Let $\varphi : X \to Y$ be an open, j-irresolute fibrewise surjection, where X and Y are fibrewise topological spaces over B. If X is fibrewise locally j-compact and fibrewise j-regular then so is Y, where $j \in \{S, P, \gamma, \alpha, \beta\}$.

Proof. Let y be a point of Y_b, where $b \in B$, and let V be a j-open set of y in Y. Pick any point x of $\varphi^{-1}(y)$. Then $\varphi^{-1}(V)$ is a j-open set of x in X. Since X is fibrewise locally j-compact there exists a nbd W of b in B and an open set U of x in X_W such that the closure $X_W \cap Cl(U)$ of U in X_W is fibrewise j-compact over W and is contained in $\varphi^{-1}(V)$. Then $\varphi(U)$ is an open set of y in Y_W, since φ is open, and the closure $Y_W \cap Cl(\varphi(U))$ of $\varphi(U)$ in Y_W is fibrewise j-compact over W and contained in V, where $j \in \{S, P, \gamma, \alpha, \beta\}$, as required.

Proposition 3.6. Let X be fibrewise locally j-compact and fibrewise j-regular over B. Let C be a j-compact subset of X_b, where $b \in B$, and let V be a j-open set of C in X. Then there exists a nbd W of b in B and an open set U of C in X_W such that the closure $X_W \cap Cl(U)$ of U in X_W is fibrewise j-compact over W and contained in V, where $j \in \{S, P, \gamma, \alpha, \beta\}$.

Proof. Since X is fibrewise locally j-compact there exists for each point x of C a nbd W_x of b in B and an open set U_x of x in X_{W_x} such that the closure $X_{W_x} \cap Cl(U_x)$ of U_x in X_{W_x} is fibrewise j-compact over W_x and contained in V. The family $\{U_x;\ x \in C\}$ constitutes a covering of the j-compact C by open sets of X. Extract a finite subcovering indexed by $x_1, \ldots, x_n$, say. Take W to be the intersection $W_{x_1} \cap \ldots \cap W_{x_n}$, and take U to be the restriction to X_W of the union $U_{x_1} \cup \ldots \cup U_{x_n}$. Then W is a nbd of b in B and U is an open set of C in X_W such that the closure $X_W \cap Cl(U)$ of U in X_W is fibrewise j-compact over W and contained in V, where $j \in \{S, P, \gamma, \alpha, \beta\}$, as required.

Proposition 3.7. Let $\varphi : X \to Y$ be a j-proper, j-irresolute fibrewise surjection, where X and Y are fibrewise topological spaces over B. If X is fibrewise locally j-compact and fibrewise j-regular then so is Y, where $j \in \{S, P, \gamma, \alpha, \beta\}$.

Proof. Let $y \in Y_b$, where $b \in B$, and let V be a j-open set of y in Y. Then $\varphi^{-1}(V)$ is a j-open set of $\varphi^{-1}(y)$ in X. Suppose that X is fibrewise locally j-compact. Since $\varphi^{-1}(y)$ j-compact, by Proposition (3.6) there exists a nbd W of b in B and an open set U of $\varphi^{-1}(y)$ in X_W such that the closure $X_W \cap Cl(U)$ of U in X_W is fibrewise j-compact over W and contained in $\varphi^{-1}(V)$. Since φ is closed there exists an open set U* of y in Y_W such that $\varphi^{-1}(U^*) \subset U$. Then the closure $Y_W \cap Cl(U^*)$ of U* in Y_W is contained in $\varphi(X_W \cap Cl(U))$ and so is fibrewise j-compact over W. Since $Y_W \cap Cl(U^*)$ is contained in V this shows that Y is fibrewise locally j-compact, where $j \in \{S, P, \gamma, \alpha, \beta\}$, as asserted.

Proposition 3.8. Let $\varphi : X \to Y$ be a continuous fibrewise function, where X is fibrewise j-compact space and Y is fibrewise Hausdorff space over B. Then φ is j-proper, where $j \in \{S, P, \gamma, \alpha, \beta\}$.

Proof. Consider the figure shown below, where r is the standard fibrewise topological equivalence and G is the fibrewise graph of φ.

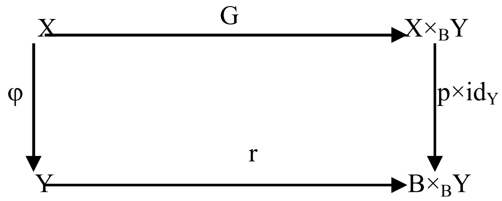

Figure 3.1.

Now G closed embedding, by Proposition (2.10) in [11], since Y is fibrewise Hausdorff. Thus G is j-proper. Also p is j-proper and so $p\times id_Y$ is j-proper. Hence $(p\times id_Y) \varnothing G = r \varnothing \varphi$ is j-proper and so φ is j-proper, since r is a fibrewise topological equivalence, where $j\in\{S, P, \gamma, \alpha, \beta\}$.

Corollary 3.9. Let $\varphi : X \to Y$ be a continuous fibrewise injection, where X is fibrewise j-compact space and Y is fibrewise Hausdorff space over B. Then φ is closed embedding, where $j\in\{S, P, \gamma, \alpha, \beta\}$.

The corollary is often used in the case when φ is surjective to show that φ is a fibrewise topological equivalence.

Proposition 3.10. Let $\varphi : X \to Y$ be a j-proper fibrewise surjection, where X and Y are fibrewise topological spaces over B. If X is fibrewise Hausdorff then so is Y, where $j\in\{S, P, \gamma, \alpha, \beta\}$.

Proof. Since φ is a j-proper surjection so is $\varphi\times\varphi$, in the following figure.

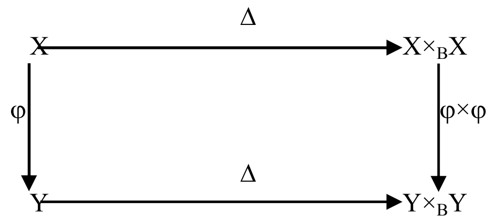

Figure 3.2.

The diagonal Δ(X) closed, since X is fibrewise Hausdorff, hence $((\varphi\times\varphi) \varnothing \Delta)(X) = (\Delta \varnothing \varphi)(X)$ is closed. But $(\Delta \varnothing \varphi)(X) = \Delta(Y)$, since φ is surjective, and so Y is fibrewise Hausdorff, where $j\in\{S, P, \gamma, \alpha, \beta\}$, as asserted.

Proposition 3.11. Let X be fibrewise j-compact and fibrewise Hausdorff space over B. Then X is fibrewise j-regular, where $j\in\{S, P, \gamma, \alpha, \beta\}$.

Proof. Let $x \in X_b$, where $b \in B$, and let U be a j-open set of x in X. Since X is fibrewise Hausdorff there exists for each point $x^* \in X_b$ such that $x^* \notin U$ an open set V_{x^*} of x and an open set $V^*_{x^*}$ of x^* which do not intersect. Now the family of open sets $V^*_{x^*}$, for $x^* \in (X - U)_b$, forms a covering of $(X - U)_b$. Since $X - U$ is j-closed in X and therefore fibrewise j-compact there exists, by Proposition (2.6), a nbd W of b in B such that $X_W - (X_W \cap U)$ is covered by a finite subfamily, indexed by $x^*_1, \ldots, x^*_n$, say. Now the intersection $V = V_{x^*_1} \cap \ldots \cap V_{x^*_n}$ is an open set of x which does not meet the open set $V^* = V^*_{x^*_1}$ Ɽ …Ɽ $V^*_{x^*_n}$ of $X_W - (X_W \cap U)$. Therefore the closure $X_W \cap Cl(V)$ of $X_W \cap V$ in X_W is contained in U, where $j\in\{S, P, \gamma, \alpha, \beta\}$, as asserted.

We extend this last result to

Proposition 3.12. Let X be fibrewise locally j-compact and fibrewise Hausdorff space over B. Then X is fibrewise j-regular, where $j\in\{S, P, \gamma, \alpha, \beta\}$.

Proof. Let $x \in X_b$, where $b \in B$, and let V be a j-open set of x in X. Let W be a nbd of b in B and let U be an open set of x in X_W such that the closure $X_W \cap Cl(U)$ of U in X_W is fibrewise j-compact over B. Then $X_W \cap Cl(U)$ is fibrewise j-regular over W, by Proposition (3.11), since $X_W \cap Cl(U)$ is fibrewise Hausdorff over W. So there exists a nbd $W^* \subset W$ of b in B and an open set U^* of x in X_{W^*} such that the closure $X_{W^*} \cap Cl(U^*)$ of U^* in X_{W^*} is contained in $U \cap V \subset V$, where $j\in\{S, P, \gamma, \alpha, \beta\}$, as required.

Proposition 3.13. Let X be fibrewise j-regular space over B and let K be a fibrewise j-compact subset of X. Let b be a point of B and let V be a j-open set of K_b in X. Then there exists a nbd W of b in B and an open set U of K_W in X_W such that the closure $X_W \cap Cl(U)$ of U in X_W is contained in V, where $j\in\{S, P, \gamma, \alpha, \beta\}$.

Proof. We may suppose that K_b is non-empty since otherwise we can take $U = X_W$, where $W = B - p(X - V)$. Since V is a j-open set of each point x of K_b there exists, by fibrewise j-regularity, a nbd W_x of b and an open set $U_x \subset X_{W_x}$ of x such that the closure $X_{W_x} \cap Cl(U_x)$ of U_x in X_{W_x} is contained in V. The family of open sets $\{X_{W_x}$

$\cap U_x$; $x \in K_b\}$ covers K_b and so there exists a nbd W^* of b and a finite subfamily indexed by $x_1, \ldots, x_n$, say, which covers K_W. Then the conditions are satisfied with

$$W = W^* \cap W_{x_1} \cap \ldots \cap W_{x_n}, \; U = U_{x_1} \cup \ldots \cup U_{x_n}.$$

Corollary 3.14. Let X be fibrewise j-compact and fibrewise j-regular space over B. Then X is fibrewise j-normal, where $j \in \{S, P, \gamma, \alpha, \beta\}$.

Proposition 3.15. Let X be fibrewise j-regular space over B and let K be a fibrewise j-compact subset of X. Let $\{V_i; i = 1, \ldots, n\}$ be a covering of K_b, where $b \in B$ by j-open sets of X. Then there exists a nbd W of b and a covering $\{U_i; i = 1, \ldots, n\}$ of K_W by open sets of X_W such that the closure $X_W \cap Cl(U_i)$ of U_i in X_W is contained in V_i for each i, where $j \in \{S, P, \gamma, \alpha, \beta\}$.

Proof. Write $V = V_2 \cup \ldots \cup V_n$, so that $X - V$ is j-closed in X. Hence $K \cap (X - V)$ is j-closed in K and so fibrewise j-compact. Applying the previous result to the j-open set V_1 of $K_b \cap (X - V)_b$ we obtain a nbd W of b and an open set U of $K_W \cap (X - V)_W$ such that $X_W \cap Cl(U) \subset V_1$. Now $K \cap V$ and $K \cap (X - V)$ cover K, hence V and U cover K_W. Thus $U = U_1$ is the first step in the shrinking process. We continue by repeating the argument for $\{U_1, V_2, \ldots, V_n\}$, so as to shrink V_2, and so on, where $j \in \{S, P, \gamma, \alpha, \beta\}$. Hence the result is obtained.

Proposition 3.16. Let $\varphi : X \to Y$ be a j-proper, j-irresolute fibrewise surjection, where X and Y are fibrewise topological spaces over B. If X is fibrewise j-regular then so is Y, where $j \in \{S, P, \gamma, \alpha, \beta\}$.

Proof. Let X be fibrewise j-regular. Let y be a point of Y_b, where $b \in B$, and let V be a j-open set of y in Y. Then $\varphi^{-1}(V)$ is a j-open set of the j-compact $\varphi^{-1}(y)$ in X. By Proposition (3.13), therefore, there exists a nbd W of b in B and an open set U of $\varphi^{-1}(y)$ in X_W such that the closure $X_W \cap Cl(U)$ of U in X_W is contained in $\varphi^{-1}(V)$. Now since φ_W is closed there exists an open set V^* of y in Y_W such that $\varphi^{-1}(V^*) \subset U$, and then the closure $X_W \cap Cl(V^*)$ of V^* in X_W is contained in V since

$$Cl(V^*) = Cl(\varphi(\varphi^{-1}(V^*))) = \varphi(Cl(\varphi^{-1}(V^*))) \subset \varphi(Cl(U)) \subset \varphi(\varphi^{-1}(V)) \subset V.$$

Thus Y is fibrewise j-regular, where $j \in \{S, P, \gamma, \alpha, \beta\}$, as asserted.

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