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ON REGULAR MODULES

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ABSTRACT

Let R be a commutative ring with 1, and let M be a unitary (left) R-module. M is called Z regular module, if every cyclic submodule (equivalently, every f.g submodule) of M is projective and direct summand . And the module M is called F regular, if every submodule N of M is pure . In this paper we study a class of modules that lies between Z regular modules and F regular modules, we call these modules regular modules .

INTRODUCTION

Let R be a commutative ring with 1, and let M be a unitary (left) R-module. M is called Z regular , if every cyclic submodule (equivalently, every f.g submodule) of M is projective and direct summand [12]. And A submodule N of M is said to be pure in M if $N \cap IM = IN$ for each ideal I of R [3]. The module M is called F regular, if every submodule N of M is pure [3] . In this paper we study a class of modules that lies between Z regular and F regular modules.

1- SOME PROPERTIES OF REGULAR MODULES:

__(1-1) Definition:__ For each natural number n, we call the R-module M an n-regular module, if each n- submodule N of M is a direct summand . And we say that M is ∞ - regular if each f.g n- submodule N of M is a direct summand.

It is clear that each n- regular module is m- regular for each $m \leq n$. Since in a semisimple module every submodule is a direct summand , then it is clear that every semisimple module is ∞- regular.

Recall that the Jacobson radical of an R-module M, denoted by J(M) is the sum of all small submodules of M [11] so we have .

(1-2) Proposition: Let M be an R- module, if M is 1- regular, then J(M)=0 .

Proof: Let $0 \neq x \in M$, Rx is a direct summand of M. so there exists a submodule K of M such that $M = Rx \oplus K$ and by the definition of the direct sum, Rx cannot be small, hence J(M)=0.

(1-3) Corollary: Let M be a 1-regular module. For each $0 \neq x \in M$, there exists maximal submodule N of M such that $x \notin N$.

Proof: If x is containd in each maximal submodule, then $x \in J(M)$, but J(M) =0 So x=0 which is contradiction with the assumption .

(1-4) Proposition: Let M be an R- module, if M is an- regular module, then M_p is also ∞- regular module for each prime ideal P of R.

Proof: Let P be a prime ideal of R and let A be an R_p f.g submodule of M_p. Thus there exists a f.g submodule N of M with $N_p = A$. By assumption N is a direct summand of M, so there exists a submodule K of M such that M=K+N . Note that $M_p = (K \oplus N)_p = K_p \oplus N_p$ [3], i.e $Mp = K_p \oplus A$, A is direct summand of M_p, this means M_p is an - regular module .

(1-5) Proposition : Let $M = M_1 \oplus M_2$ where M_1 and M_2 are two modules. If each of M_1, M_2 is n-regular then M is n-regular provided $\text{ann}\, M_1 + \text{ann} M_2 = R$.

Proof : Let N submodule of M. then by [10], $N=N_1\oplus N_2$ where N_i is a submodule of M_i i=1,2 . It is clear that if N is n –generated then N_1,N_2 are n-generated . By assumption Ni is direct summand of M_i $\forall i=1,2$ i.e there exists a submodule K_i of Mi such that $M_i=K_i \oplus N_i$, i=1,2. But $M=M_i \oplus M_2$ thus $M =(K_i \oplus N_i) \oplus (K_2 \oplus N_2) = (K_i \oplus K_2) \oplus (N_1\oplus N_2)$. This means $N_1\oplus N_2 = N$ is direct summand of M.

(1-6) ***Proposition :*** Let $M=M_1\oplus M_2$ where M_1 and M_2 are two modules. If M is n-regular then so is M_i $\forall i=1,2$.

Proof : Let N_1 submodule of M_1hence N_1 is a submodule of M is direct summand of M, that there exists a submodule K of M such that $M=K \oplus N$, i=1,2. But $M=M_i \oplus M_2$ then $M_1 =(M_2 \oplus K_1 \oplus N_1)$ N_1 is direct summand of M_1 and hence M_1 is an n- regular . .

(1-7) ***Proposition :*** If R is an ∞ - regular as R – module, then R is Bezout ring .

Proof: Let I be a f. g ideal of R. then I is direct summand of R, then there exists an epimorphism f: R $\rightarrow$ I. But R is cyclic R-module, hence I is principal .

(1-8) Proposition : Let R be a ring . The following statements are equivalent

1- R is regular ring .
2- R is ∞ - regular as an R- Module.
3- R is 1- regular as an R- module .

Proof : (1) $\rightarrow$ (2) See [1]
(2) $\rightarrow$ (3) Clear
(3) $\rightarrow$ (1) See [1]

(1-9) Remark: Every projective module over a regular ring is ∞ -regular .

Proof: See [8] .

Recall that an R-module M is said to be a multiplication module , if each submodule N of M is of the form IM for some ideal I of R [2].

(1-10) Proposition: Let M be a multiplication injective module . If M is 1-regular then every cyclic submodule of M is injective and hence End(M) is regular ring .

Proof: Let Rm be a cyclic submodule of M . By assumption, Rm is a direct summand of M. But M is an injective module , so by [4] Rm is injective and by [5:3-11] End (M) is regular .

Now it is clear that every ∞ -regular module is 1-regular, but the converse does not seen to be true, in fact we were not able to fined an example. However under some condition 1-regular became ∞ - regular as the following results .

(1-11) Theorem: Let M be a projective module. The following statements are equivalent .

1- Every homomorphic image of M is flat .
2- M is 1-regular module .
3- M is ∞-regular module .

Proof: See [8].

(1-12) Proposition: If R is a Bezout ring . Then R is 1-regular module iff it is ∞-regular module.

Proof: Let I be a f.g ideal of R . Since R is a Bezout ring so I is a principal, by assumption I is direct summand of R. The converse is clear .

Recall that an R- module M is said to be C.P.(F.G.P.), if each cyclic (f.g) submodule of M is projctive . And it is said to be C.F. (F.G.F) , if each cyclic (f.g) submodule of M is flat [6], mow we have the following proposition .

(1-13) Proposition: Let R be a ring. If R ia a Bezout ring then every multiplication C.P module over R is ∞-regular .
bold: Assume R is a Bezout ring and let M be a multiplication C.P module . Let N be a f.g submodule of M, there exists an ideal I of R such that N= IM. It is easily seen that one may choose I to be f.g [9]. Since R is a Bezout ring so there exists $a \in R$ such that I=Ra and N=aM. Define $f_a:M \rightarrow N$ by $f_a(m)=am$ $\forall m \in M$, then f_a is an R-epimorphism . But N is projective, thus the sequence splits [4], and then N is a direct summand , hence M is ∞-regular module.

2-THE RELATION BETWEEN REGULAR MODULES AND THE Z REGULAR MODULES AND THE F REGULAR MODULES .

Note that the R module M is Z regular if M is 1-regular and each cyclic submodule of M is projective [12], also M is Z regular if M is ∞-regular and each f.g submodule of M is projactive [12]. A regular module is not Z regular module in general, for example, Z_6 as Z module is 1-regular since every cyclic submodule of Z_6 is direct summand but not projective .

On the other hand every 1-regular module is F regular. In fact in regular module, every cyclic submodule is direct summand and hence is pure submodule [3].

(2-1) Proposition : Let M be a projective R-module. If M is 1-regular, then M is Z regular module and hence is ∞-regular module.

Proof : Let $m \in M$, by assumption $M = R_m \oplus N$ where N is a submodule of M . But M is projective so R_m is projective[4], hence M is Z-regular[8].

(2-2)theorem : Let M be an R-module, The following statements are equivalent .

1- M is Z regular
2- M is ∞-regular and F.G.P. module.
3- M is 1-regular and C.P. module.

Proof: (1)$\rightarrow$ (2) Let N be a f.g submodule of M. By assumption N is a direct summard of M and projective [12].

(2) $\rightarrow$(3) Clear .

(3) $\rightarrow$(1) By [12].

Recall that a submodule N of an R- module M is said to be strongly pure, if for each finite subset $\{a_i\}$ of N (equivalently for each a $\in$N), there exists an R-homomorphism $f:M \rightarrow N$ such that $f(a_i) = a_i \ \forall$ I[5]. And an R-module M is called strongly F regular if each submodule of M is strongly pure[5]. Now we have.

(2-3)theorem : Let M be an R-module.If M is ∞-regular then it is strongly F regular .

Proof: Since every direct summand of M is strongly pure [11], so M is strongly F regular .

The converse does not seem to be true, in fact we were not able to find an example. However under some conditions the converse becomes true such as the following proposition .

(2-4)Proposition: Let M be a strongly R-regular module . If M is an F regular then M is ∞ - regular.

Proof : See [5:(2-3)page 5].

(2-5)Proposition: Let M be a strongly F-regular module . If End (M) is regular then M is ∞ - regular.

Proof: Let N be a f.g submodule of M Then there exists a homomorphism $f{:}M \to N$ such that f(x)=x $\forall x \in M$. By [12:lemma 3-3]M=f(M) $\oplus$ ker f=N $\oplus$ ker f.

(2-6)Proposition: Let M be multiplication R- module . if M is a strongly F regular then it is ∞ - regular.

Proof: By assumption End (M) is regular [5], and by (2-5)we get the result .

(2-7)theorem : Let M be a multiplication R-module, The following statements are equivalent .

1- End (M) is regular .
2-M is strongly F regular module.
3- M is regular S- module, Where S=End(M).
4- M is ∞-regular module.

Proof: See [5:th.3-6].

3- RELATION OF REGULAR MODULES WITH OTHER MODULES

Recall that an R-module M is said to be a selfgenerator of type S_n where n is a natural number, if for each n-generated submodule N of M, there exists an epimorphism $f{:}M \to N$. And we say that M is a selfgenerator of type S_∞, if it is of type S_n $\forall n \in N$. Finally, we say that M is of type S, if each submodule N of M is an epimorphic image of M[7]. It is clear that every n-regular module is selfgenerator of type S_n, but the converse is false, for example the submodule

I= $\{\bar{0},\bar{1},....,\overline{10}$ is epimorphic image of the Z-module Z_{12} . In fact the map : $Z_{12} \rightarrow$ I defined by f(x)=2x is epimorphism. On the other hand it is clear that I is not direct summand of Z_{12}[14].

In this section we try to ralate the concept of modules under study with the special selfgenerator modules, C.P.(F.G.P.) and C.F(F.G.F.) modules , we start by the following proposition .

(3-1) Proposition: Let M be C.P . moldule . M is a selfgenerator of type S_1, iff it is 1-regular .

Proof: $\Rightarrow$) Let N be any cyclic submodule of M, there exists an epimorphism $f:M \rightarrow N$. But N is projective , thus the sequence is split [4]. Thus N is a direct summand .
$\Leftarrow$) Clear .

By the same way we can prove .

(3-2) Proposition: Let M be an F.G.P . moldule . M is a selfgenerator of type $S\infty$, iff it is ∞-regular .

(3-3) Proposition: Let M be a projective C.P . moldule . M is a selfgenerator module of type S_1, iff it is 1-regular .

Proof: Let N be a cyclic submodule of M, There exists an epimorphism $f:M \rightarrow N$. Since N is flat, so it is direct summand [8] thus M is ∞-regular module .
$\Leftarrow$) Clear .

By the same way we can prove .

(3-4) Proposition: Let M be a projective F.G.F. module . M is a selfgenerator of type $S\infty$, iff it is ∞-regular .

(3-5)theorem : Let M be a C.P. module . The following statements are equivalent .

1- M is Z regular module.
2-M is ∞-regular module.
3-M is strongly F regular module.
4-M is selfgeneratore of type S_1.
5-M is selfgeneratore of type $S\infty$.

Proof:

(1) $\rightarrow$ (2) Clear
(2) $\rightarrow$ (3) By (2-3)
(3) $\rightarrow$ (4) By [7:(1-6)]
(4) $\rightarrow$ (5) By [7:(1-6)]
(5) $\rightarrow$ (1) By [7:(1-6)]

(3-6)theorem : Let M be a a f. g faithful multiplication module over a P.P. ring then the following statements are equivalent .

1- End (M) is regular ring.
2-M is selfgeneratore of type S_1.
3-M is selfgeneratore of type $S\infty$.
4- M is 1- regular .

Proof:

(1) $\rightarrow$ (2) It follows from [7:(3-6)]
(2) $\rightarrow$ (3) It follows from [7:(3-6)]
(3) $\rightarrow$ (4) By (3-1)
(4) $\rightarrow$ (1) Assume (4), since M is a f.g faithful multiplication module hence End (M) is regular ring[5] .

(3-7)theorem : Let M be a an F regular multiplication module , such that annM (or annihilator of every f.g submodule of M) is generated by an idempotent element Then M is selfgeneratore of type $S\infty$ iff it is ∞-regular .

Proof: $\Rightarrow$) Let N be a f.g submodule of M . By assumption , N is projective submodule [5:(2-4)], so M is an F.G.P [6]. And by (3-2) M is an ∞-regular module.
$\Leftarrow$) Clear .

(3-8) Proposition: Let M be a multiplication R-. module . If M is 2-regular . then every f.g module of M is multiplication .

Proof: Since every 2- regular module is special selfgenerator of type S_2, so by [7: (3-3)] we get the result .
We end this paper by the following proposition .

(3-10) Proposition: Let M be a flat R- module . If M is 2-regular then is C.F. module .

Proof: Let Rx be cyclic submodule of M. By assumption Rx is direct summand of M . But M. is flat so Rx is flat [4] that is mean M is C.F. module .

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حول المقاسات المنتظمة

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الخلاصة

لتكن R حلقة ابدالية ذات عنصر محايد 1 ، وليكن M مقاسا ايسر علىR . يقال ان M مقاس منتظم من النمط Z اذا كان كل مقاس جزئي دائري (وهذا يكافيء كل مقاس جزئي منتهي التولد) من M يكون اسقاطيا وحدا مباشرا . كذلك يقال للمقاس M بانه منتظم من النمط F اذا كان كل مقاس جزئي فيه نقيا . في هذا البحث سندرس صنف من المقاسات يقع بين المقاسات المنتظمة من النمط Z وبين المقاسات المنتظمة من النمط F، نطلق على هذه المقاسات اسم المقاسات المنتظمة .