Co-small monoform modules

Muna Abbas Ahmed
Department of Mathematics
College of Science for Women
University of Baghdad
Baghdad
Iraq
drmunaabbas@gmail.com

Abstract. The concept of small monoform module was introduced by Hadi and Marhun, where a module U is called small monoform if for each non-zero submodule V of U and for every non-zero homomorphism $f \in Hom_R(V, U)$, implies that ker f is small submodule of V. In this paper the author dualizes this concept; she calls it co-small monoform module. Many fundamental properties of co-small monoform module are given. Partial characterization of co-small monoform module is established. Also, the author dualizes the concept of small quasi-Dedekind modules which given by Hadi and Ghawi. She show that co-small monoform is contained properly in the class of the dual of small quasi-Dedekind modules. Furthermore, some subclasses of co-small monoform are investigated. Other generalizations of co-small monoform are introduced.

Keywords: essential submodules, small monoform modules, Co-small monoform modules.

1. Introduction

"Throughout this paper, all rings R are commutative with identity and all modules are unitary left R-modules. A submodule V of an R-module U is called small if (denoted by $V \ll U$) if, for every proper submodule L of U, $V + L \neq U$ ([3], P.20)". "A non-zero module U is called monoform if for each non-zero homomorphism $f \in Hom_R(V, U)$ with every non-zero submodule V of U; f is monomorphism [8]". In 2014, Hadi and Marhun introduced the concept of small monoform module as a generalization of monoform module, "where an R-module U is called small monoform if for each non-zero submodule V of U and for each non-zero homomorphism $f \in Hom_R(V, U)$; ker f is small submodule of V [8]". "A submodule V of U is called essential (denoted by $V \leq_e U$ if every non-zero submodule of U has a non-zero intersection with V ([3], P.15)".

The main goal of this work is to dualize the concept of small monoform module; we call it co-small monoform module, where a non-zero R-module U is called co-small monoform if for each proper submodule V of U and for every non-zero homomorphism $f : U \to \frac{U}{V}$; $f(U)$ is essential submodule of $\frac{U}{V}$. This paper consists of five sections, in section 2; fundamental properties of co-small monoform are investigated. We prove that under the class of co-small monoform modules, the class of small submodules coincides with the class of e-small submodules, see theorem (2.13), "where a submodule V of U is called e-small if $N + W = U$ with $W \leq_e U$, implies that W = U [15]". In section 3; the relationship of co-small monoform module with coquasi-Dedekind module is studied, see proposition (3.1) and proposition (3.7). Also, we show that every coquasi-Dedekind ring is co-small monoform, see corollary (3.4). In section 4; we show that epiform, anti-hopfian and almost finitely generated modules are subclasses of co-small monoform modules. Section 5 is devoted to introduce some generalizations of co-small monoform module such as co-small quasi-Dedekind, relatively co-small monoform and E-coprime modules. Besides, partial equivalent between co-small monoform module and each one of these generalizations are obtained, see theorems (5.3) and (5.6).

2. Co-small monoform modules

In this section we introduce the dual notion of small monoform module, we start by following.

Definition 2.1. *A non-zero module R-module U is called co-small monoform module if for each proper submodule V of U and for every non-zero homomorphism $f : U \to \frac{U}{V}$; $f(U)$ is essential submodule of $\frac{U}{V}$. A ring R is called co-small monoform if R is co-small monoform R-module.*

Remarks and Examples 2.2. i. Every simple module is co-small monoform module. Since if U is small, then the only proper submodule of U is (0), and by Schur's Lemma, for each non-zero homomorphism $f \in (Hom_R(U, \frac{U}{(0)}))$; f is epimorphism ([12], P.73), hence $f(U) \leq_e \frac{U}{(0)}$.

ii. Semisimple module is not co-small monoform module, as we will see in remark (3.6)(b).

iii. The set of integer numbers Z as Z-module is not co-small monoform module. In fact if we consider the homomorphism $f : Z \to \frac{Z}{6Z}$ which is defined by $f(x) = 3x + 6Z$ for all $x \in Z$. Clearly $\frac{Z}{6Z} \cong Z_6$, and $f(Z) = \frac{3Z}{6Z} \cong (\bar{3})$ and $(\bar{3})$ is not essential submodule of Z_6.

iv. The Z-module Z_4 is a co-small monoform module. In fact the only non-zero homomorphism $f \in (Hom_R(Z_4, \frac{Z_4}{(\bar{2})}))$ is defined as: $f(x) = 2x \; \forall x \in Z$, so $f(Z4) = (\bar{2})$ is essential submodule of $\frac{Z_4}{(\bar{2})}$.

v. Z_{P^∞} is a co-small monoform Z-module, see Proposition 4.8.

vi Uniform module may not be co-small monoform, "where an R-module U is called uniform if all non-zero submodules of U are essential in U [2]". For example Z is a uniform Z-module, but it is not co-small monoform as we verified in (iii).

Proposition 2.3. *If all non-zero factor of an R-module U is uniform, then U is a co-small monoform module.*

Proof. Let $f : U \to \frac{U}{V}$ be a non-zero homomorphism, and V be a proper submodule of U. Note that $\frac{U}{V} \neq 0$, and since $\frac{U}{V}$ is a uniform module, then $f(U) \leq_e \frac{U}{V}$, and we are done. □

Remark 2.4. If we replace the word "all" in the proposition (2.3), by "there exists", then we cannot guarantee U is co-small monoform. In fact in example (2.2)(iii) we see that Z is not co-small monoform. Note that not all non-zero factor of Z are uniform modules.

Hadi and Marhun in [8] proved that an epimorphic image of small monoform is not necessarily small monoform. For co-small monoform modules we have the following.

Proposition 2.5. *The epimorphic image of co-small monoform module is co-small monoform.*

Proof. Assume that U is a co-small monoform module and $f : U \to U_1$ is an epimorphism, where U_1 is any R-module. Let $g : U_1 \to \frac{U_1}{V_1}$ be a non-zero homomorphism where V_1 is any proper submodule of U_1, so we have following compositions of homomorphisms:

$$U \xrightarrow{f} U_1 \xrightarrow{g} \frac{U_1}{V_1} \xrightarrow{h} \frac{U}{f^{-1}(V)}$$

Where h is defend as follows: for $u_1 \in U_1$, since f is an epimorphism, then there exists $u \in U$ such that $f(u) = u_1$. So set $h(u_1 + V_1) = u + f^{-1}(V_1)$ for each $u_1 \in U_1$. One can show that h is an isomorphism. Since U is co-small monoform module, then $((hgf)(U)) \leq_e \frac{U}{f^{-1}(V_1)}$. We claim that $g(U_1) \leq_e \frac{U_1}{V_1}$, to show that; $(hgf)(U) = (hg)(U_1) \leq_e \frac{U}{f^{-1}(V_1)}$. This implies that $h^{-1}((hg)(U_1)) \leq_e h^{-1}(\frac{U}{f^{-1}(V_1)})$ ([3], Prop.(1.1), P.16). But h is an isomorphism, thus $g(U_1) \leq_e \frac{U_1}{V_1}$, hence U_1 is co-small monoform. □

Corollary 2.6. *The quotient of co-small monoform module is co-small monoform.*

Corollary 2.7. *A direct summand of co-small monoform module is co-small monoform.*

Proof. Let $U = V_1 \oplus V_2$, where both of V_1 and V_2 are submodules of U. Consider the projection homomorphism $f : U \to V_1$. Since U is a co-small monoform module, then by corollary (2.6); $\frac{U}{V}_2$ is co-small monoform. But $V_1 \cong \frac{U}{V_2}$, thus V_1 is co small. □

Remark 2.8. The direct sum of co-small monoform module may not be co-small monoform; for example both of Z_2 and Z_3 are simple Z-modules, hence they are co-small monoform modules. But the Z-module $Z_2 \oplus Z_3$ is not, in fact $Z_2 \oplus Z_3 \cong Z_6$ which is semisimple, and by example (2.2)(ii) it is not co-small monoform module.

Proposition 2.9. *Let U be a co-small monoform module, then for each element $0 \neq r \in R$; either $rU = 0$ or $rU \leq_e U$.*

Proof. Let $0 \neq r \in R$ and $f \in End_R(U)$ defined by $f_r(u) = ru\, \forall u \in U$. Assume that $rU \neq 0$, then $f \neq 0$. But U is co-small monoform, then $f(U) \leq_e U$. Thus $rU \leq_e U$. □

"Recall that an R-module U is called divisible, if $rU = U$ for every non-zero divisor element $r \in R$ ([3], P.102)". This concept and proposition (2.9) led us to introduce the following concept.

Definition 2.10. *An R-module U is called essentially divisible if $rU \leq_e U$ for each non-zero divisor element $r \in R$.*

Remark 2.11. i. "It is clear that every divisible R-module is essentially divisible. The converse is not true in general", for example: Z is essentially divisible Z-module, in fact $rZ \leq_e Z \;\forall\; 0 \neq r \in Z$ while Z is not divisible.

ii. If a module U is semisimple, then we can easily show that the class of divisible module coincides with the class of essentially divisible module.

Proof (ii). Let $0 \neq r \in R$, since U is essentially divisible, then $rU \leq_e U$. But U is semisimple, therefore $rU \leq_c U$, hence $rU = U$. □

Proposition 2.12. *If U is faithful co-small monoform module then U is essentially divisible.*

Proof. Let $0 \neq r \in R$, since U is co-small monoform, then by proposition (2.9) either $rU = 0$ or $rU \leq_e U$. If $rU = 0$ and since U is faithful, then $r = 0$. But this is a contradiction, therefore $rU \leq_e U$. □

In the following proposition we use co-small monoform as a useful condition under which e-small submodule can be small.

Theorem 2.13. *Let U be a co-small monoform module, then V is small submodule if and only if V is e-small.*

Proof. The necessity is clear. Conversely, Assume that V is an e-small submodule, and let $V + L = U$ where $L \leq U$. Define $\Psi : U \to \frac{U}{V \cap L}$ as follows: $\forall u \in U; u = x + y$ where $x \in V$ and $y \in L$. Set $\Psi(u) = y + V \cap L$. It is clear that Ψ is well defined and homomorphism. If $\Psi = 0$, then $y \in V$ for all $y \in L$. This implies that $L \subseteq V$, hence $V = U$. But this is a contradiction, thus $\Psi \neq 0$. Since U is co-small monoform, then $\Psi(U) \leq_e (\frac{U}{V \cap L})$, so that $(\frac{L}{V \cap L}) \leq_e (\frac{U}{V \cap L})$. Thus

$L \leq_e U$ ([12], Exc.(1.64), P.32). On the other hand, V is an e-small submodule, therefore $L = U$ that is V is a small submodule of U. □

"An R-module U is called hollow if every proper submodule of U is small, and U is called e-hollow if every proper submodule V of U e-small [4]".

Corollary 2.14. *Let U be a co-small monoform module, then U is hollow module if and only if U is e-hollow module.*

Proof. The necessity is obvious. Conversely, assume that U is an e-hollow module and let $V \lneq U$. Since U is co-small monoform, so by theorem (2.13); V is small submodule of U, hence U is a hollow module. □

"Recall that an R-module U is called couniform if every proper submodule V of U is either zero or there exists a proper submodule W of V such that $(\frac{V}{W}) \ll (\frac{U}{W})$ [5]".

Corollary 2.15. *Let U be a co-small monoform module. If U is couniform and Artinian then U is an e-hollow module.*

Proof. Since U is a couniform and Artinian module, then U is hollow [5]. By corollary (2.14), U is e-hollow. □

3. Co-small monoform and coquasi-Dedekind modules

"Recall that an R-module U is called coquasi-Dedekind if $Hom_R(U, V) = 0$ for each proper submodule V of U. Equivalently; U is coquasi-Dedekind if every $0 \neq f \in End_R(U)$ is epimorphism [14]". The two concepts co-small monoform module and coquasi-Dedekind are independent for example; The integer numbers Z-module Z is a coquasi-Dedekind module. But it is not co-small monoform, see (2.2)(iii). On the other hand, Z_4 is co-small monoform as we have be seen in example (2.2)(iv), while it is not coquasi-Dedekind [14]. In fact we will show later that in the category of rings; every coquasi-Dedekind ring is co-small monoform.

However, this section is devoted to study how can be relate between them, before that; an R-module U is called self-generator if for every submodule V of U, $V = \Sigma_f f(U)$ where $f \in Hom_R(U, V)$ [11].

Proposition 3.1. *If U is a self-generator module, then every coquasi-Dedekind module is co-small monoform.*

Proof. Let V be a proper submodule of the self-generator module U, then$V = \Sigma_f f(U)$ where $f \in Hom_R(U, V)$. Since U is coquasi-Dedekind, then $Hom_R(U, V) = 0$, hence $f = 0$. This implies that $V = \Sigma_f f(U) = 0$, therefore U is simple. By example (2.2)(i), U is co-small monoform. □

"An R-module U is called multiplication if for every submodule V of U, there exists an ideal I of R such that $V = IU$ [11]". Since every multiplication module is self-generator [11], then we have the following.

Corollary 3.2. *If U is a multiplication module, then every coquasi-Dedekind module is co-small monoform.*

Since any ring R is multiplication R-module, so by using corollary (3.2) we deduce the following implication in the category of rings.

Corollary 3.3. *Every coquasi-Dedekind ring is a co-small monoform ring.*

Corollary 3.4. *Every cyclic coquasi-Dedekind module is a co-small monoform module.*

Now, we give a useful property for co-small monoform module.

Proposition 3.5. *Every co-small monoform module satisfies the following: condition $(*)$: For every non-zero $f \in End_R(U)$; $f(U) \leq_e U$.*

Proof. Let U be co-small monoform, and $0 \neq f \in End_R(U)$. Let V be a proper submodule of U, so we have the following composition of homomorphisms:

$$U \xrightarrow{f} U \xrightarrow{\pi} \frac{U}{V}$$

Since U is co-small monoform, then $(\pi f)(U) \leq_e (\frac{U}{V})$. This implies that $(\frac{f(U)}{V}) \leq_e (\frac{U}{V})$, hence $f(U) \leq_e U$ ([12], Exc (1.64), P.32). □

Remark 3.6.

a. The converse of proposition (3.5) is not true in general, for example the set of rational numbers Q is a uniform Z-module so every non-zero submodule of Q is essential, in particular $f(U) \leq_e U$. On the other hand Q is not co-small monoform. In fact Q is a torsion free module and $\frac{Q}{Z}$ is torsion, therefore $Hom_R(Q, \frac{Q}{Z}) = 0$, that is $f(Q) \nleq_e \frac{Q}{Z}$. Thus Q is not co-small monoform.

b. We can use proposition (3.5) to show that the semisimple module is not co-small monoform, in fact if U is co-small monoform , then by proposition (3.5), for any proper submodule V of an R-module U and any non-zero homomorphism $f \in End_R(U)$; $f(U) \leq_e U$. On the other hand, since U is semisimple, then $f(U)$ is a direct summand of U, so we have a contradiction, thus U is not co-small monoform.

"Recall that a ring R is called (Von Newoman) regular if for each $r \in R$ there exists $x \in R$ such that $r = rxr$ ([2], P.(186))".

Proposition 3.7. *Let U be an R-module, with $End_R(U)$ is regular ring, then every co-small monoform module is coquasi-Dedekind.*

Proof. Let $0 \neq f \in End_R(U)$, and V be a proper submodule of U. Consider the following composition of homomorphisms:

$$U \xrightarrow{f} U \xrightarrow{\pi} \frac{U}{V}$$

It is clear that πf is a non-zero homomorphism. Since U is co-small monoform, then $(\pi f)(U) \leq_e \frac{U}{V}$. This implies that $\frac{f(U)}{V} \leq_e \frac{U}{V}$, hence $f(U) \leq_e U$ ([12], Exe.(1.64), P.32), but $End_R(U)$ is a regular ring, then $f(U)$ is a direct summand of U [3]. Thus $f(U) = U$, that is U is coquasi-Dedekind. □

Note the condition "$End_R(U)$ is regular ring" in proposition (3.7) is necessary, because if $End_R(U)$ is not regular, we cannot guarantee that every co-small monoform module is coquasi-Dedekind. For example the Z-module Z_4 is co-small monoform as we saw in (2.2)(iv), but Z_4 is not coquasi-Dedekind [14], in fact $End_R(Z_4) \cong Z_4$ which is not regular ring.

Corollary 3.8. *Every semisimple co-small monoform module is coquasi-Dedekind.*

Proof. Assume that U is co-small monoform, since U is semisimple, then $End(U)$ is regular ([12], P.91), and the result follows by proposition (3.7). □

4. Subclasses of co-small monoform modules

In this section we show that epiform, anti-hopfian and almost finitely generated are contained in the class of co-small monoform modules." A non-zero module U is called epiform if every non-zero homomorphism $f : U \to \frac{U}{V}$ with V a proper submodule of U is an epimorphism [2]. Equivalently; U is epiform if every proper submodule of U is corational, where a submodule V of U is called corational if $Hom_R(U, \frac{V}{W}) = 0$ for all submodules W of U such that $W \subseteq V \subseteq U$ ([2], P.85)".

Remark 4.1. It can be easily show that each epiform module is co-small monoform, but the reverse is not always true; for example Z_4 is co-small monoform, see example (2.2)(iv) , but not epiform, to see this; if Z_4 is epiform, then every proper submodule of U is corational, but clearly the submodule $(\bar{2})$ is not corational, so Z_4 is not epiform.

As applications of (4.1), we have the following two propositions. The first one is about the relationship between co-small monoform and hollow modules. In fact there is no direct implication between co-small monoform and hollow modules; however, we can relate them by using the class of noncosingular module, "where a module U is called noncosingular if for any non-zero R-module T and for every nonzero homomorphism $f : U \to T$; $f(U)$ is not small submodule of T [5]".

Proposition 4.2. *Let U be a noncosingular module. If U is hollow then U is a co-small monoform module.*

Proof. Suppose that U is a hollow module. Since U is noncosingular, then U is epiform [1], and by remark (4.1), U is co-small monoform. □

"Recall that an R-module U is called cosemisimple if $Rad(\frac{U}{V}) = 0$ for all submodules V of U [2]. The second application of remark (4.1) is given by the following.

Proposition 4.3. *Let U be a couniform module. If U is Artinian and cosemisimple, then U is a co-small monoform module.*

Proof. Since U is couniform and Artinian, then U is a hollow module. But U is cosemisimple, thus U is epiform [1], hence U is a co-small monoform module. □

Theorem 4.4. *Let U be a self-generator semisimple module, then the following statements are equivalent:*

1. *U is a simple module.*
2. *U is an epiform module.*
3. *U is a co-small monoform module.*
4. *U is a coquasi-Dedekind module.*

Proof. (1) ⇒ (2) [1]
(2) ⇒ (3) It is clear.
(3) ⇒ (4) Since U is co-small monoform and semisimple, then by corollary (3.8), U is coquasi-Dedekind.
(4) ⇒ (1) Since U coquasi-Dedekind and self-generator, then by proposition (3.1), U is simple. □

"Recall that an R-module U is called anti-hopfian if U is not simple and all non-zero factor modules of U are isomorphic to U; that is U is not simple and $\frac{U}{V} \cong U$ for every proper submodule V of U [9,10]. The following proposition shows that the anti-hopfian module is subclass of co-small monoform modules.

Proposition 4.5. *Every anti-hopfian module is co-small monoform.*

Proof. Since U is anti-hopfian, then U is an epiform module [6], and by remark (4.1), U is co-small monoform. □

By using the concept of anti-hopfian we have the following.

Proposition 4.6. *If U is a coquasi-Dedekind module such that every proper submodule of U is anti-hopfian, then U is a co-small monoform module.*

Proof. Assume that U is coquasi-Dedekind module. Since every proper submodule of U is anti-hopfian, then U is an epiform ([6], Prop.(3.8)), hence U is co-small monoform. □

"An R-module U is called almost finitely generated if U is not finitely generated and every proper submodule of U is finitely generated [13]". We need the following lemma which is appeared in [13].

Lemma 4.7. *Let U be an almost finitely generated R-module. If U_1 and U_2 are almost finitely generated, then for all $0 \neq f \in Hom_R(U_1, U_2)$; f is an epimorphism.*

Proposition 4.8. *Every almost finitely generated R-module is co-small monoform.*

Proof. Suppose that U is an almost finitely generated. Let V be a proper submodule of U and $0 \neq f \in Hom_R(U, \frac{U}{V})$. Note that $\frac{U}{V}$ is almost finitely generated [13], Prop.(1.1)), by lemma (4.7), f is an epimorphism, hence we are through. □

The converse of proposition (4.8) is not true in general as the following example shows.

Example 4.9. As we saw in example (2.2)(iv), that the Z-module Z_4 is a co-small monoform module, while it is not almost finitely generated, since Z_4 is finitely generated. Thus the class of almost finitely generated is contained properly in the class of co-small monoform module.

5. Some generalizations of co-small monoform modules

In this section; some generalizations of co-small monoform are introduced. We start by the first one.

Definition 5.1. *An R-module U is called co-small quasi-Dedekind if U satisfies the condition* $(*)$ *in the proposition (3.5).*

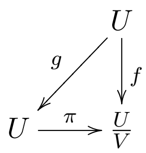
Remarks 5.2. a. It is worth mentioning to say that the class of co-small quasi-Dedekind module forms a dual of the class of small quasi-Dedekind which appeared in [7], "where an R-module U is called small quasi-Dedekind if for every $0 \neq f \in End_R(U)$; $ker f \ll U$".

b. It is clear that every uniform module is co-small quasi-Dedekind.

Under certain conditions the class of co-small monoform coincides with the class of co-small quasi-Dedekind modules as the following theorem shows.

Theorem 5.3. *Let U be a self-projective uniform module, then U is co-small monoform if and only if U is a co-small quasi-Dedekind module.*

Proof. The necessity follows by proposition (3.5). Conversely, assume that U is co-small quasi-Dedekind module, and $f : U \to \frac{U}{V}$ is a non-zero homomorphism with a proper submodule V of U. Consider the following diagram:



Where π is the natural epimorphism. Since U is a self-projective module, then there exists a homomorphism $g : U \to U$ such that $(\pi g) = f$. It is clear that $(\pi g) \neq 0$ and homomorphism. By assumption, $g(U) \leq_e U$, since U is uniform then $V \leq_e U$. This implies that $(\pi g)(U) \leq_e \pi(U)$ ([12], Prop.(2.6), P.76), hence $f(U) \leq_e \frac{U}{V}$, thus U is co-small monoform. □

We have mentioned in remark (2.2)(vi) that uniform module is not necessary co-small monoform, as consequence of theorem (5.3) we have the following.

Corollary 5.4. *If U is a uniform and self-projective module, then U is co-small monoform.*

Proof. Since U is uniform, then by (5.2)(b), U is co-small quasi-Dedekind. But U is self-projective, so by (5.3), U is a co-small monoform module. □

Now, we introduce another type for the generalizations of co-small monoform module.

Definition 5.5. *"An R-module U is called relatively co-small monoform if for every proper closed submodule V of U and every non-zero homomorphism* $f : U \to \frac{U}{V}$; $f(U) \leq_e \frac{U}{V}$*".*

"It is clear that every co-small monoform is relatively co-small monoform, but the converse is not true in general, for example": in the Z-module Z; the only closed submodule is (0), so clearly Z is relatively co-small monoform, but it is not co-small monoform, see example (2.2)(iii).

Theorem 5.6. *For a semisimple module U, the following statements are equivalent:*

i. U is a co-small monoform module.

ii. U is a relatively co-small monoform module.

Proof. (i) ⇒ (ii) It is straightforward.
(ii) ⇒ (i) Let V a proper submodule of U, and $f : U \to \frac{U}{V}$ be a non-zero homomorphism. Since U is semisimple, then V is closed, and by assumption we get the result. □

Proposition 5.7. *Every uniform module is relatively co-small monoform module.*

Proof. Let U be a uniform module, so the only proper closed submodule in U is (0), and for every non-zero homomorphism $f : U \to \frac{U}{(0)}$; $f(U) \leq_e \frac{U}{(0)}$. That is U is relativly co-small monoform.. □

It is well-known that every non-zero extending and indecomposable module is uniform, so we have the following.

Corollary 5.8. *If U is a non-zero extending and indecomposable module, then U is relatively co-small monoform..*

Following [14], a non-zero module U is called coprime if for each element of R, either $rU = 0$ or $rU = U$. This motivates us to introduce the third generalizations of co-small monoform module; we call it an E-coprime module.

Definition 5.9. *A non-zero module U is called E-coprime if for each non-zero element r of R, either $rU = 0$ or $rU \leq_e U$.*

It is clear that every a coprime module is E-coprime.

Remark 5.10. i. Every co-small monoform module is E-coprime.

ii. If $ann(\frac{U}{V}) = ann(U)$ for each proper submodule V of U, then U is E-coprime.

Proof. i. The result follows directly by proposition (2.9).

ii. Let $r \in R$, and assume that $rU \nleq_e U$, then rU is a proper submodule of U. By assumption $ann(U) = ann(\frac{U}{V})$. Now, $r(\frac{U}{rU}) = 0$, this implies that $r \in ann(U)$, hence $rU = 0$. □

The following theorem gives a partial characterization of E-coprime module.

Theorem 5.11. *Let U be a semisimple module, then U is E-coprime if and only if $ann(\frac{U}{V}) = ann(U)$ for each proper submodule V of U.*

Proof. Suppose that U is an E-coprime module, and V be a proper submodule of U. Let $t \in ann(\frac{U}{V})$, then $tU \subseteq V$. But V is proper, so $tU \neq U$. Since U is semisimple, then $tU \nleq_e U$. But U is E-coprime, therefore $tU = 0$, hence $t \in ann(U)$. On the other hand, clearly $ann(U) \subseteq ann(\frac{U}{V})$, thus $ann(U) = ann(\frac{U}{V})$. The sufficiency follows by remark (5.10)(ii). □

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