The Inverse Problem of Delay Integral Equations and Its Application in Population Growth

A Thesis
Submitted to the College of Education Ibn-AL-Haitham of Baghdad University in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics

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بسم الله الرحمن الرحيم

﴿ وَعَلَّمَكَ مَا لَمْ تَكُنْ
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SUPERVISOR CERTIFICATION

We certify that this thesis was prepared under our supervision at the Department of Mathematics, College of Education, Ibn-Al-Haitham, University of Baghdad as partial fulfillment's of the requirement for the degree of Master of Science in Mathematics.

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ABSTRACT

The main theme of this thesis could be divided into three objectives :

The first is to define and classify integral equations with one and multiple delay, including Fredholm, Volterra and integro-differential equations (Retarded, Neutral and Mixed types).

While the second and popular objective of this work is to study the inverse problems related to delay integral equations by using non-classical variational formulation method. Some examples are given for each of the discussed type of delay integral equations.

Also, a study to the direct and inverse problems related to integral equations with multiple delay, also considered as a third objective. Several examples are given for each type of these equations.

Finally, the suggested approach of the inverse problems of delay integral equations is applied on the population growth model.

CONTENTS

Chapter One : Integral Equations and its Generalization to Delay Integral Equations

Chapter Two : Inverse Problem of Delay Integral Equations

Chapter Three : Inverse Problem Application of Delay Integral Equations In Population Growth

INTRODUCTION

Integral equations are one of the most useful mathematical tools in both pure and applied analysis. This is particularly true of problems in mechanical vibrations and the related fields of engineering and mathematical physics, [Tricomi, 1957].

The calculus of variation was studied in the 18th centery by L.Euler and J. L. Lagrange and extensively developed by other mathematicians in the 19th century. At present, it is one of the most important division of theoretical and applied mathematics, [Myskis, 1975].

The basic problem in the calculus of variation is to determine a function such that certain definite integral may involved that function and of its derivatives takes on a minimum or maximum value, [Hildebrand, 1952].

The delay differential equations play an important role in the theory of functional differential equations.

In recent years, the theory of this class of equations had become an independent trend and the literature on this subject comprises over 1000 titles, [Bainov, 1991].

The simplest variational problem with deviating arguments is posed in the following form; which is required to determine the extremum of the functional :

$$\int_{t_0}^{t_1} F(t, x(t), x(t-\tau), x'(t), x'(t-\tau))dt$$

under the conditions that $x(t) = \phi(t)$, for to $\tau \leq t \leq 0$ and $x(t_1) = x_1$. Here τ is a positive constant, the integrand function F will be supposed to be two times differentiable, and the function ϕ differentiable also.

This will correspond to another form, which is called the necessary condition, which is the same delay differential equation, and called the Euler's equation, [Marie, 2001].

Because of the importance of integral equations and delay differential equations in the scientific applications, therefore in this thesis we will discuss and study the inverse problem of delay integral equations.

The aim of this thesis is to study and solve the inverse problem of delay integral equations by using variational approach.

This thesis consists of three chapters :

The first chapter deals with delay integral equations, in which we define delay (and multiple delay) integral equations and give classification of these equations. Also, a relation between these equations and differential equations had been given. After that, we will refer to some methods of the solution and some applications on them.

In the second chapter, we will discuss a noval approach called non-classical variational approach, which solve a delay integral equation. First, the direct problems of integral equations with multiple delay are solved by using variational technique then the inverse problems of one and multiple delay integral equations are discussed and solved by using variational approach as well as we give some examples for each type are discussed.

In the third chapter, the inverse problem of delay integral equations is applied in application for model of population growth, and solved by non-classical variational technique.

The computer programs are written in Quick Basic language and an algorithm had been given to explain these programs.

CHAPTER ONE

Integral Equations and its Generalization to Delay Integral Equations

CHAPTER ONE

Integral Equations and its Generalization to Delay Integral Equations

1.1 INTRODUCTION

One of the most important and applicable subjects of applied mathematics, and in developing modern mathematics is the integral equation.

The integral equation formulation of physical problems is more elegant and compact than the differential equation formulation, since the boundary conditions can be satisfied and embeded in the integral equation. Also, the form of the solution of an integral equation is often more stable for today's extremely fast machine computation, [AL-Mayahi, 1999].

In this chapter, we will define and classify integral equations with delay (or multiple delay), with some applications. Further more, the relation between differential and integral equations with delay is illustrated.

Finally, we refer some methods to solve integral equations and delay integral equations.

1.2 LITERATURY SURVEY

In recent years, there has been a growing interest in the formulation of many physical problems in terms of integral equations, [Golberg, 1979].

The area of integral equations is quite old going back almost to 300 years ago, but most of its theory dates form the 20th century, [Atkinson, 1997].

The name integral equation means an unknown function involved in the integrand which had introduced by Du Bois-Reymond in 1888. However, the early history of integral equations goes back a considerable time before that to Laplace who, in 1782, used the integral transform :

$$f(x) = \int_0^\infty e^{-xs}\,\phi(s)\,ds$$

to solve linear differential equations.

In 1826 Abel solved the integral equation named after him having the form

$$f(x) = \int_a^x (x-s)^{-\alpha}\,\phi(s)\,ds\,,$$

where f(x) is a continuous function satisfying $f(a) = 0$ and $0 < \alpha < 1$, [Moiseiwitsch, 1977].

Delay differential equations have been studied during the last two centuries. The significance of these equations lies in their ability to describe processes with retarded time. The importance of these equations in various branches of technology, economics, biology, and medical science, has been recognized recently and has caused mathematicians to study them with increasing interest. In the last two decades enormous number of papers have been devoted to differential equations involving time delay, [Namik, 1966].

Delay integral equations had been studied by several researchers as [Al-Kiffa'i, 2000], [Al-Safi, 2001] and [Al-Shakhaly, 2001].

Al-Shakhaly in 2001, discussed and studied delay integral equations as well as their solutions using the subject of calculus of variation which will be refered later in chapter two. Also, she studied its application to the model of population growth which is of great importance for future planning throughout the world.

Many researchers worked at different methods to solve and apply delay integral equations, [Burton, 1996], [Cahlon, 1990], [Chen, 1981], [Krueger, 1979], [Rus, 1989], [Smith, 1977], [Werbowki, 1980].

1.3 SOME APPLICATIONS FOR INTEGRAL EQUATIONS AND DELAY DIFFERENTIAL AND INTEGRAL EQUATIONS

The subject of integral equations is used in several domains of applied sciences, such as : population dynamics, the surge in birth rates, the mortality of equipment and the rate of replacement, biological species living together, the torsion of a wire or rod, the control of a rotating shaft, the propation of a nerve impulse, the smoke filtration in a cigarette, the chance of crossing dense traffic, the shape of a hanging chain, the deflection of a rotating rod, and the shape of a wire that allows a bead to descend on it a predetermined time (Abel's problem), [Jerri, 1985].

Another applications of integral equations is in finding the speed of the seismic waves in the layers of the ground, [Tawfeek, 1994]. And inverse problem of integral equations had been discussed as application on radiation heattransfer [Al-Safi, 2001], and on population growth, [Al-Kiffa'i, 2000].

While delay differential equations can be applied actively to prey-predator population models, nuclear reactors, neutron shielding, electron energy distribution in a gas discharge, liquid fuel rocket engines, transistor circuits, electromagnetic vibrators, transmission lines, elasticity theory, the spread of infectious diseases, photo cmulsions, neurology, the respiratory system, business cycles and economic growth, inventory maintenance, the production and death of red blood cells, metal rolling control system [Driver, 1977].

The most recent kind of equations that worth studying is the delay integral equations. These equations have many applications like : A model to explain the observed periodic outbreaks of certain infectious diseases, [Smith, 1977]. Another application is the electromagnetic inverse scattering problem in a medium with discontinuous changes in conductivity and permittivity. We are using time delay in this problem, see [Krueger, 1976, 1979].

1.4 DELAY DIFFERENTIAL EQUATIONS

Delay differential equations had found many applications in mechanics, physics, engineering, economics, biology and especially in the theory of automatic control, [Marie, 2001].

We briefly describe what is meant by delay differential equations and mention the types of these equations because of their importance in constructing delay integral equations.

A delay differential equation or differential difference equation mean an unknown function and certain of its derivatives, evaluated at argument which differ by any of a fixed number of values, [Bellman, 1963].

The general form of the first order delay differential equation is given by

$$a_0\, y'(x) + a_1\, y'(x-\tau) + b_0\, y(x) + b_1\, y\,(x-\tau) = f(x) \quad \text{.........................} \quad (1.1)$$

As a classification, this equation is said to be of retarded type if $a_0 \neq 0$ and $a_1 = 0$ (i.e the delay comes in y only) and is said to be of neutral type if $a_0 \neq 0$, $a_1 \neq 0$, $b_1 = 0$ (i.e. the delay comes in y' only), also it is said to be of mixed type (sometimes it is called an advance type , [Bellman, 1963]) when the delay comes in both y and y', [Al-Shakhaly, 2001], [Marie, 2001].

There exist another types of Eq. (1.1) such as: if $a_0 = a_1 = 0$, the equation is a pure difference equation. If $b_0 = b_1 = 0$, it reduces to a pure difference equation. If $a_0 = b_0 = 0$ or $a_1 = b_1 = 0$ it is an ordinary differential equation, [Bellman, 1963].

τ usually represents here the time lag, and in a special case if $\tau = 0$, then the delay differential equation reduces to an ordinary equation, [Driver, 1977].

Moreover, Eq. (1.1) is called a differential equation with deviating argument [Driver, 1977], or differential-difference equation [Bellman, 1963], or a functional differential equation [Bellman, 1963], or an equation with time lag [Halanay, 1966].

There exist many approximation and numerical methods to solve delay differential equations, but the best well know analytical methods for solving these equations is the method of steps (step by step), [Bellman, 1963] and Laplace transform method, [Marie, 2001].

1.5 DELAY INTEGRAL EQUATIONS

In this section, we are discussing several kinds of delay integral equations. The delay may appears in the unknown function f(x) involved in the integrand, or may appear in the unknown function f(x) in the left hand side of the equation, or may appear in one of limits of the integration.

These three cases can be clarified in the following forms :

- The first type, when the delay appears in the integrand function, may take the form:

$$h(x)\, f(x) = g(x) + \lambda \int_{a}^{b(x)} k(x, y)\, f(y-\tau)\, dy ,$$

.... (1.2), [Al-Shakhaly, 2001].

- The second type may take the form:

$$h(x)\,f(x-\tau) = g(x) + \lambda \int_{a}^{b(x)} k(x,y)\,f(y)\;dy \text{ .... } (1.3)$$

- The third type may take the form :

$$h(x)\,f(x) = g(x) + \lambda \int_{a}^{\tau} k(x,y)\,f(y)\;dy \text{ .... } (1.4)$$

or

$$h(x)\,f(x) = g(x) + \lambda \int_{\tau}^{b(x)} k(x,y)\,f(y) \text{ .... } (1.5)$$

where h, g, k are given functions, λ is a scalar parameter (we will take $\lambda=1$), f(x) is unknown function.

The given function k(x, y) is a kernel of this equation, g(x) is a driving term or free term, and τ is a positive integer ($\tau > 0$) called delay or time lag.

There can be more than one delay in the integral equation. Which are called integral equations with multiple delay, [Corduneanu, 1969].

The integral equation with multiple delay (which have two delays τ_1 and τ_2 such that τ_1 and $\tau_2 > 0$) having the following cases :

- If τ_1 appears in the unknown function f(x) inside the integral sign, and τ_2 appears in the unknown function f(x) outside the integral sign :

$$h(x)\, f(x-\tau_1) = g(x) + \lambda \int_{a}^{b(x)} k(x, y)\, f(y-\tau_2)\, dy \quad \text{.........................} (1.6)$$

- If τ_1 appears in the unknown function f(x) inside the integral sign, and τ_2 appears in one of the limits of integration :

$$h(x)\, f(x) = g(x) + \lambda \int_{a}^{\tau 1} k(x, y)\, f(y-\tau_2)\, dy \quad \text{...........................} (1.7)$$

or

$$h(x)\, f(x) = g(x) + \lambda \int_{\tau_1}^{b(x)} k(x, y)\, f(y-\tau_2)\, dy \quad \text{...........................} (1.8)$$

- If τ_1 appears in the unknown function f(x) outside the integral sign, and τ_2 appears in one of the limits of integration :

$$h(x)\, f(x-\tau_1) = g(x) + \lambda \int_{a}^{\tau_2} k(x, y)\, f(y)\, dy \quad \text{.................................} (1.9)$$

or

$$h(x)\, f(x-\tau_1) = g(x) + \lambda \int_{\tau_2}^{b(x)} k(x, y)\, f(y)\, dy \quad \text{...............................} (1.10)$$

1-5-1 Classification of Integral Equations with Delay and Multiple Delay:

In this subsection, we are classifying several kinds of delay integral equations. All kinds of delay integral equations (either of one or two delays) , already have the same classification of integral equations.

Consider the delay integral equation of the different kinds equations (1.2) – (1.5) :-

When b(x) = x, the delay integral equation is called delay Volterra integral equation with the following types :

- ***The first type***

$$h(x)\, f(x) = g(x) + \lambda \int_{a}^{x} k\,(x, y)\, f\,(y-\tau)\; dy \quad \text{..................................} \quad (1.11)$$

- ***The second type***

$$h(x)\, f(x-\tau) = g(x) + \lambda \int_{a}^{x} k\,(x, y)\, f\,(y)\; dy \quad \text{..................................} \quad (1.12)$$

- ***The third type***

$$h(x)\, f(x) = g(x) + \lambda \int_{\tau}^{x} k\,(x, y)\, f\,(y)\; dy \quad \text{......................................} \quad (1.13)$$

If h(x) = 0, the equations (in all the different types above) are called delay Volterra integral equations of the first kind.

If h(x) = 1, the equations are called delay Volterra integral equations of the second.

When b(x) = b which is a constant, the delay integral equation is called delay Fredholm integral equation, with the following types :

- ***The first type***

$$h(x)\, f(x) = g(x) + \lambda \int_a^b k(x, y)\, f(y - \tau)\, dy \qquad \text{.................................. (1.14)}$$

- ***The second type***

$$h(x)\, f(x - \tau) = g(x) + \lambda \int_a^b k(x, y)\, f(y)\, dy \qquad \text{.................................. (1.15)}$$

- ***The third type***

$$h(x)\, f(x) = g(x) + \lambda \int_a^\tau k(x, y)\, f(y)\, dy \qquad \text{...................................... (1.16)}$$

or

$$h(x)\, f(x) = g(x) + \lambda \int_\tau^b k(x, y)\, f(y)\, dy \qquad \text{...................................... (1.17)}$$

If h(x) = 0, the equations (in all the different types above) are called delay Fredholm integral equations of the first kinds. While if h(x) = 1, the equations are called delay Fredholm integral equations of the second kind.

If $g(x) = 0$ then equations (1.11) - (1.17) are called homogeneous integral equations with delay, while if $g(x) \neq 0$ then equations (1.11) - (1.17) are called non-homogeneous integral equations with delay.

Also, if $h(x)$ is a variable then equations (1.11) - (1.17) are called delay integral equations of the third kind (Fredholm/Volterra).

All previous equations are considered linear integral equations with delay, while non-linear delay integral equation having the unknown function $f(x)$ is non-linear.

Using the same approach in classifying delay integral equations (which contain one delay), we can classify integral equations with two delays. In this case, the integral equations are called integral equations with multiple delay.

1-5-2 Integro-Differential Equations without and Delay :

Integro-differential equation is an equation involving one (or more) unknown function f, together with both differential and integral operations on f. This means that it is an equation contains derivative of the unknown function $f(x)$ which is appear outside the integral sign, [Delves, 1985], [Chambers, 1976].

If the derivative are always taken with respect to a single variable, the integro-differential equations are called ordinary, and of order n. Other integro-differential equations, on the contrary, which often occur in questions of physics and mathematics, contain derivatives with respect to different variables; these equations are called partial integro-differential equations, [Vito Volterra, 1959].

Similarly we can classify delay integro-differential equations as follows:

$$\frac{df}{dx} = g(x) + \lambda \int_a^{b(x)} k(x, y) f(y - \tau)\, dy \quad \text{.......} \quad (1.18)$$

$$\frac{df}{dx} = g(x) + \lambda \int_a^{\tau} k(x, y) f(y)\, dy \quad \text{.......} \quad (1.19)$$

$$\frac{df(x - \tau)}{dx} = g(x) + \lambda \int_a^{b(x)} k(x, y) f(y)\, dy \quad \text{.......} \quad (1.20)$$

whatever it is linear or non-linear, Fredholm or Volterra.

In a similar manar, we can introduce the three types of delay integro-differential equations (which contain two delays) as follows :

- ***Retarded integro-differential equation:***

$$\frac{df}{dx} = g(x) + \lambda \int_a^{\tau_1} k(x, y) f(y - \tau_2)\, dy \quad \text{.......} \quad (1.21)$$

- ***Neutral integro-differential equation :***

$$\frac{df(x - \tau_1)}{dx} = g(x) + \lambda \int_a^{\tau_2} k(x, y) f(y)\, dy \quad \text{.......} \quad (1.22)$$

- ***Mixed integro-differential equation :***

$$\frac{df(x - \tau_1)}{dx} = g(x) + \lambda \int_a^{b(x)} k(x, y) f(y - \tau_2)\, dy \quad \text{.......} \quad (1.23)$$

1-5-3 Transformation of Delay Differential Equations to Delay Integral Equations:

One of the reasons for using a time lag in differential equation is to seek for a continuous extension of a function in this equation into future by previous interval which has a delay. Also, the delay is useful to know the future solution of the equation by means of the previous solution which is defined on an interval which has a delay, [Driver, 1977].

It is necessary to make use of the Libeneze formula :

$$\frac{d}{dx}\int_{A(x)}^{B(x)} F(x,y)\,dy = \int_{A(x)}^{B(x)} \frac{\partial F(x,y)}{\partial x}\,dy + F(x,B(x))\,\frac{dB}{dx} - F(x,A(x))\,\frac{dA}{dx}$$

For differentiating an integral involving a parameter. This equation is valid if both of F and $\frac{\partial F}{\partial x}$ are continuous functions of x and y, [Hildebrand, 1952].

Now, we will transform delay differential equation of retarded type into delay integral equation as follows :

Consider a delay differential equation of the retarded type :

$$\frac{dy(x)}{dx} = f(x, y(x), y(x-\tau)),\ \tau > 0 \quad \text{...} (1.24)$$

In which the right-hand side depends not only on the position y(x), but also on y(x-τ), the position at τ units back. Integration both sides in Eq. (1.24) from x_0 to x, we have:

$$\int_{x_0}^{x} \frac{dy(t)}{dt}\, dt = \int_{x_0}^{x} f(t, y(t), y(t-\tau))\, dt$$

$$y(x) - y(x_0) = \int_{x_0}^{x} f(t, y(t), y(t-\tau))\, dt$$

then,

$$y(x) = y(x_0) + \int_{x_0}^{x} f(t, y(t), y(t-\tau))\, dt \quad \text{...} \quad (1.25)$$

which is equivalent to (1.24).

Equation (1.25) is called delay integral equation.

Now, conversely we can transform equation (1.25) with delay to delay differential equation (1.24) as follows :

Differentiated both sides in Eq.(1.25) with respect to x, we have :

$$y'(x) = \frac{d}{dx} \int_{x_0}^{x} f(t, y(t), y(t-\tau))\, dt$$

and by using Libeneze formula, we get :

$$y'(x) = f(x, y(x), y(x-\tau))$$

By the same approach, we can transfer delay differential equation of the neutral or mixed type to delay integral equation, [Bainov, 1991].

Also, we can find an equivalence between the solution of delay differential equation and delay integral equations, [Al-Shakhaly, 2001].

Theorem (1.1) [Al-Shakhaly, 2001]

A function ϕ_0 is a solution of delay differential equation :

$y'(x) = f(x, y, y(x-\tau))$,

such that $y(x) = \phi_0(x)$ where x belongs to the interval $[x_0-\tau, x_0]$ if and only if it is a solution of the integral equation .

$$y = y_0 + \int_{x_0}^{x} f(t, y(t), y(t-\tau))\, dt$$

Proof:

Let ϕ_0 be a real-value differentiable functional. Suppose ϕ_0 is a solution of the delay differential equation:

$$y'(x) = f(x, y(x), y(x-\tau))$$

such that $y(x) = \phi_0(x)$ where $x \in [x_0-\tau, x_0]$.

Then :

$$\phi'_0(x) = f(x, \phi_0(x), \phi_0(x-\tau)), \text{ for } x \in [x_0-\tau, x_0] \text{ . ... (1.26)}$$

Since ϕ_0 is continuous on the interval $[x_0-\tau, x_0]$. The function F defined by :

$F(x) = f(x, \phi_0(x), \phi_0(x-\tau))$ is continuous on $[x_0-\tau, x_0]$. Integration (1.26) from x_0 to x, we obtain:

$$\phi_0(x) = \phi_0(x_0) + \int_{x_0}^{x} f(t, \phi_0, (t), \phi_0(t-\tau))\, dt \quad \text{..............................} \quad (1.27)$$

Since $\phi_0(x)=y$. We see that ϕ_0 is a solution of delay integral equation :

$$y = y_0 + \int_{x_0}^{x} f(t, y(t), y(t-\tau))\, dt$$

Conversely, since ϕ_0 satisfy the integral equation on $[x_0-\tau, x_0]$ and by using Libeneze formula to differentiate (1.27), we get :

$$\phi'_0(x) = f(x, \phi_0(x), \phi_0(x-\tau))$$

It is clearly that $y(x) = \phi_0(x)$ for $x \in [x_0-\tau, x_0]$. Thus ϕ_0 is a solution to the delay differential equation.

Similar argument could be made to the next time steps, which ensure the equivalence of solutions.

1-5-4 Methods for Solving Delay Integral Equations:

The solutions of the integral equations depend on the type of integral equation Fredholm or Volterra, first kind or second kind, linear or non-linear, homogeneous or non-homogenous.

Also, solutions of the integral equations depend on the kernel of the integral equation, whether it is degenerate, symmetric, difference, resolvent, iterated type, [Al-Shakhaly, 2001].

Sometimes the integral equation do not have exact, closed-form solutions, or it is difficult to solve it, also the explicit solution may not be reasonable, therefore, we must resort to the numerical or approximate methods, where the integral representations more suitable, [Al-Shakhaly, 2001], [Al-Saady, 2000], [Pinney, 1959].

There are many approximate and numerical methods to solve integral equations like successive approximations, neumann series, collecation method, Rayleigh-Ritz method for estimating eigen values, the Galerkin approximate method, the least square methods, see [Atkinson, 1997], [Chambers, 1976], [Delves, 1985], [Delves, 1974], [Golberg, 1979], [Hildebrand, 1952], [Jerri, 1985].

There are some methods to solve delay integral equations such that collecation method and variational approach see [Burgstaller, 2000] and [Al-Shakhaly, 2001].

CHAPTER TWO

Inverses Problem OF Delay Integral Equations

CHAPTER TWO

Inverse Problem of Delay Integral Equations

2.1 INTRODUCTION

Our aim in this chapter is to study and discuss the direct and inverse problems related to delay integral equations using variational approach.

Al-Shakhaly in 2001, studied in details variational formulation and solutions of delay integral equations of Fredholm, Volterra and integro-differential equations.

We will give a summery of representation to the variational formulation of linear non-homogenous (homogenous) integral equations with one and two delays (which is the direct problem) of the two types, Fredholm and Vollterra integral equations of the first and second kinds, also to integro-differential equations (retarded, neutral, and mixed).

Also, we present the subject of inverse problem which is related to integral equations with one and two delays, and solved by using variational approach in connection with least sequare methods.

Several examples are discussed also in this chapter in order to illustrate the inverse problem.

2.2 INVERSE PROBLEM

In the past two decades, the theory and practice of inverse problems have been developed in several domains of applied science: medical diagnosis, atmospheric sounding, radar and sonar target estimation, seismology, radioastronomy, microscopy and so on.

Nowdays, operational applications are in used every day, for instance, in seismic data processing for geophysical exploration, and in radiometric data processing for meteorological forecasts and monitoring, and in X-ray or MR (magnetic resonance) tomography, [Mahlol, 1993].

In general, the mathematical model of a problem is governed by a rule that defines a mapping M from a set of parameters A into a set of results B. Therefore; a mapping $M : A \to B$ is called the direct problem, and solving the direct problem means that for a given elements of A, the model could be solved to obtain the elements B. While the inverse problem of a mapping M (direct problem) which could be denoted by $M^{-1} : B \to A$ is called an inverse problem, and solving the inverse problem means that for a given element of B "solution of the problem" with some unknown parameters of A one have to find the remaining unknown elements "parameters of A", [Ali, 1994] and [Al-Soudany, 2001].

Many researchers had the inverse problem. Mahlol in 1993, concerned with the problem of medical diagnosis, which is the main goal to locate brain tumors using electron-cephalogram records, [Mahlol, 1993]. Ali in 1994, studied the problem of inverse acoustic wave scattering using the direct variational methods. Tawfeek in 1994, studied the speed of seismic waves in underground. This problem is formulated as an inverse mathematical problem involving a non-linear integral equation. Also, Al-Ani in 2001, solved the inverse problem formulation for the seepage problem with singularity, where the solution is given in advance and to evaluate the width of the dam, using the variational approach.

Al-Kiffa'i in 2000, studied the direct and inverse problems related to Fredholm integral equations by using variational approach with its application on population growth when the integral equation involve the unknown survival population at time t as a driving term g, which is solved by the non-classical variational approach. While Al-Safi in 2001, studied the direct and inverse problems related to Volterra integral equations and integro-differential equations by using the non-classical variational technique, taking the application of radiation heattransfer to find the surface reflectivity. This problem is formulated as a mathematical inverse problem. In addition, Al-Soudany in 2001, sloved the direct and inverse problem of Sturm-Liouville ordinary differential equations using multiobjective criteria and modified an approach to slove the direct and inverse problems in partial differential equations to determine the shape and radius of the region of definition of wave equation.

Krueger in 1979, studied the application of the electromagnetic inverse scattering problem in a medium with discontinuous changes in conductivity and permittivity. We are using time delay in this problem, see [Krueger, 1979].

Several well known methods could be used to solve the inverse problems such as least sequare method, ε-method, the pulse-Spectrum technique method, Bellman-Adomian method and direct variational method, [Mahlol, 1993], and [Ali, 1994].

2.3 THE DIRECT AND INVERSE PROBLEMS OF CALCULUS OF VARIATION

2-3-1 The Direct Problem of Calculus of Variation :

The basic idea of the subject of calculus of variation is the derivation of a necessary condition formulated as an equation which is called the Euler-Lagrange equation that represents the linear problem which have to be solved in operator form, [Myskis, 1975].

The derivation of this equation depends on the so called the first variation of a functional, [El'sgolc, 1962].

Definition 2.3.1 :-

A first variation of a functional F(u) defined on some normed linear space is defined by :

$$\delta F[u] \equiv F[u + \delta u] - F[u]$$

where, the symbol δ is a customary symbol representing variation in the functional, [Magri, 1974] and [Vito Volterra, 1959].

The element $u_0 \in U$ is called the critical point of functional F, and the functional F is said to be stationary at u_0 if $\delta F[u_0]|_{\text{linear part}} = 0$, [Al-A'asam, 1987].

The most developed branch of calculus of functionals is concerned with finding the critical points of functionals and it is called calculus of variation. Also, problems of such types are called variational problems, [Gelfand, 1963].

The main problem in calculus of variation is to find the maximum or minimum values of a given functional F(f), its necessary condition is called the Euler-Lagrange equation, and the solution of this problem is called the direct problem of calculus of variation, [El’sgolc, 1962].

2-3-2 The Inverse Problem of Calculus of Variation :

The most difficulty in the calculus of variation is in the construction a variational formulation, which corresponds to the linear equation:

$Lu = f$.. (2.1)

and the most popular approach for evaluating this functional F[u] is the non-classical variational approach which could be summarized as follows:

consider the linear equation (2.1) where $u \in U$, f is a vector valued function and L denotes a linear operator, with domain D(L) in a linear space U and any range R(L) in a second linear space V, [Magri, 1974]. The problem is to find a functional F[u] defined on the domain of the linear operator L, whose critical points are the solutions of the given equation (2.1).

This problem may be called the inverse problem of calculus of variation, while the usual problem of finding the critical points of a pre assigned functional may be called the direct problem, [Al-Safi, 2001] and [Al-Shakhaly, 2001].

The basic theorem of the theory of the inverse problem of calculus of variation is as follows:

Theorem 2.3.1

The solution of equation (2.1) is the critical points of the functional :

$$F[u] = \frac{1}{2} \langle Lu, u\rangle - \langle f, u\rangle \quad \text{.......} \quad (2.2)$$

If and only if the given linear operator L is symmetric with respect to the chosen bilinear form <u, v> which is non-degenerate.

Therefore; instead of solving the linear equation (2.1), one can find the critical points of (2.2), [Magri, 1974].

Remark 2.3.1

The linear operator L must be symmetric with respect to the chosen non-degenerate bilinear form <u, v>. But if the linear operator L is not symmetric with respect to the chosen bilinear form <u, v>, the problem is to obtain the variational formulation, using the transformation:

$$(u, v) = \langle u, Lv\rangle \quad \text{.......} \quad (2.3)$$

where $v \in V$ and $u \in D(L)$.

The bilinear form (2.3) makes the given linear operator symmetric since:

$$(Lu_1, u_2) = \langle Lu_1, Lu_2\rangle = \langle Lu_2, Lu_1\rangle = (Lu_2, u_1) \quad \text{.......} \quad (2.4)$$

Therefore, in general we will use the bilinear form (2.3) to obtain a variational formulation because of the symmetry of L.

Since L is symmetric and by using theorem (2.3.1), the solution of equation $Lu = f$ is a critical point to the functional :

$$F(u) = \frac{1}{2}\,(Lu, u) - (f, u)$$

$$= \frac{1}{2}\,\langle Lu, Lu\rangle - \langle f, Lu\rangle \text{ .... } (2.5)$$

The functional (2.5) is a variational formulation for the linear equation Lu = f, [Magri, 1974].

2.4 VARIATIONAL FORMULATION OF DELAY INTEGRAL EQUATIONS

2-4-1 Variational Formulation of Integral Equations with One Delay :

In this section, we will find a functional F(f) corresponding to the linear delay integral equation in operator form Lf = g.

The first step is to define the operator L related to the delay integral equation for each type that had been discussed above with an examination to the linearity and similarity of this operator with respect to the non-degenerate bilinear form (u, v) = <u, Lv>.

Then applying theorem (2.3.1), to find the functional F(f) variational formulation corresponding to the linear operator L, [El’sgolc, 1962].

Finally, this functional will be minimized using the direct Ritz methods in connection with optimization methods and Gaussian integration method of degree 7, [Burden, 1985].

Al-Shakhaly in 2001, solved the delay integral equation with one delay of the following types:

♦ ..Delay Fredholm integral equations of the first and second kind.

♦ ..Delay Volterra integral equations of the first and second kind.

♦ ..Delay integro-differential equations.

In [Al-Shakhaly, 2001] the variational formulation of delay integral equations was discussed in details which we could be summarized as follows:

∢ The variational formulation of delay Fredholm (Volterra) integral equation of the first kind as follows :

$$F(f) = \int_0^T \left[\frac{1}{2} \left(\int_a^{b(x)} k(x,y) f(y-\tau)\, dy \right)^2 - g(x) \left(\int_a^{b(x)} k(x,y) f(y-\tau)\, dy \right) \right] dx$$

....(2.6)

∢ The variational formulation of delay Fredholm (Volterra) integral equation of the second kind ad follows :

$$F(f) = \int_0^T \left[\frac{1}{2} \left(f(x) - \int_a^{b(x)} k(x,y) f(y-\tau)\, dy \right)^2 - g(x) \right.$$

$$\left. \left(f(x) - \int_a^{b(x)} k(x,y) f(y-\tau)\, dy \right) \right] dx \quad \text{..(2.7)}$$

∢ The variational formulation of delay Fredholm (Volterra) integro-differential equation as follows :

$$F(f) = \int_{0}^{T} \left[\frac{1}{2} \left(\frac{df}{dx} - \int_{a}^{b(x)} k(x,y)\, f(y-\tau)\; dy \right)^{2} - g(x) \right.$$

$$\left. \left(\frac{df}{dx} - \int_{a}^{b(x)} k(x,y)\, f(y-\tau)\; dy \right) \right] dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (2.8)$$

[Al-Shakhaly, 2001].

<u>Remark 2.4.1</u>

The conditions of existence and uniqueness of solutions of delay integral equations will be assumed here to be already satisfied.

Now, variational formulation of delay integral equations with multiple delay will be discussed.

2-4-2 Variational Formulation of Integral Equations with Multiple Delay:

Delay integral equations with multiple delay may occur frequently in different applications. The basic for such type of problems is similar to that discussed in one delay.

The following type of problems will be discussed :

1) When the integral equation takes the form:

$$f(x-\tau_1) = g(x) + \lambda \int_{a}^{b(x)} k(x,y)\, f(y-\tau_2)\; dy \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (2.9)$$

2) The second type with multiple delay takes the form:

$$f(x-\tau_1) = g(x) + \lambda \int_a^{\tau_2} k(x,y)\, f(y)\, dy \quad \text{...} \quad (2.10)$$

3) The third type occurs when the time lag appears under the integral sign and one of the limits of it :

$$f(x) = g(x) + \lambda \int_a^{\tau_1} k(x,y)\, f(y-\tau_2)\, dy \quad \text{.......................................} \quad (2.11)$$

whatever Fredholm or Volterra.

The first step, is to define an operator L related to the delay integral equation corresponding to each type discussed above, with the examination to its linearity and similarity with respect to the non-degenerate bilinear form.

At the beginning, the first type of integral equations with multiple delay whenever the two delays τ_1, τ_2 are given outside and inside the integration of homogenous and non- homogenous, will be considered.

Consider for simplicity and without loose of generality the integral equation with multiple delay :

$$f(x-\tau_1) = g(x) + \lambda \int_a^{b(x)} k(x,y)\, f(y-\tau_2)\, dy \quad \text{................................} \quad (2.12)$$

which have the integral operator of the form :

$$L = D_1 - \int_a^{b(x)} k(x,y)\, D_2 \, dy \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.13)$$

where D_1 is the shift operator with respect to τ_1 defined by :

$$D_1 f(x) = f(x-\tau_1)$$

and D_2 is the shift operator with respect to τ_2 defined by :

$$D_2 f(x) = f(x-\tau_2)$$

The operator L is linear since it is easily seem that :

$$L(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 L(u_1) + \alpha_2 L(u_2)$$

where u_1, u_2 belong to U and α_1, α_2 are any scalar, since :

$$L(\alpha_1 u_1 + \alpha_2 u_2) = \left(D_1 - \int_a^{b(x)} k(x,y)\, D_2 \, dy \right)(\alpha_1 u_1 + \alpha_2 u_2)$$

$$= \alpha_1 D_1 u_1(x) - \int_a^{b(x)} k(x,y)\, D_2 u_1(y)\, dy + \alpha_2 D_1 u_2(x) - \int_a^{b(x)} k(x,y)\, D_2 u_2(y)\, dy$$

$$= \alpha_1 u_1(x-\tau_1) - \int_a^{b(x)} k(x,y) u_1(y-\tau_2)\, dy + \alpha_2 u_2(x-\tau_1) - \int_a^{b(x)} k(x,y) u_2(y-\tau_2)\, dy$$

$$= \alpha_1 \left[D_1 u_1(x) - \int_a^{b(x)} k(x,y) D_2 u_1(y)\, dy \right] + \alpha_2 \left[D_1 u_2(x) - \int_a^{b(x)} k(x,y) D_2 u_2(y)\, dy \right].$$

$$=\alpha_1\left[D_1-\int_a^{b(x)}k(x,y)D_2\,dy\right]u_1(x)+\alpha_2\left[D_1-\int_a^{b(x)}k(x,y)D_2dy\right]u_2(x).$$

$$=\alpha_1\,L\,(u_1(x))+\alpha\,L\,(u_2(x))$$

$$=\alpha_1\,Lu_1+\alpha_2\,Lu_2$$

Therefore, the operator L is a linear.

In order to prove the symmetry of the operator L, the bilinear form $(u_1, u_2) = <u_1, Lu_2>$ could be used to ensure that $(Lu_1, u_2) = (Lu_2, u_1)$.

Because of the symmetry of the linear operator L and by using theorem (2.3.1), the solution of eq. (2.12) is the critical point of the functional F :

$$F(f)=\frac{1}{2}<Lf, Lf>-<g, Lf>$$

$$=\frac{1}{2}\int_0^T L\,f(x)\,L\,f(x)\,dx-\int_0^T g(x)\,L\,f(x)\,dx$$

$$=\int_0^T\left[\frac{1}{2}\left[\left(D_1-\int_a^{b(x)}k(x,y)\,D_2\,dy\right)f(x)\right]^2-g(x)\left(D_1-\int_a^{b(x)}k(x,y)\,D_2\,dy\right)f(x)\right]dx$$

$$= \int_0^T \left[\frac{1}{2} \left(f(x-\tau_1) - \int_a^{b(x)} k(x,y) f(x-\tau_2)\, dy \right)^2 - g(x) \right.$$

$$\left. \left(f(x-\tau_1) - \int_a^{b(x)} k(x,y) f(x-\tau_2)\, dy \right) \right] dx \text{ .... } (2.14)$$

which is the variational formulation of multiple delay integral equation of the second kind (Fredholm or Volterra).

Example (2.1) :

Consider the following integral equation :

$$f(x-\tau_1) = g(x) + \int_{-1}^{1} k(x,y) f(y-\tau_2)\, dy,$$

such that $g(x) = -x^3 + 3x - 3$, $k(x,y) = x^3 - y$, where $\tau_1 = 1$ and $\tau_2 = 2$.

For comparison purpose, the analytical solution corresponding to this problem is given by :

$$f(x) = 3x + 3$$

and hence the operator is given by:

$$L = D_1 - \int_{-1}^{1} (x^3 - y) D_2\, dy.$$

Therefore, the variational formulation (2.14) takes the form :

$$F(f) = \int_{0}^{T=1} \left\{ \frac{1}{2} \left[f(x-1) - \int_{-1}^{1} (x^3 - y) f(y-2)\, dy \right]^2 - \right.$$

$$\left. (-x^3 + 3x - 3) \left[f(x-1) - \int_{-1}^{1} (x^3 - y) f(y-2)\, dy \right] \right\} dx \text{ .... } (2.15)$$

By using the direct Ritz method with the following approximation for f(x) :

$$f_a(x) = a_0 + a_1 x$$

Then $f_a(x-1) = a_0 + a_1 (x-1)$ .... (2.16)

$f_a(x-2) = a_0 + a_1 (x-2)$ .... (2.17)

Substituting (2.16) and (2.17) back into the functional (2.15), we get :

$$F(f) = \int_{0}^{1} \left\{ \frac{1}{2} [2a_0 + a_1(x^3 + x - 3)]^2 - (-3 + 3x - 3x^3) \right.$$

$$\left. [2a_0 + a_1(x^3 + x - 3)] \right\} dx \text{ .... } (2.18)$$

Using the computer program, we get the following results which are presented in table (2.1) with its comparison with the exact solution.

Table (2.1)

Results of the direct problem for evaluating ai in example 2.1

Coefficient of Basic	***Exact Solution***	***Approximate Solution***	***Basic Function***
a_0	3	3.003985	
a_1	3	3.002942	x

The second type of integral equations with multiple delay occur whenever the two delays τ_1 and τ_2 are given outside and on the limit of integration of homogenous, non- homogenous, Fredholm or Volterra.

Consider the following integral equation:

$$f(x-\tau_1) = g(x) + \lambda \int_a^{\tau_2} k(x, y) f(y)\, dy \quad \text{.......................................} (2.19)$$

The operator L corresponding to this equation is defined by:

$$L = D_1 - \int_a^{\tau_2} k(x, y)\, dy \quad \text{...} (2.20)$$

where D_1 is the shift operator with respect to τ_1 defined by :

$$D_1 f(x) = f(x - \tau_1)$$

We can prove similarly as in the first case the linearity and symmetry of the operator L.

Therefore, the variational formulation is given by:

$$F(f) = \frac{1}{2} <Lf, Lf> - <g, Lf>$$

$$= \int_{0}^{T} \left[\frac{1}{2}(Lf)^2 - g(Lf) \right] dx$$

$$= \int_{0}^{T} \left[\frac{1}{2} \left[\left(D_1 - \int_{a}^{\tau_2} k(x,y)dy \right) f(x) \right]^2 - g(x) \left[\left(D_1 - \int_{a}^{\tau_2} k(x,y)dy \right) f(x) \right] dx \right.$$

$$= \int_{0}^{T} \left[\frac{1}{2} \left(f(x-\tau_1) - \int_{a}^{\tau_2} k(x,y) f(y)\, dy \right)^2 - g(x) \left[\left(f(x-\tau_{1)} - \int_{a}^{\tau_2} k(x,y) f(y)\, dy \right) \right] \right] dx$$

....(2.21)

The next example illustrate this type of problems :

Example (2.2) :

Consider the integral equation:

$$3\,(x - \tau_1) = g(x) + \int_{0}^{\tau_2} (xy+1) f(y)\; dy$$

where $g(x) = -\,(5x + 9)$ and $\tau_1 = 1$, $\tau_2 = 2$.

The analytical solution corresponding to this problem is given by :

$f(x) = 3x$

The linear operator takes the form :

$$L = D_1 - \int_0^2 (xy+1)\,dy$$

Then (2.21) leads to :

$$F(f) = \int_0^1 \left\{ \frac{1}{2}\left[f(x-1) - \int_0^2 (xy+1)\,f(x)\,dy \right]^2 + \right.$$

$$\left. (5x+9)\left[f(x-1) - \int_0^2 (xy+1)\,f(x)\,dy \right] \right\} dx \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (2.22)$$

By approximating f(x) by :

$$f_a(x) = a_0 + a_1 x \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (2.23)$$

and substituting (2.23) back into the functional (2.22), resulting the following functional :

$$F(f) = \int_0^1 \left\{ \frac{1}{2}\left[(a_0 - a_1) + a_1 x - \left(2a_0(x+1) + 2a_1\left(1 + \frac{4}{3}x\right)\right)\right]^2 + \right.$$

$$\left. (5x+9)\left[(a_0 - a_1) + a_1 x - \left(2a_0(x+1) + 2a_1\left(1 + \frac{4}{3}x\right)\right)\right] \right\} dx$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.24)$$

By carrying out the computer program, we get the results presented in table (2.2) and their comparison with the exact solution.

Table (2.2)

Results of the direct problem for evaluating ai in example (2.2)

Coefficient of Basic	***Exact Solution***	***Approximate Solution***	***Basic Function***
a_0	0	6.910414 E–07	
a_1	3	3	x

The third type of integral equations with multiple delay occur when the two delays τ_1 and τ_2 appear inside the integral sign and one of the limits of integration.

Consider the following integral equation:

$$f(x) = g(x) + \lambda \int_a^{\tau_1} k(x, y)\, f(y - \tau_2)\, dy \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (2.25)$$

Which have the integral operator of the form :

$$L = I - \int_a^{\tau_1} k(x, y)\, D_2\, dy \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (2.26)$$

where D_2 is the shift operator with respect to τ_2 defined by :

$D_2\, f(x) = f(x - \tau_2)$

We can prove similarly as in the first case the linearity and symmetry of the operator L.

Therefore, the variational formulation is given by :

$$F(f) = \int_0^T \left\{ \frac{1}{2} \left[f(x) - \int_a^{\tau_1} k(x,y) f(x-\tau_2) dy \right]^2 - g(x) \right.$$

$$\left. \left[f(x) - \int_a^{\tau_1} k(x,y) f(x-\tau_2) dy \right] \right\} dx \quad \text{.............................. (2.27)}$$

Similar case could be carried for the first kind integral equations with multiple delay.

We can explain this type in the following example .

Example (2.3) :

Consider the following integral equation:

$$2x^2 + x = g(x) + \int_0^{\tau_1} (x^2 y + 2) f(y-1)\ dy$$

where $\tau_1 = 1$ and $\tau_2 = 1$, and $g(x) = 2x^2 + x - \frac{1}{3}$

The analytical solution corresponding to this problem is given by :

$$f(x) = 2x^2 + x$$

Then (2.27) leads to :

$$
\begin{aligned}
F(f) = \int_0^1 \Bigg\{ & \frac{1}{2}\Bigg[(a_1 x + a_2 x^2) - \Bigg[(a_2 - a_1)\left(\frac{1}{2}x^2 + 2\right) \\
& + \frac{1}{3}(a_1 - 2a_2)x^2 + a_2\left(\frac{1}{3}x^2 - \frac{4}{3}\right) + a1 \Bigg]\Bigg]^2 - g(x) \\
& \Bigg[(a_1 x + a_2 x^2) - \Bigg[(a_2 - a_1)\left(\frac{1}{2}x^2 + 2\right) + \frac{1}{3}(a_1 - 2a_2)x^2 \\
& + a_2\left(\frac{1}{3}x^2 - \frac{4}{3}\right) + a_1 \Bigg]\Bigg] \Bigg\}\, dx \quad \ldots\ldots (2.28)
\end{aligned}
$$

Now, applying the computer program, we get the results written them in table (2.3) and its comparison with the exact solution.

Table (2.3)

Results of the direct problem for evaluating ai in example (2.3)

Coefficient of Basic	***Exact Solution***	***Approximate Solution***	***Basic Function***
a_1	1	1	x
a_2	2	2	x^2

Variational formulation relating to multiple delay integro-differential equations of Frdholm or Volterra types, homogeneous or non-homogeneous, could be given also.

- Retarded integro-differential equations :

The retarded integro-differential equation takes the form :

$$\frac{df}{dx} = g(x) + \int_{a}^{\tau_1} k(x,y)\, f(y-\tau_2)\, dy \quad \text{.........} \quad (2.29)$$

which have the integral operator of the form :

$$L = \frac{d}{dx} - \int_{a}^{\tau_1} k(x,y)\, D_2\, dy \quad \text{.........} \quad (2.30)$$

The operator L is also linear and symmetric.

Therefore, the variational formulation is given by :

$$F(f) = \frac{1}{2} <Lf, Lf> - <g, Lf>$$

$$= \int_{0}^{T} \left\{ \frac{1}{2} \left[\left(\frac{d}{dx} - \int_{a}^{\tau_1} k(x,y)\, D_2\, dy \right) f(x) \right]^2 \right.$$

$$\left. - \, g(x) \left[\left(\frac{d}{dx} - \int_{a}^{\tau_1} k(x,y)\, D_2\, dy \right) f(x) \right] \right\} dx$$

$$= \int_{0}^{T} \left\{ \frac{1}{2} \left[\frac{df}{dx} - \int_{a}^{\tau_1} k(x,y)\, f(y-\tau_2)\, dy \right]^2 - g(x) \right.$$

$$\left. \left[\frac{df}{dx} - \int_{a}^{\tau_1} k(x,y)\, f(y-\tau_2)\, dy \right] \right\} dx \quad \text{.........} \quad (2.31)$$

- Neutral integro-differential equation:

The governing equation of this problem is given by :

$$\frac{df\,(x-\tau_1)}{dx}=g(x)+\int_a^{\tau_2} k(x,y)\,f(y)\,dy \quad\text{...}\quad (2.32)$$

with the operator :

$$L=D_1\,\frac{d}{dx}-\int_a^{\tau_2} k(x,y)\;dy \quad\text{...}\quad (2.33)$$

where D_1 is the shift operator with respect to τ_1 defined by:

$$D_1\,\frac{d}{dx}\,f(x)=\frac{df\,(x-\tau_1)}{dx}$$

This operator is obviously linear and symmetric, and the variational formulation is given by :

$$F(f)=\frac{1}{2}<Lf, Lf>-<g, Lf>$$

$$=\int_0^T\left\{\frac{1}{2}\left[\left(D_1\,\frac{d}{dx}-\int_a^{\tau_2} k(x,y)\;dy\right)f(x)\right]^2-g(x)\right.$$

$$\left.\left[\left(D_1\,\frac{d}{dx}-\int_a^{\tau_2} k(x,y)\;dy\right)f(x)\right]\right\}dx$$

$$=\int_0^T\left\{\frac{1}{2}\left[\frac{df\,(x-\tau_1)}{dx}-\int_a^{\tau_2} k(x,y)\,f(y)\,dy\right]^2-\right.$$

$$g(x)\left[\frac{df(x-\tau_1)}{dx} - \int_a^{\tau_2} k(x,y)\,f(y)\,dy\right]\Bigg\}\,dx \quad \text{.................} \quad (2.34)$$

- Mixed integro-differential equation :

The mixed integro-differential equation has the form :

$$\frac{df(x-\tau_1)}{dx} = g(x) + \lambda \int_a^{b(x)} k(x,y)\,f(y-\tau_2)\,dy \quad \text{...............................} \quad (2.35)$$

with the operator :

$$L = D_1 \frac{d}{dx} - \int_a^{b(x)} k(x,y)\,D_2\,dy \quad \text{...} \quad (2.36)$$

where D_1 is the shift operator with respect to τ_1 and D_2 is the shift operator with respect to τ_2, such that :

$$D_1 \frac{d}{dx} f(x) = \frac{df(x-\tau_1)}{dx}$$

$$D_2\, f(x) = f(x-\tau_2)$$

Also, the operator is linear and symmetric. The variational formulation of the mixed integro-differential equation has the form :

$$F(f) = \frac{1}{2} <Lf, Lf> - <g, Lf>$$

$$= \int_0^T \left\{ \frac{1}{2} \left[\left(D_1 \frac{d}{dx} - \int_a^{b(x)} k(x,y)\, D_2 dy \right) f(x) \right]^2 \right.$$

$$\left. - g(x) \left(D_1 \frac{d}{dx} - \int_a^{b(x)} k(x,y)\, D_2 dy \right) f(x) \right\} dx$$

$$= \int_0^T \left\{ \frac{1}{2} \left[\frac{df\,(x-\tau_1)}{dx} - \int_a^{b(x)} k(x,y)\, f(y-\tau_2)\, dy \right]^2 \right.$$

$$\left. - g(x) \left[\frac{df\,(x-\tau_1)}{dx} - \int_a^{b(x)} k(x,y)\, f(y-\tau_2)\, dy \right] \right\} dx \;\ldots\ldots\ldots\ldots (2.37)$$

The following example is given to illustrate the situation in integro-differential equations with multiple delay.

Example (2.4) :

Consider the following equation:

$$\frac{df}{dx} = g(x) + \int_0^{\tau_1} x\, y\, f(y-\tau_2)\, dy\,,$$

where $g(x) = 1 - \frac{17}{6} x$ and $\tau_1 = 1$, $\tau_2 = 2$.

The analytical solution of this problem is given by:

$f(x) = x + 7$

The functional (2.31) reduces to :

$$F(f) = \int_0^1 \left\{ \frac{1}{2}\left[\frac{df}{dx} - \int_0^1 xy\, f(y-2)\, dy\right]^2 - \left(1 - \frac{17}{6}x\right) \left[\frac{df}{dx} - \int_0^1 xy\, f(y-2)\, dy\right]\right\} dx \quad \text{.......................................} \quad (2.38)$$

Approximating the function f by the following :

$f_a(x) = a_0 + a_1 x$

Hence, $\frac{d f_a}{dx} = a_1$ .... (2.39)

and $f_a(x\text{-}2) = a_0 + a_1 (x-2)$ .... (2.40)

Substituting (2.39) and (2.40) back into the functional (2.38), we have:

$$F(f) = \int_0^1 \left\{ \frac{1}{2}\left[a_1 - \left(\frac{1}{2}(a_0 - 2a_1) + \frac{1}{3}a_1\right)x\right]^2 - g(x)\left[a_1 - \left(\frac{1}{2}(a_0 - 2a_1) + \frac{1}{3}a_1\right)x\right]\right\} dx \quad \text{......................} \quad (2.41)$$

When we carry out the computer program, we get the following results and their comparison with the exact solution :

Table (2.4)

Results of the direct problem for evaluating ai in example (2.4)

Coefficient of Basic	***Exact Solution***	***Approximate Solution***	***Basic Function***
a_0	7	7.00008	
a_1	1	1	x

2.5 INVERSE PROBLEM OF DELAY INTEGRAL EQUATIONS

2-5-1 Inverse Problem of Integral Equations with One Delay :

Several kinds of the inverse problems which are related to the integral equations with one delay of all types, could be considered. These inverse problems could be summerized as follows :

The First Inverse Problem

Suppose that the functions f, g and k are known in advance, but the upper (or lower) limit of the integration is to be estimated. This case is illustrated in the following example.

Example (2.5) :

Consider the following integro-differential equation :

$$\frac{df}{dx} = 0.5291667 + \int_0^b (xy + x)\, f(y-1)\, dy,$$

where the analytical solution related to this problem is given by :

$$f(x) = 0.2\,x + 0.25\,x^2$$

The upper limit b is unknown, and the inverse problem is to find such b.

The approximate solution is written as a linear combination of some basic, by letting:

$$f_a(x) = a_1 x + a_2 x^2 \qquad (\text{fix } a_0 = 0 \quad \text{since } f(0) = 0) \text{ .... } (2.42)$$

Substituting (2.42) back into the suitable functional, we get :

$$F(f) = \int_0^1 \left[\frac{1}{2} \left[\left(\frac{df_a}{dx} - m(x, a_1, a_2, b) \right)^2 - 0.5291667\,x \right.\right.$$

$$\left.\left. \left(\frac{df_a}{dx} - m(x, a_1, a_2, b) \right) \right] \right] dx \text{ .... } (2.43)$$

where $\frac{df_a}{dx} = a_1 + 2a_2 x$

and $m(x, a_1, a_2, b) = \int_0^b (xy + x)\, f_a(y-1)\, dy$

$$= \left[(a_2 - a_1)\left(\frac{1}{2}b^2 + b\right) + (a_1 - 2a_2)\left(\frac{1}{3}b^3 + \frac{1}{2}b^2\right) \right.$$

$$\left. + a_2 b^3 \left(\frac{1}{4}b + \frac{1}{3}\right) \right] x$$

Now, applying the computer program, we get the approximate value of b to be $b^* = 1.0100001$ which has a good result in comparison with the exact value $b = 1$.

The Second Inverse Problem

Suppose that the limits of integration are known as well as the solution f(x) is given, while one of the two functions g and /or k are unknown, which are to be estimated.

In order to solve this problem, one must express each of the two functions, as a linear combination of functions belong to a complete sequence of chosen function.

This case can be explained in the following examples.

Example (2.6) :

Consider the following equation:

$$x = g(x) + \int_{0}^{1} xy\, f(y-1)\, dy ,$$

where g(x) is unknown function to be determined. Letting the approximate function of g(x) to be

$$g(x, b_1, b_2) = b_1 + b_2\, x$$

where the analytical solution of this problem is given by :

$$f(x) = x$$

In order to formulate the inverse problem we let

$f_a(x) = a_0 + a_1 x$

Carrying out the same steps in brivous examples, we get the following functional :

$$F(f) = \int_1^1 \left[\frac{1}{2}\left(f_a(x) - \left(\frac{1}{2} a_0 - \frac{1}{6} a_1 \right) \right)^2 - g(x, b_1, b_2) \right.$$

$$\left. \left(f_a(x) - \left(\frac{1}{2} a_0 - \frac{1}{6} a_1 \right) \right) \right] dx \text{ .... } (2.44)$$

Executing the computer program, we get the following results presented in table (2.5) and their comparison with the exact solution of g(x), which is $g(x) = \frac{7}{6} x$.

Table (2.5)

Results of the inverse problem for evaluating g(x) in example (2.6)

Coefficient of Basic	***Exact Solution***	***Approximate Solution***	***Basic Function***
b_1	0	$-1.490116119384766 \times 10^{-8}$	
b_2	1.666667	1.166999101638794	x

Example (2.7) :

Consider this problem:

$$g(x) = \int_0^1 k(x, y)\ f(y-1)\ dy$$

where $g(x) = -\frac{5}{6}x - \frac{7}{60}$

and consider the approximate solution of $k(x, y)$ to be $k(x, y, b_1, b_2) = b_1 x + b_2 y^2$.

And suppose the given analytical solution corresponding this problem is given by :

$$f(x) = x - x^2$$

Then, we write the function f as a linear combination of some basic, by letting :

$$f_a(x) = a_1 x + a_2 x^2 \qquad (\text{fix } a_0 = 0 \text{ since } f(0) = 0)$$

By some substitutings, the following functional will be obtained :

$$F(f) = \int_0^1 \left[\frac{1}{2}(m(x, a_1, a_2, b_1, b_2))^2 - g(x)\, m(x, a_1, a_2, b_1, b_2)\right] dx \qquad \text{.......................................(2.45)}$$

where $m(x, a_1, a_2, b_1, b_2) = \int_0^1 k(x, y, b_1, b_2)\, f_a(y-1)\, dy$

$$= (a_2 - a_1)\left(b_1 x + \frac{b_2}{3}\right) + (a_1 - 2a_2)\left(\frac{1}{2}b_1 x + \frac{1}{4}b_2\right) + a_2\left(\frac{b_1}{3}x + \frac{b_2}{5}\right).$$

Applying the computer program, we get the results presented in table (2.6) and their comparison with the exact solution $k(x, y) = x + y^2$.

Table (2.6)

Results of the inverse problem for evaluating k(x, y) in example (2.7)

Coefficient of Basic	***Exact Solution of k(x,y)***	***Approximate Solution of k(x, y)***	***Basic Function***
b_1	1	1	x
b_2	1	.9990001	y^2

The Third Inverse Problem:

We can clarify the last inverse problem as follows:

Suggest that the limits (if they are not delay) of integration are known, also the two functions k and g are known as well as the solution f(x) is given, while the delay which is given inside or outside of the given equation, or it upper (lower) limit of integration of this equation, is unknown which is to be estimated.

The inverse problem on time lag can be carried, as the following examples show :

Example (2.8) :

Consider the integral equation:

$$5x + 1 = \left(\frac{-25}{6}x^3 + 6x^2 + 5x + 1\right) + \int_0^x (x+y)\,f(y-\tau)\,dy$$

with unknown time lag τ.

The analytical solution of this problem is given by :

$$f(x) = 5x + 1$$

while the approximate solution used for the sake of the inverse problem is given by :

$$f_a(x) = a_0 + a_1 x$$

We carry out some substitutions to the following variational formulation:

$$F(f)= \int_0^1 \left[\frac{1}{2}(f_a(x) - m(x, a_0, a_1, \tau))^2 - g(x)\,(f_a(x) - m(x, a_0, a_1, \tau))\right] dx$$

....(2.46)

where $m(x, a_0, a_1, \tau) = \int_0^x (x+y)\,f(y-\tau)\,dy$

$$= 1.5\,(a_0 - a_1\tau)\,x^2 + \frac{5}{6}\,a_1 x^3$$

Applying the computer program, we get the following approximate value of τ which is $\tau^* = 1.00090014$ in which the exact value is $\tau = 1$.

Example (2.9) :

Consider another problem:

$$\frac{df}{dx} = \left(1 + x^2 - \frac{x^4}{3}\right) + \int_0^x (xy - x)\, f(y - \tau)\, dy,$$

this integro-differential equation with delay τ. The analytical solution corresponding to this problem is given by :

$$f(x) = x + 2$$

and hence the variational formulation with $f_a(x) = a_0 + a_1 x$ is given by :

$$F(f) = \int_0^1 \left[\frac{1}{2}\left(\frac{df_a}{dx} - m(x, a_0, a_1, \tau)\right)^2 - g(x)\left(\frac{df_a}{dx} - m(x, a_0, a_1, \tau)\right)\right] dx$$

....(2.47)

where $m(x, a_0, a_1, \tau) = \int_0^x (xy - x)\, f_a(y - \tau)\, dy$

$$= \frac{1}{3} a_1 x^4 + \left(\frac{1}{2} a_0 - \frac{1}{2} a_1 \tau - \frac{1}{2} a_1\right) x^3 + (a_1 \tau - a_0) x^2$$

The result upon applying the computer program produces the approximate value of τ to be 1.058619 which has a good result in comparison with $\tau = 1$.

Example (2.10) :

Consider the homogeneous delay integral equation:

$$\int_0^1 \left(\frac{1}{2}xy - \frac{1}{3}x\right) f(y-\tau)\,dy = 0$$

In this problem, the analytical solution is given by :

$f(x) = x + 1$

while the approximate solution could be taken to be :

$f_a(x) = a_0 + a_1 x$

Therefore, the general form of the variational formulation is given by :

$$F(f) = \int_0^1 \left[\frac{1}{2}\left(\int_a^{b(x)} k(x,y)\, f(y-\tau)\; dy \right)^2 \right] dx$$

and hence :

$$F(f) = \int_0^1 \left[\frac{1}{2}\left(\frac{1}{12}(a_1\tau - a_0)\,x \right)^2 \right] dx \quad \text{...} \quad (2.48)$$

Carrying the computer program, we get the approximate value of τ to be equals to 1 which is very accurate in comparison with the exact solution $\tau = 1$.

Modified Approach

We can use another approach to solve the inverse problem which is related to delay integral equations, by using variational approach. This approach is similar to the previous approach except when we use the least sequare method; at each minimization process, we carry a comparison between the parameters a_i^* which we get them from minimizing the parameters of the approximate solution $f_a(x, a_i^*)$ of the direct problem and the parameters a_i which will be obtained by minimizing the parameters of the approximate solution $f_a(x, a_i)$ of the inverse approach.

That is; the least sequare objective function is to be minimized the form :

$$Z = \underset{\substack{\text{unknown} \\ \text{term}}}{\text{Min}} \sum_{i=1}^{n} (a_i^* - \text{Min}\, ai)^2$$

For illustration purpose, the following example will be taken.

<u>Example (2.11) :</u>

Consider the following integral equation which is given in example (2.5)

$$\frac{df}{dx} = 0.5291667 + \int_0^b (x\,y + x)\; f(y-1)\; dy\,.$$

where b is unknown parameter to be evaluated.

The results of the first approach when the least sequare method related with the solutions (f_e, f_a) is presented in table (2.7) while the results of the second approach when the least sequare method related with the parameters (a_i^*, a_i) is presented in table (2.8).

Table (2.7)

Results of the inverse problem for evaluating b by using the first approach in example (2.11)

Upper limit of Integration	***Exact Value***	***Approximate Value***
b	1	0.99900007247248

Which is obtained by applying a program.

Table (2.8)

Results of the inverse problem for evaluating b by using the second approach in example (2.11)

Upper Limit of Integration	***Exact Value***	***Approximate Value***
b	1	1.009

Which is obtained by applying a program.

2-5-2 Analysis of the Inverse Problem :

In the previous kinds of the inverse problems, the aim is to find the unknown term (parameter of function or constant); that is, the minimum value of the unknown term, which converges to the exact term, by using Hooke and Jeeves optimization method and Gaussion quadrature integration method of degree 7.

The analysis of the inverse problem has its basis on using the least sequare method which minimizes the residual error between the exact and approximate value of the results. Therefore, this minimization will be carried numerically using optimization methods.

At each step of the minimization problem one carry out a computer comparison between the exact and approximate solution of f by using the least sequare approximation method until the approximate solution converges to the exact solution which must be satisfied only when the approximate term (unknown parameter) tends to the exact value; that is, when the following Z is equal to zero or converges to zero.

$$Z = \underset{\substack{\text{unknown}\\\text{term}}}{\text{Min}} \sum_{i=1}^{n} (f_e(x) - \underset{ai}{\text{Min}}\, f_a(x, ai))^2$$

where n is the number of data points in which the approximate solution is given as:

$f_a(x) = a_0 + a_1 x + \ldots.. + a_n x^n$,

$f_e(x)$ is the exact solution of the functional F that measured analytically, while the approximate solution $f_a(x)$ of the functional F is estimated by Hooke and Jeeves minimization method.

2-5-3 Inverse Problem of Integral Equation with Multiple Delay:

The approach of the inverse problem which is related to the integral equations with two delays is to evaluate the two delays when they are unknown, at the same time the other terms of this equation are known, as well as the solution f(x) of it is known.

We notice that this approach is also successful when the integral equation is Fredholm (Volterra) or homogeneous (non-homogeneous), also with the integro-differential equations (retarded, neutral, mixed).

The solution of the inverse problem related to multiple delay also satisfied, as the following examples show :

Example (2.12) :

Consider the following integral equation :

$$f(x-\tau_1) = g(x) \int_{-1}^{1} (x^3 - y)\, f(y-\tau_2)\, dy,$$

where $g(x) = -x^3 + 3x - 3$.

with unknown delays τ_1 and τ_2 to be determined.

where the analytical solution related to this problem is given by :

$f(x) = 3x + 3$.

The variational formulation related to this problem is given by :

$$F(f) = \int_0^1 \left\{ \frac{1}{2} [f_a(x-\tau_1) - m(x, a_0, a_1, \tau_2)]^2 - g(x) \right.$$

$$\left. (f_a(x-\tau_1) - m(x, a_0, a_1, \tau_2)) \right\} dx \qquad (2.49)$$

where $m(x, a_0, a_1, \tau_2) = \int_{-1}^{1} (x^3 - y)\, f_a(y - \tau_2)\, dy$

$$= (-a_0 + a_1 \tau_2) + a_1 x^3$$

while $f_a(x - \tau_1) = (a_0 - a_1 \tau_1) + a_1 x$

and $f_a(x - \tau_2) = (a_0 - a_1 \tau_2) + a_1 x$

Carrying out the computer program, we get the approximate values of τ_1 and τ_2 to be $\tau_1^* = 1$ and $\tau_2^* = 2$ which are the exact values.

<u>Example (2.13) :</u>

Consider the integral equation :

$$3\,(x - \tau_1) = g(x) + \int_0^{\tau_2} (xy + 1)\, f(y)\, dy,$$

where $g(x) = -(5x + 9)$

with unknown delays τ_1 and τ_2 to be estimated.

Where the analytical solution related to this problem is given by :

$f(x) = 3x$

The functional of this inverse problem is :

$$F(f) = \int_0^1 \left\{ \frac{1}{2}[f_a(x-\tau_1) - m(x, a_0, a_1, \tau_2)]^2 - g(x) \right.$$

$$\left. [f_a(x-\tau_1) - m(x, a_0, a_1, \tau_2)] \right\} dx \quad \text{.................................... (2.50)}$$

where $m(x, a_0, a_1, \tau_2) = \int_0^{\tau_2} (xy+1)\ f_a\ (y)\ dy$

$$= \frac{1}{2} a_0\ \tau_2^2\ x + \frac{1}{3}\ a_1\ \tau_2^3\ \ x + a_0\ \tau_2 + \frac{1}{2} a_1\ \tau_2^2$$

while $f_a(\ x - \tau_1) = (a_0 - a_1\ \tau_1) + a_1\ x$

Using computer program, we get the results presented in table (2.9) and its comparison with the exact solution.

Table (2.9)

Results of the inverse problem for evaluating two delays in example (2.13)

Delays	***Exact Value***	***Approximate Value***
τ_1	1	1.09000015
τ_2	2	2.01000022

<u>Example (2.14) :</u>

Consider the integral equation :

$$2\ x^2 + x = g(x) + \int_0^{\tau} (x^2 y + 2)\, f(y - \tau_2)\ dy,$$

where $g(x) = 2x^2 + x - \frac{1}{3}$

with unknown delays τ_1 and τ_2 to be determined. While the functional of this inverse problem is given by :

$$F(f) = \int_0^1 \left\{ \frac{1}{2}[f_a(x) - m(x, a_1, a_2, \tau_1, \tau_2)]^2 - g(x) \right.$$
$$\left. [f_a(x) - m(x, a_1, a_2, \tau_1, \tau_2)] \right\} dx \text{.......................................} (2.51)$$

$$\text{where } m(x, a_1, a_2, \tau_1, \tau_2) = \int_0^{\tau_1} (x^2 y + 2)\ f_a\ (y - \tau_2)\ dy$$

$$= (a_2\ \tau_2^2 - a_1\ \tau_2) \left(\frac{\tau_1^2}{2} x^2 + 2\tau_1 \right) + \frac{1}{3}\ (a_1 - 2\ a_2)\ \tau_1^3\ x^2$$

$$+ (\ a_1 - 2\ a_2)\,\tau_1^2 + \left(\frac{1}{4} a_2 \tau_1^4 x^2 + \frac{2}{3} a_2 \tau_1^3 \right)$$

while $f_a(\ x - \tau_2) = (a_2\ \tau_2^2\ - a_1\ \tau_2) + (\ a_1 - 2a_2)\ x + a_2\, x^2$.

Applying the computer program ,we get the approximate values of τ_1^* and τ_2^* to be equal to 1.05 and 1 respectively, which are very accurate in comparison with the exact solution $\tau_1 = 1$ and $\tau_2 = 1$.

Example (2.15)

Consider the following equation :

$$\frac{df}{dx} = \left(1 - \frac{17}{6}x\right) + \int_0^{\tau_1} x\, y\, f(y - \tau_2)\, dy,$$

with unknown delays τ_1 and τ_2 to be estimated. Where the analytical solution corresponding to this problem is given by :

$$f(x) = x + 7$$

The variational formulation of this inverse problem is given by :

$$F(f) = \int_0^1 \left\{ \frac{1}{2}\left[\frac{d f_a}{dx} - m(x, a_0, a_1, \tau_1, \tau_2)\right]^2 - g(x) \left[\frac{d f_a}{dx} - m(x, a_0, a_1, \tau_1, \tau_2)\right]\right\} dx \quad \text{.....................................} (2.52)$$

where $m(x, a_0, a_1, \tau_1, \tau_2) = \int_0^{\tau_1} x\, y\ f_a\,(y - \tau_2)\, dy$

$$= \frac{1}{2}(a_0 - a_1\, \tau_2)\ \tau_1^2\, x + \frac{1}{3} a_1\, \tau_1^3\ x$$

while $\frac{df_a}{dx} = a_1$ and $f_a\,(x - \tau_2) = (\,a_0 - a_1\, \tau_2) + a_1\, x.$

Carrying out the computer program, we get the approximate values of τ_1^* and τ_2^* to be equal to 1 and 2 respectively which is very accurate in comparison with the exact values $\tau_1 = 1$ and $\tau_2 = 2$.

CHAPTER THREE

Inverse Problem Application of Delay Integral Equations in Population Growth

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Inverse Problem Application of Delay Integral Equations in Population Growth

3.1 INTRODUCTION

The model of population growth which will be studied in this chapter aims to answer the following inverse problem equation :

What is the number of individuals in the population at time t depending on a life time lag τ ?.

One can notice that, the reason for this change in population size is caused by births and deaths only, [Myer, 1990].

Although the communities of human, animals and plants are growing but in some cases decreasing relative to time. Therefore, the ability to forecasting the size of community future wise according to present and past charts is something preferable, [Myer, 1990].

As it is mentioned earlier in chapter one, there exist a number of problem from many different fields which will be classified according to whether they are formulated directly in terms of integral equations , or are represented in terms of differential equations that can be reduced to integral equations. The one of these problems is the population growth. The study of population growth includes the forecasting of any future surge in birth rates, which is of great importance for future planning, [Jerri, 1985].

Many models governing the population growth have been proposed both with and without time lag. For example Cooke and Yorke in 1972 have studied the following delay differential equation:

$$N'(t) = g(N(t)) - g(N(t-\tau)),$$

where g is a given continuous, positive function and τ is the members life time of the species, [Driver, 1977]. Therefore, in this chapter we will construct the model of population growth depending on past values.

First, we will discuss the direct approach (problem) corresponding to this problem which was solved by using variational approach, [Al-Shakhaly, 2001], then we shall discuss and study the inverse problem related to this direct problem which will be solve also using variational approach.

3.2 THE MODEL OF POPULATION GROWTH

Let N(t) denote the number of individuals in a population at time t, and let us suppose that every member from the population has a life time τ. Now, let us assume that the number of births per unit time is a function of N(t) only, namely $g_1(u)$ at time t and $g_2(u)$ at time $t-\tau$.

Then we have the simplest model of population growth:

$$N'(t) = g_1(N(t)) - g_2(N(t-\tau)) \quad \text{.......} \quad (3.1)$$

where g_1 and g_2 are given continuous positive functions and τ is the life time of members of the species, [Gorecki, 1989].

We can make the above model more realistic by assuming that the number of deaths per unit time t to be:

$$g_2(N(t-\tau)) = -\int_0^\tau g_2(N(t-s))\,b'(s)\,ds,\ \ 0 \le t \le \tau$$

where b(s) represents the probability of survival to age s.

As in the previous model, we can state the following, more complicated relation :

$$N'(t) = g_1(N(t)) + \int_0^\tau g_2(N(t-s))\,b'(s)\,ds,\ \ \text{for } 0 \le t \le \tau \text{ ....} \tag{3.2}$$

This equation (3.2) gives more precise model governing the population growth, [Gorecki, 1989].

3.3 VARIATIONAL FORMULATION FOR THE MODEL OF POPULATION GROWTH

The variational formulation corresponding to eq. (3.2) will be found, which represents the model of population growth with time lag τ. Suppose that continuous positive functions $g_1(u)$ and $g_2(u)$ in (3.2) are given by :

$$g_1(u) = u^2,\ g_2(u) = u$$

Therefore, eq. (3.2) becomes:

$$N'(t) = (N(t))^2 + \int_0^\tau N(t-s)\,b'(s)\,ds,\ \ \ 0 \le t \le \tau \text{ ....} \tag{3.3}$$

Eq. (3.3) represents the delay integro-differential equation with delay as an upper limit of the integration and we consider the time delay τ is constant.

Now, define an operator L corresponding to eq. (3.3) of the form:

$$L = \frac{d}{dt} - \int_0^{\tau} M\, b'(s)\, ds,$$

where M is defined by MN (s) = N (t–s).

The operator L is a linear operator since it satisfied :

$$L(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 L(u_1) + \alpha_2 L(u_2)$$

where $u_1, u_2 \in U$ and α_1, α_2 are any two scalars.

The symmetry of the linear operator L with respect to the bilinear form is satisfied since $(Lu_1, u_2) = (Lu_2, u_1)$, see [Al-Shakaly, 2001].

Because of the symmetry of the linear operator L and by using theorem (2.3.1), the solution of eq. (3.3) isthe critical points of the functional :

$$F[N] = \frac{1}{2} <LN, LN> - <N, LN>$$

$$= \int_0^T \left\{ \frac{1}{2}(N'(t) - \int_0^{\tau} N(t-s)b'(s)\, ds)^2 - \right.$$

$$\left. \left[(N(t))^2 (N'(t) - \int_0^{\tau} N(t-s)\, b'(s)\, ds) \right] \right\} dt \quad \text{.........................} \; (3.4)$$

Hence, eq. (3.4) represents the variational formulation of the delay integro-differential equation (3.3) which is the variational formulation of the model of population growth with delay, [Al-Shakhaly, 2001].

3.4 APPROXIMATE SOLUTION OF THE MODEL OF POPULATION GROWTH

An example about the population growth will be taken to illustrate the direct and inverse problem of this problem. The survival function in eq. (3.3) will be taken, which represents the probability of survival of age S, to be of the following form:

$$b(s) = s^2$$

Hence,

$$b'(s) = 2S \quad \text{.......} \quad (3.5)$$

Therefore, eq. (3.3) becomes :

$$N'(t) = (N(t))^2 + \int_0^{\tau} 2S\ N(t-s)\ ds, \quad 0 \le t \le \tau \quad \text{.......} \quad (3.6)$$

3-4-1 The Direct Problem of the Model of Population Growth :

The solution of the direct problem was found by Al-Shakhaly in 2001, which could be summarized as follows :

The critical points of eq. (3.4) represents the solution of the delay integro-differential equation (3.6). Also, the critical points of the functional F could be evaluated using Hooke and Jeeves optimization method that minimizes the functional using the bilinear form and with 7-degree Gaussian quadrature integration method.

In connection with the direct Ritz method, the function N(t) as a linear combination of some basic functions is approximated, by letting :

$$N(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \quad \text{..........} \quad (3.7)$$

Hence,

$$N'(t) = a_1 + 2\, a_2 t + 3\, a_3 t^2 \quad \text{..........} \quad (3.8)$$

Also,

$$N(t-s) = a_0 + a_1 (t - s) + a_2 (t-s)^2 + a_3 (t - s)^3 \quad \text{..........} \quad (3.9)$$

Substituting (3.7), (3.8) and (3.9) back into the functional (3.4), we get:

$$\begin{aligned} F(N) = \int_0^T \Bigg\{ \Bigg[\Bigg[\frac{1}{2}\left(-a_0 + \frac{5a_1}{3} - \frac{a_2}{2} + \frac{2a_3}{5}\right) + \left(-a_1 + \frac{10a_2}{3} - \frac{3a_3}{2}\right) t \\ + (-a_2 + 5\, a_3)\, t^2 - a_3 t^3 \Bigg] - [\, (a_0 + a_1 t + a_2 t^2 + a_3 t^3)^2\,] \\ \left(\left(-a_0 + \frac{5a_1}{3} - \frac{a_2}{2} + \frac{2a_3}{5}\right) + \left(-a_1 + \frac{10a_2}{3} - \frac{3a_3}{2}\right) t \right. \\ \left. + \left(-a_2 + 5a_3\right) t^2 - a_3 t^3 \right) \Bigg] \Bigg\} dt \quad \text{..........} \quad (3.10) \end{aligned}$$

The critical points of (3.10) could be found using the computer program such that we suppose that the population size at time t = 0 to be equals to $a_0 = 0.1$, and carrying out the minimization on F[N] using Hooke and Jeeves optimization method with the initial value for a_1, a_2 and a_3 to be equals to zero in order to ensure that the population size at time t = 0 equals to a_0, we have the following results :

$a_1 = 0.0773$, $a_2 = 5.29 \times 10^{-3}$ and $a_3 = -2.7 \times 10^{-2}$ with functional minimum equals to -3.57041×10^{-4}.

Also, a graphical representation to the results of population growth is presented in Fig. (3.1), [Al-Shakhaly, 2001].

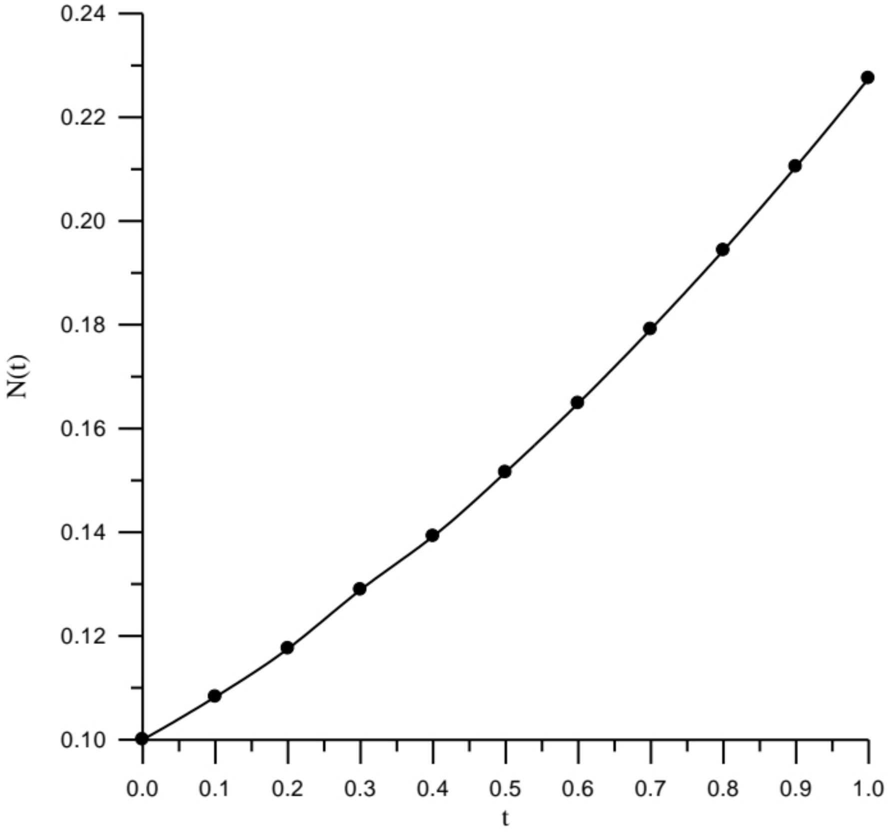

Fig.(3.2) Population growth for the first time step [0,1].

3-4-2 Inverse Problem of the Model of Population Growth :

Corresponding to the direct problem of population growth, we can set two inverse problems, these are:

The First Inverse Problem

Consider that the survival function $b'(s)$ in eq. (3.6) is an unknown and has to be determined when the function N(t) is given. Then, to estimate the survival function $b'(s)$, we must express the function $b'(s)$ as a linear combination of functions, as follows :

$$b'(s) = b_0 + b_1 s \quad \text{...} (3.11)$$

and take τ to be equals to 1.

By using the direct Ritz method, the function N(t) is written as a linear combination of some basic functions. Then, substituting (3.7), (3.8) and (3.9) back into the functional (3.4), we get :

$$F[N] = \int_0^T \left\{ \frac{1}{2}\left(N'(t) - \int_0^{\tau=1} N(t-s)\, b'(s, b_0, b_1)\ ds \right)^2 - (N(t))^2 \left(N'(t) - \int_0^{\tau=1} N(t-s)\, b'(s, b_0, b_1)\ ds \right) \right\} dt$$

$$\text{.......................................}(3.12)$$

where

$$\int_0^{\tau} N(t-s) b'(s, b_0, b_1)\ ds = b_0 \left(a_0 - \frac{1}{2} a_1 + \frac{1}{3} a_2 - \frac{1}{4} a_3 \right) +$$

$$b1\left(\frac{1}{2}a_0-\frac{1}{3}a_1+\frac{1}{4}a_2-\frac{1}{5}a_3\right)+b_0\,(a_1-a_2+a_3)\,t+$$

$$b_1\left(\frac{1}{2}a_1-\frac{2}{3}a_2+\frac{4}{3}\,a_3\right)t+\,b_0\left(a^2-\frac{3}{2}\,a_3\right)t^2\;+$$

$$b_1\left(\frac{1}{2}a_2-a_3\right)t^2+(b_0\,a_3+b_1\,a_3)\;t^3.$$

By using the method of Hooke and Jeeves optimization method that minimizes the survival function $b'(s)$ and using the least sequare method to change the inverse problem into unconstrained non-linear minimizing problem which is of minimizing :

$$Z=\underset{b_0,b_1}{\mathrm{Min}}\left[\sum_{i=1}^{n}\left(\mathrm{Ne(t)}-\underset{a_1,a_2,a_3}{\mathrm{Min}}\;\;\mathrm{Na(t)}\right)^2\right] \text{..................................} (3.13)$$

where n is the number of data points in which the exact solution Ne(t) is given as :

$$\mathrm{Ne(t)} = .1 + 0.0773\ t + 5.29\times 10^{-3}\,t^2 - 2.7\times 10^{-2}\,t^3$$

while the approximate solution of the functional is estimated by minimizing the functional by using the method of Hooke and Jeeves optimization method and 7-degree Gaussian quadrature integration method.

Carrying out the computer program to solve this inverse problem, we get the results presented in table (3.1).

Table (3.1)

Coefficient of Basis	*Exact Solution for* $b'(s)$	*Approximate Solution for* $b'(s)$	*Basic Function*
b_0	0	0	
b_1	2	1.60000002	s

The Second Inverse Problem

Consider that the survival function $b'(s)$ as well as the function N(t) are known, while the life time τ is unknown parameter which has to be estimated by using the method of Hooke and Jeeves optimization method.

The direct Ritz method, then substituting (3.7), (3.8) and (3.9) back into the functional (3.4), we get :

$$F[N] = \int_0^{T=1} \left\{ \frac{1}{2} \left(N'(t) - \int_0^{\tau} N(t-s)\, b'(s)\, ds \right)^2 - (N(t))^2 \right.$$

$$\left. \left(N'(t) - \int_0^{\tau} N(t-s)\, b'(s)\, ds \right) \right\} dt \quad \text{.................................} \ (3.14)$$

where $\int_0^{\tau} N(t-s)\, b'(s)\, ds = 2\,(a_0\,\tau^2/2 + a_1\,\tau^2\,t/2 - a_1\,\tau^3/3 + a_2\,\tau^2\,t^2/2 -$

$2\,a_2\,\tau^3\,t/3 + a_2\,\tau^4/4 + a_3\,\tau^2\,t^3/2 - a_3\,\tau^3\,t^2 + 3\,a_3\,\tau^4\,t/4 - a_3\,\tau^5/5).$

Also, using the least sequare method to change the inverse problem into unconstrained non-linear minimizing problem which is of minimizing :

$$Z = \underset{\tau}{\mathrm{Min}} \left[\sum_{i=1}^{n} \left(Ne(t) - \underset{a_1,a_2,a_3}{\mathrm{Min}} \; Na(t) \right)^2 \right] \text{..................................... (3.15)}$$

where n is the number of data points in which the exact solution Ne(t) is given as :

$$Ne(t) = .1 + 0.0773\, t + 5.92 \times 10^{-3}\, t^2 - 2.7 \times 10^{-2}\, t^3$$

While the approximate solution Na(t) of the functional is estimated by minimizing the functional by using Hooke and Jeeves optimization method and 7-degree Gaussian quadrature integration method.

Applying the computer program, we get the approximate value of τ to be equal to 1.00000011, which is very accurate in comparison with the exact value $\tau = 1$.

CONCLUSIONS AND RECOMMENDATIONS

From the present study, we can conclude the following:

1- Solving the inverse problem of delay integral equations using variational approach, is an accurate procedure since the absolute error often approaches zero between the exact and approximate results.

2- It may be difficult to solve the delay integral equations analytically as a direct problem whenever one or more of the parameters of the problem is unknown. Therefore the variational methods could be used to find such parameters approximately using the inverse problem approach, and using these results to solve the problem under consideration analytically.

Also, for future work, we can introduce the following open problems:

1- Solving non-linear delay integral equations by using variational approach, as a direct and inverse problem, [Al-Doory, 2002].

2- Solving non-linear multiple delay integral equations by using variational approach, as a direct and inverse problem, [Burgstaller, 2000].

3- Applying multiple delay integral equations as a direct and inverse problem to real world problems.

4- Studying the random integral equations, also random delay integral equations. And solving such types of problems using variational approach as a direct and inverse problem, [Bharucha-Reid, 1972], and [Tsokos, 1974].

5- Solving direct and inverse problem of delay integral equations of more than one unknown variables.

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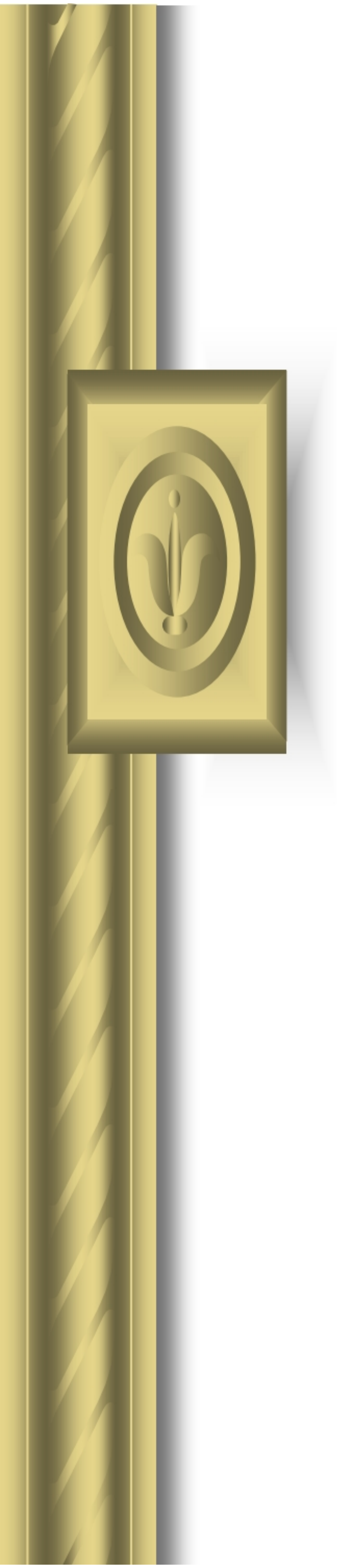

APPENDIX

APPENDIX A

In this appendix, for the sake of completion, some general concepts and definitions that will be needed in this thesis are introduced and discussed.

Definition A1

Let U and V be two linear spaces. A mapping L from U into V is called an operator or a transformation, where domain $D(L) \subseteq U$ and range $R(L) \subseteq V$. Also, L is called a linear operator or linear transformation if the following conditions are satisfied :

1- $L(u + v) = L(u) + L(v)$, $\forall u, v \in U$

2- $L(\alpha u) = \alpha L(u)$, $\forall u \in U$ and real α, [Erwin, 1978].

Definition A2

A linear space U is said to be normed linear space if it is endowed with a non-negative real valued function $||.||$, called a norm which satisfies :

1- $||u|| = 0$ if and only if $u = 0$.

2- $||\alpha u|| = |\alpha| \, ||u||$, α is scalar (real or complex)

3- $||u_1 + u_2|| \le ||u_1|| + ||u_2||$

for any vectors $u_1, u_2 \in U$, [Erwin, 1978].

Definition A3

A functional is an operator whose range lies on the real line R.

A functional F on a linear space U is called linear if :

1) $F(u_1 + u_2) = F(u_1) + F(u_2)$

2) $F(\alpha u_1) = \alpha F(u_1)$

For any u_1, $u_2 \in U$ and any scalar α, [Erwin, 1978].

Definition A4

A functional F(u, v) depending on two elements u and v belonging some normed linear space U, is said to be bilinear form if it satisfies :

(1) $F(u + w , v) = F(u, v) + F(w, v)$

(2) $F(\alpha u , v) = \alpha F(u, v)$

(3) $F(u, w + v) = F(u, v) + F(u, w)$

(4) $F(u, \alpha v) = \alpha F(u, v)$.

For any u, v and w $\in$ U and any real number α, [Erwin, 1978].

Definition A5

The bilinear form <u, v> is called non-degenerate on U and V if the following two conditions are satisfied :

(1) $< u, \bar{v} > = 0$ then $\bar{v} = 0$ for every $u \in U$

(2) $< \bar{u}, v> = 0$ then $\bar{u} = 0$ for every $v \in V$

one of non-degenerates bilinear forms are given by :

$$<u, v> = \int_0^T u(x)\, v(x)\; dx\,, \text{ [Magri, 1974].}$$

Definition A6

An inner product on X is a mapping of X × X into the scalar K on X; that is, with every pair of vectors x and y there is associated a scalar which is written $<x, y>$ and is called the inner product of x and y, such that for all vectors x, y, z and scalars α we have

(IP_1) $<x+y, z> = <x, z> + <y , z>$

(IP_2) $<\alpha x, y> = \alpha <x, y>$

(IP_3) $<x, y> = \overline{< y, x >}$

(IP_4) $< x, x> \geq 0$ and $<x, x> = 0 \Leftrightarrow x = 0.$

An inner product space (or pre-Hilbert space) is a vector space X with an inner product defined on X.

A Hilbert space is a complete inner space, [Erwin, 1978].

Remark

It is important to notice that the space under consideration in this thesis is the Hilbert space, which is due to the usage of the bilinear form (., .).

Definition A7

Linear operator L is said to be symmetric with respect to the chosen bilinear form <u, v> if L satisfies the condition :

$\langle Lu_1, u_2 \rangle = \langle Lu_2, u_1 \rangle$

for every point of elements u_1 and u_2 belong to D(L), [Magri, 1974].

APPENDIX B

Algorithm for Evaluating the Solution of Delay Integral Equations Using Variational Formulation Method

The general algorithm that could be applied to all programs might be summarized as follows:

1- Approximate the unknown function or constants ((limits of integration a or b) and / or f(x) and / or k(x, y) and/ or g(x)), by direct Ritz-method as :

$\varphi + \sum_{i=1}^{n} a_i \varnothing_i$ where φ satisfying the non-homogeneous conditions and $\varnothing$ satisfying the homogeneous conditions.

That is, approximate the unknown function by a linear combination of a complete sequence of functions.

Also, we can similarly estimate the unknown time lag constant τ in the given problem.

2- Substitute the approximation in the functional F to get a new functional

$$F(f) = \frac{1}{2} \langle Lf, f\rangle - \langle g, f\rangle$$

When L is the linear operator corresponding to the integral equation and g(x) is the deriving term, to get a new functional F which depended on the constants a_i, i=1, 2, ….. , n.

3- Solving the direct problem on minimizing $F(a_1, a_2, \dots , a_n)$ which depends on the results of the inverse problem.

4- Carrying out the inverse calculation depending on minimizing the difference between the exact and approximate results (obtained from step3) using the least sequare method on the function.

$$Z = \sum_{i=1}^{n} \left(f_e(x) - \min_{a_i} f_a(x, a_i) \right)^2$$

Where f_e is the exact results associated with the problem, while f_a is the approximate results obtained from step3.

```
REM **** The General Inverse program for solving****
********* Integral equation evaluate unknown term ***

    CLS
    ? “number of varables for unknown term” : input n
    'PRINT "number of varables for f": INPUT n1
    DIM x1(n1), b1(n1), y1(n1), p1(n1)
    PRINT "initial point for the f"
    FOR i1 = 1 TO n1: INPUT x11(i1): x1(i1) = x11(i1): NEXT i1
    h11 = .11: 'INPUT "step length h11 for f  "; h11

'**************************************************************************
```

Functions Definitions Corresponding to The Variational Formulation

$$F(f)=\frac{1}{2}<Lf, Lf>-<g, Lf>$$

```
****************************************************************************
   REM the roots and the coefficients of the legender polynomial of degree 7
    a(1) = .1294849661688697#: a(7) = a(1)
     a(2) = .279053914892767#: a(6) = a(2)
     a(3) = .3818300505051189#: a(5) = a(3)
     a(4) = .4179591836734694#
     r(1) = .9491079123427585#
     r(2) = .7415311855993945#
     r(3) = .4058451513773972#
     r(4) = 0
     r(7) = -r(1)
     r(6) = -r(2)
     r(5) = -r(3)

50   PRINT "INITIAL POINT for unknown term"
60   FOR i = 1 TO n: INPUT x(i): NEXT i
70   h = .1: 'PRINT "STEP LENGTH": INPUT h
80   K = h
90   FOR i = 1 TO n
100  y(i) = x(i): p(i) = x(i): b(i) = x(i): NEXT i
110  GOSUB 2000: FI = z
120  PRINT "INITIAL VALUE"; z
130  FOR i = 1 TO n: PRINT x(i); "     "; : NEXT i: PRINT ""
140  PS = 0: BS = 1
150  REM EXPLORE ABOUT BASE POINT
180  J = 1: FB = FI
200  x(J) = y(J) + K
210  GOSUB 2000
220  IF z < FI THEN GOTO 280
230  x(J) = y(J) - K
240  GOSUB 2000
250  IF z < FI THEN GOTO 280
260  x(J) = y(J)
270  GOTO 290
```

```
280  y(J) = x(J)
290  GOSUB 2000
300  FI = z
310  PRINT "EXPLORATION STEP"; z
320  FOR i = 1 TO n: PRINT x(i); "      "; : NEXT i: PRINT ""
330  IF J = n THEN GOTO 360
340  J = J + 1
350  GOTO 200
360  IF FI < FB - 1E-08 THEN GOTO 540
370  REM AFTER 360 MAKE A PATTERN MOVE IF FUNCTION HAS BEEN
REDUCED
380  IF PS = 1 AND BS = 0 THEN GOTO 420
390  REM BUT IF EXPLORATION WAS ABOUT APATTERN PT.
395  REM AND NO REDUCTION WAS MADE CHANGE BASE AT 420
400  REM OTHERWISE REDUSE STEP LENGTHE AT 490
410  GOTO 490
420  FOR i = 1 TO n: p(i) = b(i): y(i) = b(i): x(i) = b(i): NEXT i
430  GOSUB 2000: BS = 1: PS = 0
440  FI = z: FB = z
450  PRINT "BASE CHANGE"; z
460  FOR i = 1 TO n: PRINT x(i); "     "; : NEXT i: PRINT ""
470  REM (FOLLOW ON FROM 395)AND EXPLOR ABOUT NEW BASE POINT
480  J = 1: GOTO 200
490  K = K / 10
500  PRINT "CONTRACT STEP LENGTHE"
510  IF K < 1E-08 THEN GOTO 700
520  REM IF WE HAVE NOT FINISHED MAKE NEW
525  REM EXPLORATION ABOUT LATEST BASE POINT
530  J = 1: GOTO 200
535  REM PATTERN MOVE
540  FOR i = 1 TO n: p(i) = 2 * y(i) - b(i)
550  b(i) = y(i): x(i) = p(i): y(i) = x(i)
560  NEXT i
570  GOSUB 2000: FB = FI: PS = 1: BS = 0: FI = z
580  PRINT "PATTERN MOVE"; z
590  FOR i = 1 TO n: PRINT x(i); "    "; : NEXT i: PRINT ""
600  REM THEN EXPLORE ABOUT LATEST PATTERN POINT
610  J = 1: GOTO 200
700  PRINT "        MINIMUM FOUND"
710  FOR i = 1 TO n: PRINT "X"; i; "="; p(i): NEXT i: PRINT ""
750  PRINT "FUNCTION MINIMUM="; FB
760  PRINT "NO. OF FUNCTION EVALUATIONS="; fe
   END

2000  PRINT "************************************************************"
2010  fe = fe + 1
    h1 = h11
    FOR i1 = 1 TO n1: x1(i1) = x11(i1): NEXT i1
    k1 = h1: fe1 = 0
```

```
     FOR i1 = 1 TO n1
     y1(i1) = x1(i1): p1(i1) = x1(i1): b1(i1) = x1(i1): NEXT i1
     GOSUB 3640: FI1 = z1
     PS1 = 0: BS1 = 0
     j1 = 1: FB1 = FI1
3140  x1(j1) = y1(j1) + k1
     GOSUB 3640
     IF z1 < FI1 THEN GOTO 3220
     x1(j1) = y1(j1) - k1
     GOSUB 3640
     IF z1 < FI1 THEN GOTO 3220
     x1(j1) = y1(j1)
     GOTO 3230
3220  y1(j1) = x1(j1)
3230  GOSUB 3640
     FI1 = z1
     IF j1 = n1 THEN GOTO 3300
     j1 = j1 + 1
     GOTO 3140
3300  IF FI1 < FB1 - 1E-08 THEN GOTO 3510
     IF PS1 = 1 AND BS1 = 0 THEN GOTO 3370
     GOTO 3440
3370  FOR i1 = 1 TO n1
     p1(i1) = b1(i1): y1(i1) = b1(i1): x1(i1) = b1(i1): NEXT i1
     GOSUB 3640: BS1 = 1: PS1 = 0
     FI1 = z1: FB1 = z1
     j1 = 1: GOTO 3140
3440  k1 = k1 / 10
     IF k1 < 1E-08 THEN GOTO 3590
     j1 = 1: GOTO 3140
3510  FOR i1 = 1 TO n1
     p1(i1) = 2 * y1(i1) - b1(i1)
     b1(i1) = y1(i1): x1(i1) = p1(i1): y1(i1) = x1(i1): NEXT i1
     GOSUB 3640:  PS1 = 1: BS1 = 0: FI1 = z1: FB1 = z1
     j1 = 1: GOTO 3140
3590  PRINT "       MINIMUM FOUND of f"
     FOR i1 = 1 TO n1
     PRINT "a"; i1; p1(i1): x1(i1) = p1(i1): NEXT i1
     PRINT "FUNCTION MINIMUM for a="; FB1
     PRINT "NO. OF FUNCTION EVALUATIONS for a="; fe1
    Input i, q
     sum3 = 0
     FOR xx = 0 TO 1 STEP .01
     sum3 = sum3 + (fnne(xx) - fnna(xx, x1(1), x1(2) , …, x1(n1))^ 2
     NEXT xx
     z = sum3
     PRINT "zzzzz="; z

3630  RETURN
```

```
3640  sum1 = 0
      FOR jj = 1 TO 7
      x0 = r(jj) / 2 + 1 / 2
      sum1 = sum1 + a(jj) * fns(x0, x1(1), …, x1(n1), x(1), …, x(n))
      NEXT jj
      z1 = sum1 / 2
3650  fe1 = fe1 + 1
      RETURN
```

المسألة العكسية للمعادلات التكاملية التباطؤية وتطبيقها في النمو السكاني

رسالة مقدمة إلى
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علوم في الرياضيات

من قبل
مها عبد الجبار محمد
(بكالوريوس كلية التربية أبن الهيثم، 2000)

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