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Estimating server utilization rate in single server queuing models using an approximate solution of stiff fluid flow model

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ABSTRACT

Modeling real-life scenarios as differential equation models have displayed numerous advantages as the model is able to infuse more realistic features of the scenario under consideration. Adopting a differential equation flow model to estimate server utilization rate in queuing systems has not attracted much attention due to the challenges to obtain an exact solution to the model. However, this concern is bypassed by adopting approximate methods. Thus, this article adopts a first order differential equation model to calculate the rate at which work arrives and the capacity of workstation. The values were obtained as the independent variable becomes large, which expresses the values of the utilization of resources which is now infused into conventional queuing expressions to obtain the required information about the queuing systems. A new novel and convergent one-step second-derivative method is developed in this article to obtain the required value of the server utilization rate.

1. Introduction

Queues are evident in our day-to-day activities, such as, in basic operations like making payment at the checkout counter of a supermarket or bank [1,2]. Several studies to date have explored various areas where a queueing problem can exist or where a waiting line is expected to occur. Some of these fields include analyzing hospital patients during peak congestion [3,4], adoption of the queueing concept in emergency departments [5], supply and demand through logistical loading operations [6,7], amongst others.

In every queuing system, there are certain inputs required to describe or define the queue. These inputs include the basic variables which are arrival rate, service rate and the number of servers. Although, depending on the situation, other inputs may be required to describe the extent of variability in arrivals and service. These input variables are fundamental in any queuing system in order to obtain certain required outputs which vary from information about the waiting line itself or the queuing system as a whole. Specifically, the first output that is required to be computed in a queuing system is the utilization of resource. This is equal to the ratio of the rate at which work arrives and the capacity of the station. From this value, other outputs such as the average number in the waiting line, the average waiting time in the waiting line, average time spent in the queuing system, and the average number in the queuing system can be obtained with adoption of conventional rules such as Little's law [8].

Various studies have considered obtaining the outputs of a queuing model with specific variations unique to the system being considered. Such research includes but not limited to the following; estimation and interpretation of the waiting time for customers arriving to a non-stationary queueing system by [9], minimisation of the average passenger waiting time in personal rapid transit systems [10], develop a new mathematical model to construct membership functions of the fuzzy single channel queueing model [11], analysing congestion at vessel terminals by determining the number of berths that minimize total cost, including waiting time cost and berth's construction costs [12], and optimizing the execution time while minimizing average waiting time in cloud computing environment using queuing model [13]. Other recent studies include [14–18]. As extensive as the research to obtain these queuing parameter outputs, the adoption of differential equation models is quite scarce due to the constraint of the

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rigorous process involved to obtain the exact solution of differential equations or even its non-existence in certain cases.

Approximate solutions have been introduced and widely adopted to solve differential equation models and the specific first order ordinary differential equation (ODE) model in this article is not an exemption. Related studies that have considered approximate solutions to first order ODEs include Dhage [19] whose study approximated solutions of nonlinear first order ODEs, Mall and Chakraverty [20] with the application of Legendre neural network, and Omar and Adeyeye [21] who presented a numerical solution of first order ODEs using block method. Other studies concerned with approximate solution of first order ODEs include [22–25], among other studies. In their various investigations, these authors have explored obtaining approximate solutions to the first order models, but little or no investigation exists with respect to utilizing these differential equation models in queuing systems.

Hence, the rest of this paper has organized the introduction of a differential equation model, the development of a new approximate method to obtain its solution, the utilization of the obtained solution to estimate the server utilization rates in queuing models and further calculate the basic outputs of the queuing system. The article introduces the differential equation flow model in the next section, with details of the developed method and its properties in Section 3. Section 4 discusses the implementation of the method while its application to queuing models is displayed in Section 5. The article is concluded in Section 6.

2. Differential equation flow model

Various processing scenarios can be depicted by a set of flow diagrams that displays the order in which the processing progression follows. A sample flow diagram as seen in the early work of Vandergraft [26] is shown in the following figure where the links between the boxes describe the flow of information from one box to the other.

The flow through a network of queues can be interpreted in terms of fluid flow and thus modeled as a set of differential equations. It has been grounded that the applicability of such modeling approach is seen to be suitable for describing flow of work. To derive the equations to define this flow, a system of ordinary differential equations is presented to approximate the flow graph in Fig. 1.

Given nodes $n = 1, 2, 3, \ldots$ as shown in Fig. 1, at time t, let $\tau_n(t)$, $\omega_n(t)$, and $\xi_n(t)$ denote the number of tasks, incremental arrivals, and the average number of hours to process one item, respectively. The flow of work is defined approximately by the set of first order differential equations,

$$\dot{\tau}_n(t) = \dot{\gamma}_n(t) + \sum_i \alpha_{in}\xi_i^{-1}(t) - \xi_n^{-1}(t), \quad n = 1, 2, \ldots \tag{1}$$

The initial condition can be imposed at $\tau_n(0) = 0$ such that there are no tasks at the beginning of the workflow, with $\gamma_n(t) = 0$ for most nodes n, asides node 1 where $\gamma_1(t)$ is an increasing function of t. Note that the summation in Eq. (1) is over all nodes that feed into node n, and α_{in} is the ration of the flow out of node i that goes to node n as shown in Fig. 1. Since $\xi_i(t)$ denotes the time to process one item at node i, thus $\xi_i^{-1}(t)$ is the rate at which the items are processed. Moreover, the case of $\xi_i^{-1}(t)$ being undefined does not appear as a problem as $\xi_i(t) > 0$. However, the challenge arises in finding an approximation to $\dot{\gamma}_n(t)$ which requires a rearrangement of Eq. (1) as

$$\dot{z}_n(t) = \sum_i \alpha_{in}\xi_i^{-1}(t) - \xi_n^{-1}(t) \tag{2}$$

where $z_n(t) = \tau_n(t) - \gamma_n(t)$, which makes the expression in Eq. (2) solvable by any suitable approximate approach such as the method introduced in this article.

3. Development of method and properties

The approximate method proposed for the solution of the flow model is derived using the newly introduced linear block approach by Adeyeye and Omar [27]. To obtain the proposed method, consider the reduced model in Eq. (2) for the linear block method form expressed as

$$\psi_{\eta+x} = \sum_{j=0}^{m-1} \frac{(xh)^j}{j!}\psi_\eta^{(j)} + \sum_{j=0}^{k} \left(\phi_{xj} f_{\eta+j} + \varphi_{xj} g_{\eta+j}\right), \quad x = 1, 2, \ldots, k, \tag{3}$$

where m is the order of the differential equation flow model, x is the step number of the approximate method, $f_{\eta+j} = f(t_{\eta+j}, \psi_{\eta+j})$, and $g_{n+j} = \frac{df(t_{\eta+j}, \psi_{\eta+j})}{dt}$. Specifically, for suitable comparison with other one-step methods which are the conventional approaches for solving equations of the form of Equation (2), x is also chosen to be 1. Substituting the values of m and x transforms Equation (3) to the form

$$\psi_{\eta+1} = \psi_\eta + \phi_{10} f_\eta + \phi_{11} f_{\eta+1} + \varphi_{10} g_\eta + \varphi_{11} g_{\eta+1} \tag{4}$$

Still following the steps from the algorithm of the linear block approach, the unknown coefficients in Eq. (4) are obtained from

$$\begin{bmatrix} \phi_{10} \\ \phi_{11} \\ \varphi_{10} \\ \varphi_{11} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & h & 1 & 1 \\ 0 & h^2/2 & 0 & h \\ 0 & h^3/6 & 0 & h^2/2 \end{bmatrix}^{-1} \begin{bmatrix} h \\ h^2/2 \\ h^3/6 \\ h^4/24 \end{bmatrix} = \begin{bmatrix} h/2 \\ h/2 \\ h^2/12 \\ h^2/12 \end{bmatrix} \tag{5}$$

which presents the proposed approximate method for the flow model to be

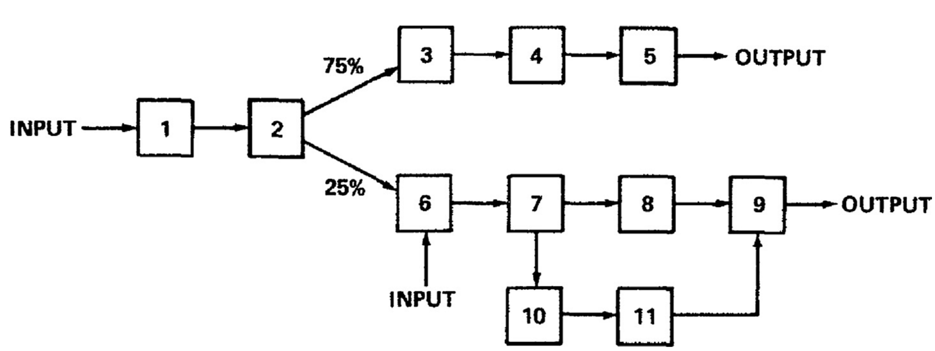


Fig. 1. Conventional flow diagram [26].

$$\psi_{\eta+1} = \psi_\eta + h/2\left(f_\eta + f_{\eta+1}\right) + h^2/12\left(g_\eta - g_{\eta+1}\right). \tag{6}$$

The developed method in the equation preceding is coined a one-step second-derivative method. A basic property that is demanded of an acceptable linear multistep method, such as the proposed method in Eq. (6), is the property of convergence. The developed method is said to be a convergent linear multistep method, if the solution ψ_η generated by the method, converges to the theoretical solution $\psi(t)$, as the $\Delta t \to 0$. Thus, the following theorem is considered.

Theorem 3.1 (*Dahlquist Equivalence Theorem*). *A linear multistep method is convergent if and only if it is consistent and zero-stable.*

The definitions of the hypotheses in Theorem 3.1 is as given.

Definition 3.1 (*Consistency*). A linear multistep method is said to be consistent if it has order $p \geqslant 1$.

Definition 3.2 (*0-stability*). A linear multistep method is said to be 0-stable if no root of the first characteristic polynomial has modulus greater than one, and if every root with modulus one is simple.

To imply the acceptability of the developed method in Eq. (6) with respect to its convergence property, the following propositions are stated.

Proposition 3.1 (*Convergence of a one-step second-derivative method*). *A linear multistep method of the form* $\psi_{\eta+1} = \psi_\eta + h/2\left(f_\eta + f_{\eta+1}\right) + h^2/12\left(g_\eta - g_{\eta+1}\right)$ *is consistent and 0-stable, hence convergent.*

Proof. Considering the first premise of the proposition, it is required to show that the developed method is consistent with respect to Definition 3.1. Following the approach in Lambert [28], associate the linear difference operator $\Im$ with the developed method, such that $h = \Delta t$ and

$$\begin{aligned}\Im[\psi(t);\Delta t] = -\psi(t_\eta) + \psi(t_\eta + \Delta t) - \Delta t/2\left(\psi'(t_\eta) + \psi'(t_\eta + \Delta t)\right) \\ -(\Delta t)^2/12\left(\psi''(t_\eta) - \psi''(t_\eta + \Delta t)\right).\end{aligned} \tag{7}$$

Expanding $\psi(t_\eta + \Delta t)$ and its derivatives, $\psi'(t_\eta + \Delta t)$ and $\psi''(t_\eta + \Delta t)$ as Taylor series about t_η gives

$$\begin{aligned}\Im[\psi(t);\Delta t] = -\psi(t_\eta) - \tfrac{\Delta t}{2}\left[\psi'(t_\eta)\right] \\ +\left[\psi(t_\eta) + \Delta t\psi'(t_\eta) + \tfrac{(\Delta t)^2}{2!}\psi''(t_\eta) + \tfrac{(\Delta t)^3}{3!}\psi'''(t_\eta) + \tfrac{(\Delta t)^4}{4!}\psi^{(4)}(t_\eta) + \tfrac{(\Delta t)^5}{5!}\psi^{(5)}(t_\eta) + \ldots\right] \\ -\tfrac{\Delta t}{2}\left[\psi'(t_\eta) + \Delta t\psi''(t_\eta) + \tfrac{(\Delta t)^2}{2!}\psi'''(t_\eta) + \tfrac{(\Delta t)^3}{3!}\psi^{(4)}(t_\eta) + \tfrac{(\Delta t)^4}{4!}\psi^{(5)}(t_\eta) + \ldots\right] \\ -\tfrac{(\Delta t)^2}{12}\left[\psi''(t_\eta)\right] + \tfrac{(\Delta t)^2}{12}\left[\psi''(t_\eta) + \Delta t\psi'''(t_\eta) + \tfrac{(\Delta t)^2}{2!}\psi^{(4)}(t_\eta) + \tfrac{(\Delta t)^3}{3!}\psi^{(5)}(t_\eta) + \ldots\right].\end{aligned} \tag{8}$$

Collecting terms in Eq. (8) simplifies the expression to

$$\Im[\psi(t);\Delta t] = \frac{(\Delta t)^5}{720}\psi^{(5)}(t_\eta), \tag{9}$$

where the coefficient of $(\Delta t)^5\psi^{(5)}(t_\eta)$ is the error constant, and the order of the method $p = m - 1$ where m is the number of the highest derivative in Equation (9). Thus $p = 4$ and hence the developed method is consistent.

Considering the second premise in the proposition with respect to Definition 3.2, the first characteristic polynomial takes the form $\rho(\zeta) = \zeta - 1$ with principal root $\zeta = 1$. Since the method has no root of the first characteristic polynomial with modulus greater than one with the root being simple. Hence, the developed method is said to be 0-stable.

On satisfying both hypotheses, it is concluded that a linear multistep method of the form $\psi_{\eta+1} = \psi_\eta + h/2\left(f_\eta + f_{\eta+1}\right) + h^2/12\left(g_\eta - g_{\eta+1}\right)$ is convergent. □

4. Implementation of the method

The flow model is specifically considered under the condition where the flow graph consists of one workstation with constant arrival of $\lambda = \dot{\gamma}(t)$ items per hour, a processing time of s hours per item per person and w workers assigned to the station. The differential equation model then takes the form

$$\dot{\tau}(t) = \lambda - \frac{w}{s}; \quad \tau(0) = 0. \tag{10}$$

Considering a case where many workers are available to be assigned at the workstation, then the number of workers assigned is equivalent to the number of work items per time. Hence, the differential in Eq. (10) becomes

$$\dot{\tau}(t) = \lambda - \frac{\tau}{s}; \ \tau(0) = 0 \tag{11}$$

with exact solution

$$\tau(t) = \lambda s - \lambda s e^{-\frac{t}{s}} \tag{12}$$

The differential equation in Eq. (11) is considered a stiff differential equation because its exact solution in Eq. (12) has a term of the form e^{-ct}, where c is a large positive constant. This implies that when the processing time in hours per item per person becomes very small, then it is seen that c becomes large. However, the developed method will adequately solve the flow differential equation model by choosing the step size to be small in order to avoid the solution being numerically unstable. Details of comparison of the developed method in this article to other existing approaches will be displayed in the next section.

It is known that τ approximates the ratio of the rate at which work arrives and the workstation capacity. This implies that as t becomes large $(t \to \infty)$, τ tends towards the value that expresses the utilization of resources (ρ). Specifically, in analyzing any queuing system, one is basically interested in obtaining the values for the number of work items waiting to be attended to, and the time that an average customer waits before being attended to. One may also be keen to know more details about the whole queuing system by considering the time the customer spends in the system, and the number of customers in the system. Since the scenario under consideration in this model is that of a single workstation, one can easily obtain the number of work items waiting to be attended to in the system, denoted L_q as [29]

$$L_q = \frac{\rho^2}{1-\rho}, \tag{13}$$

while Little's law [8] provides the expression to obtain the time that an average customer waits before being attended to, which is denoted W_q and defined as

$$W_q = \frac{L_q}{\lambda}. \tag{14}$$

The time the customer spends in the system and the number of customers in the system, are denoted W_s and L_s respectively. The former is obtained by adding W_q to the service time, while the latter can be obtained using another application of Little's law as $L_s = \lambda W_s$.

5. Application of developed method to queuing models

In this section, the developed method will be used to estimate the server utilization rate in various single server queuing model scenarios. The accuracy will be compared to both the exact solution and other methods, before implementing the method to analyze the required parameters of the queuing models. It is assumed in the problems considered that models assume a certain type of variability; Poisson distribution for number of arrivals and exponential distribution for service times.

Problem 1: Consider a bank's ATM with a single machine, where customers arrive at the rate of one every other minute and each customer spends an average of 90 s completing his/her transactions.

The differential equation model in Eq. (11) is solved using the developed method in Eq. (6) to estimate ρ for Problem 1 using parameters $\lambda = 0.5/\text{min}$ and $s = 9/6$. The obtained solutions in comparison to the exact solution in Eq. (12) as $t \to \infty$ is given in Table 1.

From Table 1, as $t \to \infty$, $\tau \approx 0.7490 = \rho$ which implies the value of the ATM machine utilization rate. The other parameters are obtained as

$L_q = \frac{0.7490^2}{1-0.7490} = 2.2350637450$. The average number in line to use the ATM is approximately 2 customers.

$W_q = \frac{L_q}{\lambda} = \frac{2.2350637450}{0.5} = 4.47012749$. The average waiting time for each customer is approximately 4.5 min.

$W_s = 4.5 + 1.5 = 6$. The average waiting time-in-system is 6 min.

$L_s = 0.5(6) = 3$. The average number of customers in the system are 3 customers.

Problem 2: Consider a queue with a single server, the arrival rates of 5 per hour and service rates are 10 per hour.

The parameters to estimate ρ for Problem 2 are $\lambda = 5/\text{h}$ and $s = 1/10$. The obtained solutions in comparison to the exact solution as $t \to \infty$ is given in Table 2.

From Table 2, as $t \to \infty$, $\tau = 0.5 = \rho$ which implies the value of the server utilization rate. The other parameters are obtained as

$L_q = \frac{0.5^2}{1-0.5} = 0.5$.

$W_q = \frac{L_q}{\lambda} = \frac{0.5}{5} = 0.1$. The average waiting time for each customer is approximately 6 min.

Table 1
Exact Solution and Absolute Error (AE) for Solving Problem 1.

t	Exact Solution	AE($\Delta t = \frac{1}{2}$)	AE($\Delta t = \frac{1}{4}$)	AE($\Delta t = \frac{1}{6}$)	AE($\Delta t = \frac{1}{8}$)	AE($\Delta t = \frac{1}{10}$)
1	0.3649371607	2.388333e−03	1.778817e−03	1.317400e−03	1.037719e−03	8.541191e−04
2	0.5523021464	2.460027e−03	1.830769e−03	1.355065e−03	1.067002e−03	8.780114e−04
3	0.6484985376	1.900412e−03	1.413181e−03	1.045357e−03	8.228330e−04	6.769293e−04
4	0.6978874115	1.304981e−03	9.696400e−04	7.168310e−04	5.640351e−04	4.639104e−04
5	0.7232445050	8.401049e−04	6.237279e−04	4.608299e−04	3.624700e−04	2.980550e−04
6	0.7362632708	5.192009e−04	3.851694e−04	2.844043e−04	2.236194e−04	1.838358e−04
7	0.7429473281	3.119638e−04	2.312463e−04	1.706466e−04	1.341259e−04	1.102375e−04
8	0.7463790375	1.836198e−04	1.360015e−04	1.003009e−04	7.880639e−05	6.475515e−05
9	0.7481409359	1.063890e−04	7.873609e−05	5.803273e−05	4.557964e−05	3.744379e−05
10	0.7490455246	6.088083e−05	4.502044e−05	3.316244e−05	2.603668e−05	2.138407e−05

Table 2
Exact Solution and Absolute Error (AE) for Solving Problem 2.

t	Exact Solution	AE($\Delta t = \frac{1}{2}$)	AE($\Delta t = \frac{1}{4}$)	AE($\Delta t = \frac{1}{6}$)	AE($\Delta t = \frac{1}{8}$)	AE($\Delta t = \frac{1}{10}$)
1	0.4999773000	6.046879e−02	1.599674e−02	6.306521e−03	3.149507e−03	1.833349e−03
2	0.4999999989	7.318441e−03	5.132439e−04	8.011704e−05	2.012477e−05	6.888808e−06
3	0.5000000000	8.854069e−04	1.644379e−05	1.014170e−06	1.276864e−07	2.557571e−08
4	0.5000000000	1.071192e−04	5.268408e−07	1.283781e−08	0.000000e+00	0.000000e+00
5	0.5000000000	1.295960e−05	1.687939e−08	0.000000e+00	0.000000e+00	0.000000e+00
6	0.5000000000	1.567891e−06	5.407967e−10	0.000000e+00	0.000000e+00	0.000000e+00
7	0.5000000000	1.896881e−07	1.732647e−11	0.000000e+00	0.000000e+00	0.000000e+00
8	0.5000000000	2.294903e−08	5.550560e−13	0.000000e+00	0.000000e+00	0.000000e+00
9	0.5000000000	2.776443e−09	1.776357e−14	0.000000e+00	0.000000e+00	0.000000e+00
10	0.5000000000	3.359024e−10	5.551115e−16	0.000000e+00	0.000000e+00	0.000000e+00

Table 3
Exact Solution and Absolute Error (AE) for Solving Problem 3.

t	Exact Solution	AE($\Delta t = \frac{1}{2}$)	AE($\Delta t = \frac{1}{4}$)	AE($\Delta t = \frac{1}{6}$)	AE($\Delta t = \frac{1}{8}$)	AE($\Delta t = \frac{1}{10}$)
1	0.2500000000	1.378226e−01	7.732905e−02	4.412358e−02	2.558516e−02	1.506595e−02
2	0.2500000000	7.598025e−02	2.391913e−02	7.787563e−03	2.618402e−03	9.079311e−04
3	0.2500000000	4.188718e−02	7.398573e−03	1.374461e−03	2.679690e−04	5.471537e−05
4	0.2500000000	2.309200e−02	2.288499e−03	2.425845e−04	2.742412e−05	3.297355e−06
5	0.2500000000	1.273039e−02	7.078697e−04	4.281480e−05	2.806602e−06	1.987111e−07
6	0.2500000000	7.018143e−03	2.189556e−04	7.556570e−06	2.872295e−07	1.197509e−08
7	0.2500000000	3.869034e−03	6.772650e−05	1.333692e−06	2.939526e−08	0.000000e+00
8	0.2500000000	2.132961e−03	2.094890e−05	2.353890e−07	3.008330e−09	0.000000e+00
9	0.2500000000	1.175881e−03	6.479835e−06	4.154483e−08	0.000000e+00	0.000000e+00
10	0.2500000000	6.482517e−04	2.004318e−06	7.332428e−09	0.000000e+00	0.000000e+00

$W_s = 6 + 6 = 12$. The average waiting time in the system is 12 min.
$L_s = 5(0.2) = 1$. The average number of customers in the system is 1 customer.

Problem 3: Consider a single server queue system for which customers arrive at a rate of 10 per hour and the average processing time is 1.5 min.

The parameters to estimate ρ for Problem 3 are $\lambda = 10$ and $s = 0.025$. The obtained solutions in comparison to the exact solution as $t \to \infty$ is given in Table 3.

From Table 3, as $t \to \infty$, $\tau = 0.25 = \rho$ which implies the value of the server utilization rate. The other parameters are obtained as

$$L_q = \frac{0.25^2}{1-0.25} = \frac{1}{12}.$$
$$W_q = \frac{L_q}{\lambda} = \frac{1/12}{10} = \frac{1}{120}.$$
$$W_s = \frac{1}{120} + \frac{1.5}{60} = \frac{1}{30}.$$
$$L_s = 10\left(\frac{1}{30}\right) = \frac{1}{3}.$$

6. Conclusion

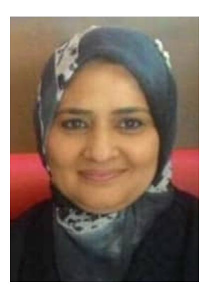
This article has considered a new approach of adopting a differential equation model to estimate the server utilization rate, in order to obtain relevant queuing parameters in single server queuing models. For the specific case of the model being utilized for single workstations or servers, the expression transformed to a stiff differential equation whose solution is usually unstable in cases of large step sizes, hence the reason to adopt the developed method with adequate stepsize choices. The step sizes were chosen to display the convergence property of the developed method in terms of its closeness in accuracy to the exact solution as the stepsize tends to zero. From the considered problems, the developed method accurately converged to the required value of the server utilization rate and then conventional expressions from Little's law were adopted to obtain the required queuing parameters. The article achieves its main objective of presenting a new insight into the estimation of the server utilization rate. It has additionally developed a new approximate approach coined the one-step second-derivative method for approximating models in the form of first-order ordinary differential equations. Therefore, subsequent research aims to extend the model to the solution for multiple-server queues and also apply the one-step second-derivative method to models in other fields outside queuing theory.

Declaration of Competing Interest

The author has no competing interests to declare.

References

[1] Gross D. Fundamentals of queueing theory. Canada: John Wiley & Sons; 2008.
[2] Bhat UN. An introduction to queueing theory: modeling and analysis in applications. Birkhäuser; 2015.
[3] Bastani P. A queueing model of hospital congestion Doctoral dissertation. Dept. of Mathematics-Simon Fraser University; 2009.
[4] Madelbaum A, Momcilovic P, Tseytlin Y. On fair routing from emergency departments to hospital wards: QED queues with heterogeneous servers. Manage Sci 2012;58(7):1273–91.
[5] Brahma PK. Queuing theory and customer satisfaction: a review of terminology, trends, and applications to pharmacy practice. Asia Pac J Market Manage Rev 2013;2(6):83–9.
[6] White JA, Schmidt JW, Bennett GK. Analysis of queueing systems. New York: Academic Press; 2012.
[7] Newell C. Applications of queueing theory. New York: Springer Science & Business Media; 2013.
[8] Little JD. A proof for the queuing formula: L= λ W. Oper Res 1961;9(3):383–7.
[9] Smith JS, Nelson BL. Estimating and interpreting the waiting time for customers arriving to a non-stationary queueing system. Proceedings of 2015 winter simulation conference (WSC), December 2015, 2015.
[10] Lees-Miller JD. Minimising average passenger waiting time in personal rapid transit systems. Ann Oper Res 2016;236(2):405–24.
[11] Mueen Z, Ramli R, Zaibidi NZ. A single server fuzzy queues with priority and unequal service rates. Int J Math Oper Res 2017;11(3):422–34.
[12] Saeed N, Larsen OI. Application of queuing methodology to analyze congestion: a case study of the Manila International Container Terminal, Philippines. Case Stud Transp Policy 2016;4(2):143–9.
[13] Pal S, Pattnaik PK. A Simulation-based approach to optimize the execution time and minimization of average waiting time using queuing model in cloud computing environment. Int J Electr Comput Eng 2016;6(2):743–50.
[14] Joshi PK, Gupta S, Rajeshwari KN. Analysis and comparative study of various performance measures of M/G/1 and M/G/S queuing models. J Comput Math Sci 2019;10(1):112–20.
[15] Findley DJ, Anderson TJ, Bert SA, Nye T, Letchworth W. Evaluation of wait times and queue lengths at ferry terminals. Res Transp Econ 2018;71:27–33.
[16] Koko MA, Burodo MS, Suleiman S. Queuing theory and its application analysis on bus services using single server and multiple servers model. Am J Oper Manage Inf Syst 2018;3(4):81–5.
[17] Singhal K, Singhal J, Kumar S. The value of the customer's waiting time for general queues. Decis Sci 2019;50(3):567–81.
[18] Xavier G, Crane J, Follen M, Wilcox W, Pulitzer S, Noon C. Using Poisson modeling and queuing theory to optimize staffing and decrease patient wait time in the emergency department. Open J Emerg Med 2018;6(3):54–72.
[19] Dhage SB. Approximating solutions of nonlinear first order ordinary differential equations. Global J Math Sci 2015;2(2):25–35.
[20] Mall S, Chakraverty S. Application of Legendre neural network for solving ordinary differential equations. Appl Soft Comput 2016;43:347–56.
[21] Omar Z, Adeyeye O. Numerical solution of first order initial value problems using a self-starting implicit two-step Obrechkoff-type block method. J Math Stat 2016;12(2):127–34.
[22] Wu X, Wang B. Exponential Fourier collocation methods for first-order differential equations. Recent developments in structure-preserving algorithms for oscillatory differential equations. Singapore: Springer; 2018.
[23] Shokri A, Shokri AA. The new class of implicit L-stable hybrid Obrechkoff method for the numerical solution of first order initial value problems. Comput Phys Commun 2013;184(3):529–31.
[24] Sunday J, Odekunle MR, Adesanya AO. Order six block integrator for the solution of first order ordinary differential equations. Int J Math Soft Comput 2013;3(1):87–96.
[25] Badmus AM, Fatunsin LM. A new ninth order hybrid block integrator for solution of first order ordinary differential equations. Asian J Math Appl 2015:1–8 (Article ID ama0238).
[26] Vandergraft JS. A fluid flow model of networks of queues. Manage Sci 1983;29(10):1198–208.
[27] Adeyeye O, Omar Z. New generalized algorithm for developing k-step higher derivative block methods for solving higher order ordinary differential equations. J Math Fundam Sci 2018;50(1):40–58.
[28] Lambert JD. Computational methods in ordinary differential equations. Wiley; 1973.
[29] Ross SM. Introduction to probability models. USA: Academic Press; 2014.



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