Pure Graph of a Commutative Ring

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Abstract

A new definition of a graph called Pure graph of a ring denote *Pur(R)* was presented , where the vertices of the graph represent the elements of R such that there is an edge between the two vertices α and β if and only if $\alpha = \alpha\beta$ *or* $\beta = \beta\alpha$, denoted by *pur(R)* . In this work we studied some new properties of *pur(R)* finally we defined the complement of *pur(R)* and studied some of it is properties .

Keywords: Graph theory, commutative ring.

1- Introduction

There is a a lot of research linking between graph theory and algebraic ring theory. Ali Majidinya et.al. studied Ring in which the annihilator of an ideal is pure [1]. Bhavanari S. etal defined Prime Graph of a Ring [3] Mohammad Habibi etal. They studied clean graph of a ring [7]. Dhiren K.Basnet and Jayanta Bhattacharyya defined nil clean graph of rings [4], Jafari A. and Sahebi S., studied Vonneumann regular graphs associated with rings[6], A graph G is defined by an ordered pair $(V(G),E(G))$, where $V(G)$ is a nonempty set whose elements are called vertices and $E(G)$ is a set (may be empty) of unordered pairs of distinct vertices of $V(G)$. the element of $E(G)$ are called edges of the graph G . we denote by $\overline{\alpha\beta}$, an edge between two end vertices α and β [8] .

In this paper we give new definition named Pure graph of ring and denoted by *pur(R)* with some properties of this new graph .

Basic concept :

Definition 1.1:[1] An element p in R is called pure element if there exist q in R such that $p=pq$.

Definition 1.2:[2]Let H be a graph , $V(G)$ the set of vertices of G and $S \subseteq V(H)$, the set S is said to be a dominating set if the following condition is satisfy ; $a \in V(H)$ implies either $a \in S$ or there exists $k \in S$ such that a and k are adjacent .

Definition 1.3:[8] A cycle graph with n vertices denoted by C_n , obtained by joining the two end vertices of a path graph and then each vertex of a cycle have degree two .

Definition 1.4:[5] The complete tripartite graph $K_{1,1,p}$. It is a graph consisting of p triangles sharing a common edge is called triangular book.

Theorem 1.5:[2] A connected graph G is Euler if and only if its edge set can be decomposed into cycles.

2- **Main Result :**

Definition 2.1: let R be a ring. A graph $K(V , E)$ where $V(K) = R$ and $E(K)=\{ \overline{\alpha\beta} / \alpha = \alpha\beta$ or $\beta = \beta\alpha$ and $\alpha \neq \beta \}$ is called Pure graph of R and denoted by *pur(R)*

Example :

$Z_2=\{0,1\}$

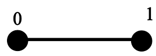

Fig 1: *pur* (Z_2)

$Z_3=\{0,1,2\}$

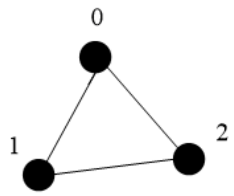

Fig 2: *pur*(Z_3)

$Z_4=\{0,1,2,3\}$

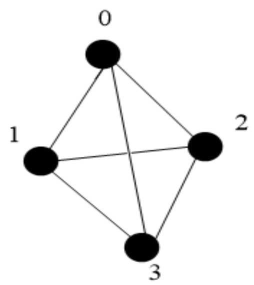

Fig 3:$pur(Z_4)$

Z_5={0,1,2,3,4}

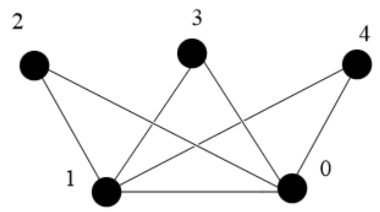

Fig 4:$pur(Z_5)$

Z_6={0,1,2,3,4,5}

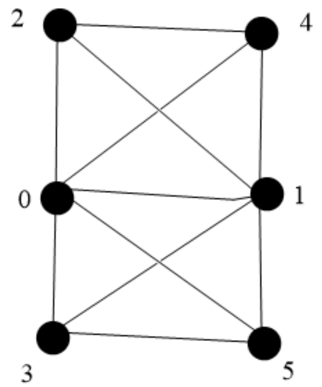

Fig 5: $pur(Z_6)$

Z_7={0,1,2,3,4,5,6}

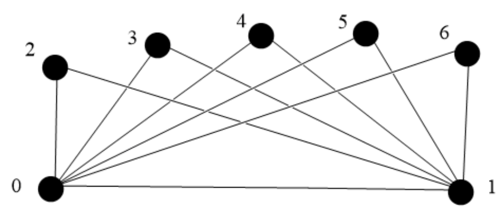

Fig 6: $pur(Z_7)$

Z_8={0,1,2,3,4,5,6,7}

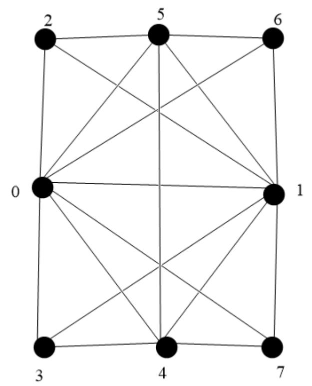

Fig 7: $pur(Z_8)$

Remarks 2.2: Let *Pur(R)* be Pure graph where $R = Z_n$ then

1- *Pur(R)* has no self loops
2- Since $0 = 0\mu$ and $\mu = \mu 1$ for all $0 \neq \mu \neq 1 \in R$ there is an edge from 0 and 1 to μ for all $\mu \in V(G) = R$ so degree (0)= degree (1) = $|R| - 1$
3- For any two non-zero elements a,b inR there are edge one from 0 and 1 to a and another edge from 0 and 1 to b this show that the graph *pur(R)* is connected graph. $d(0,a) = d(a, 1) = 1$ and $d(a,b) \leq 2$ for any two non-zero elements $a,b \in R$
4- If there are two non-zero elements a,b in R such that $a = ab$ or $b = ba$, then the subgraph produced by {0,1,a,b} is K_4 graph , note that the graph *Pur(R)* where$R = Z_6$ as fig 5, Sub graph produced by {0,1,2,4} is K_4.
5- If $R = Z_n$ then max $Pur(Z_n)$= n-1 and min $Pur(Z_n) \geq 2$.

Remark 2.3:

1- $\boldsymbol{v_1 = v_1 v_2\ or\ v_2 = v_2 v_1}$ **if and only if the distance between $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ equal 1.**
Proof:
Suppose that $v_1 = v_1 v_2\ or\ v_2 = v_2 v_1$ and $v_1 \neq v_2 \neq 0\ or\ 1$ then $v_1 v_2 \in E(Pur(R))$ and so by definition of *Pur*(R) then $d(v_1, v_2)$ =1.
Conversely , suppose $d(v_1, v_2)$ =1, if (v_1=0 or v_2=0) or (v_1=1 or v_1=1) then $v_1 = v_1 v_2\ or\ v_2 = v_2 v_1$
if $d(v_1, v_2)$ =1 and $v_1 \neq v_2 \neq 0\ or\ 1$ then $v_1 v_2 \in E(Pur(R))$ which implies
$v_1 = v_1 v_2\ or\ v_2 = v_2 v_1$

2- $\boldsymbol{u_1 \neq u_1 u_2\ or\ u_2 \neq u_2 u_1}$ **if and only if the distance between $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$ equal 2**.
Proof:
Let $u_1 \neq u_1 u_2\ or\ u_2 \neq u_2 u_1$ then there is no edge between $u_1$andu_2 , so the distance between $u_1$and u_2 is largest than 1 . since $0 = 0u_1$, $0 = 0u_2$, $\overline{u_1 0}$, $\overline{u_2 0} \in E(Pur(R))$, hence the distance between u_1 and u_2 equal 2.
Conversely , let the distance between u_1 and u_2 equal 2 since $d(u_1,u_2) \neq 1$, there is no edge between u_1 and u_2 so $u_1 \neq u_1 u_2\ or\ u_2 \neq u_2 u_1$

Theorem 2.4 : If *R*= Z_p, and $p \geq 3$, p (prime number), then *Pur(R)* is a triangular book graph.

Proof :

It is clear that 0, 1 adjacent to all remaining vertices in *Pur* (Z_p) by definition and there is no edge between any other two vertices α and β where ($\alpha\ and\ \beta \neq 0\ or\ \alpha\ and\ \beta\ \neq 1$) since $\alpha \neq \alpha\beta\ mod(p)\ or\ \beta \neq \beta\alpha\ mod(p)$.

Theorem 2.5: let *R*= Z_p, and $p \geq 3$ (*p* isprime number),then *Pur(R)* has $p-2$ of cycle C_3 .

Proof:

By definition of pure graph of a ring it is clear that 0 and 1 adjacent to all ramming vertices then $\forall\ a \in V\left(Pur\left(Z_p\right)\right), 0 \neq a \neq 1$ then we have a cycle of length 3 { 0,1,a}, that is the number of cycle C_3 is *p*-2.

Theorem 2.6: If $R = Z_p$, p is prime number then $Pur(Z_p)$ is Euler graph.

Proof :

By theorem (2.5) the graph $Pur(Z_p)$ that is the set edges can be decomposed into cycles then $Pur(Z_p)$ is Euler graph by theorem (1.5)

Theorem 2.7 : If *R*= Z_p, and $p \geq 3$ (*p* is prime number), then *Pur(R)* has $\sum_{i=3}^{p}(p-i)$ of C_4

Proof :

Suppose that $0 \neq v_1 \neq 1$ be a vertex in *Pur*(*R*) then we have (*p*-3) of C_4 start from the vertex v_1 where *p*-3 is the number of ramming vertices , now we take another vertex v_2 it is clear that is the number of ramming vertices is *p*-4 then we have (*p*-4) of C_4 start from the vertex v_2 Repeat the process for the rest of the vertices that is we have

$$(p-3) + (p-4) + \cdots + 1 = \sum_{i=3}^{p-1}(p - i\ ).$$

Corollary 2.8: The graph $\mathrm{Pur}(Z_n)$, $n > 3$ not prime number has at least one of K_4.

Proof :

By definition of *Pur* (R) for any two vertices $0 \neq v_1, v_2 \neq 1$ we have a cycle C4 { 0,$v_1, 1, v_2$} and 0 and 1 are adjacent that is if $v_1 = v_1v_2$ or $v_2 = v_2v_1$ then we have K_4 sub graph of $\mathrm{Pur}(\mathrm{Z_n})$, since n not prime then there are another vertices so by definition of $Pur(R)$ has another K_4.

3- **Invariants of Pure graph**:

In this part, we studied some results related to invariants of graph theory. The girth of *Pur*(R) is compute in the following theorem.

3.1 Girth of *Pur* (*R*)

In a graph G, the girth of G is the length of the shortest cycle inG. We have following results on girth of *Pur(R)*.

Theorem 3.1.1 : If *R*= Z_n, and $n \geq 3$, then the Girth of *Pur(R)* is equal to 3 .

Proof :

It is clear that ,Since 0 and 1 adjacent to all remaining vertices in *Pur(R)* and also 0 and 1 are adjacent to other , that is the shortest cycle in *Pur(R)* is of length 3

3.2 Dominating set of Pure graph :

Let *G* be a graph, a subset $S \subseteq V(G)$ is said to be dominating set for G if for all $x \in V(G)$, $x \in S$ or there exists $y \in S$ such that x is adjacent to y. Following theorem shows that for a finite commutative ring dominating number is1,where dominating number is the carnality of smallest dominating set.

Theorem 3.2.1 : The dominating number of *Pur* (Z_n) is 1 .

Proof :

Since the smallest dominating set in $Pur(Z_n)$ graph is {0} and {1} because 0 and 1 are adjacent to all vertices in $Pur(Z_n)$ graph then the dominating number is 1.

4- **The complement of *pur(R).***

Definition 4.1: let *R* be a ring. A graph $Pur^c(R)$ is said to be the complement of *pur(R)* where the vertex set is the ring R and the edge set equal to { $\overline{\alpha\beta}/\alpha \neq \alpha\beta$ or $\beta \neq \beta\alpha$ and $\alpha \neq \beta$ }.

Example :Consider Z_n (the ring of integers modulo n).

1- Where $n = 2,3,4$ $Pur^c(Z_n)$ is empty graph since, where *n*=2 then $V(Pur(Z_2)) = \{0,1\}$ and $E(Pur(Z_2)) = \{\overline{01}\}$ since 01=0 , and hence there no edges in $Pur^c(Z_2)$

,also where n=3 then $V(Pur(Z_3)) = \{0,1,2\}$ and $E(Pur(Z_3)) = \{\overline{01}, \overline{02}, \overline{12}\}$ since 01=0 ,02=0 , 21=2 , and hence there no edges in $Pur^c(Z_3)$, now if n=4 then $V(Pur(Z_4)) = \{0,1,2,3\}$ and $E(Pur(Z_4)) = \{\overline{01}, \overline{02}, \overline{03}, \overline{12}, \overline{13}, \overline{23}\}$ since 01=0 ,02=0 ,03=0, 21=2, 31=3 , 23=2 and hence there no edges in $Pur^c(Z_4)$.

2- Where $n = 5$ $Pur^c(Z_5)$ is disconnected graph since $V(Pur(Z_5)) = \{0,1,2,3,4\}$ and $E(Pur(Z_5)) = \{\overline{01}, \overline{02}, \overline{03}, \overline{04}, \overline{12}, \overline{13}, \overline{14}\}$ since 01=0 ,02=0,03=0,04=0 , 21=2,31=3 and 41=4 and hence the E($Pur^c(Z_5)) = \{\overline{23}, \overline{24}, \overline{34}\}$ that is 0 and 1 don't adjacent to any vertex in $Pur^c(Z_5)$ the blew figure show that

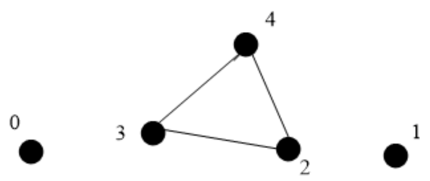

Fig 8 : $Pur^c(Z_5)$

Note: the $Pur^c(Z_n)$is disconnected graph since 0,1 is adjacent to all vertices in $Pur\ (Z_n)$. then $Pur^c(Z_n)$ has at least three components.

Theorem 4.2: A graph $Pur^c(Z_p)$, p is prime number is disconnected graph has three components two isolated vertices 0 and 1 and K_{p-2} complete graph .

Proof :

By theorem 2.4 $Pur\ (Z_p)$ is triangle book graph and 0,1 adjacent to all vertices, and there is no edge between any two vertices that is 0 and 1 are isolated vertices in $Pur^c(Z_p)$, now since the rest of the vertices (their number is p-2) are not adjacent to each other in $Pur\ (Z_p)$ they will be adjacent to each other in $Pur^c(Z_p)$, this means that $Pur^c(Z_p)$ contains K_{p-2} complete sub graph .

5- Conclusion :

The definition of pure graph of commutative ring $Pur(\mathrm{R})$ was introduced in this work and the number of cycle C_3 , C_4 *and* K_4 in $Pur(\mathrm{R})$ was found where $\mathrm{R} = \mathrm{Z_p}$, also gave girth and the dominating number of $Pur(\mathrm{R})$, in addition, a definition is provided the complement of pure graph denoted by $Pur^{\mathrm{c}}(\mathrm{R})$.

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