Int. J. Contemp. Math. Sciences, Vol. 7, 2012, no. 17, 805 - 820

Pre-Topology Generated by the Short Path Problems

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Abstract

Let G be a graph, each edge e of which is given a weight w(e). The shortest path problem is a path of minimum weight connecting two specified vertices a and b, and from it we have a pre-topology. Furthermore, we study the restriction and separators in pre-topology generated by the shortest path problems. Finally, we study the rate of liaison in pre-topology between two subgraphs. It is formally shown that the new distance measure is a metric

Mathematics Subject Classification: 05C78, 54C05.

Keywords: Graph theory, Short path problem, Closure operators, j-pre-topology; j = 1, 2, j-dependency; j = 1, 2., Restriction, Separators, Rate of liaison.

1 Introduction and Preliminaries

The field of graph theory has undergone tremendous growth during the past century. As recently as fifty years ago, the graph theory community has few members and most were in Europe and North America; today there are hundreds of graph theorists and they span the globe. By the min-1970s, the field had reached the point where we perceived the need for a collection of surveys of the areas of graph theory: the result was our three-volume series selected Topics in Graph Theory, comprising articles written by distinguished experts in a common style. During the past quarter-century, the transformation of the subject has continued, with individual areas (such as topological graph theory) expanding to the point of having important sub-branches themselves. This inspired us to conceive of a new series of papers, each a collection of articles within a particular area written by experts within that area and this paper one of these series. The basic idea is that such structure can get pre-topological spaces by using closure operators on the short path problem.

A *graph* [10], G = (V(G), E(G)) consists of a vertex set V(G) and an edge set E(G) of unordered pairs of elements of V(G). To avoid ambiguities, we assume that the vertex and edge sets are disjoint. We say that two vertices *v* and *w* of a graph G are *adjacent* if there is an edge of the form *vw* joining them, and the vertices *v* and *w* are then *incident* with such an edge. A *subgraph* [8], of a graph G is a graph, each of whose vertices belong to V(G) and each of whose edges belong to E(G). Two graphs G and G* are said to be *isomorphic* [10] if there are bijections correspondence between the vertices of G and those of G* such that the number of edges joining any two vertices of G is equal to the number of edges joining the corresponding vertices of G*. A *walk* [5], is a "way of getting from one vertex to another", and consists of a sequence of edges, one following after another. A walk in which no vertex appears more than once is called a *path*. A graph is *connected* [3], if its cannot be expressed as the union of two graphs, and *disconnected* otherwise For other notions or notations in topology not defined here we follow closely [3, 9].

The Short Path Problem 1.1. [1]

With each edge e of G let there be associated a real number w(e), called its weight. Then G, together with these weights on its edges, is called a weighted graph.

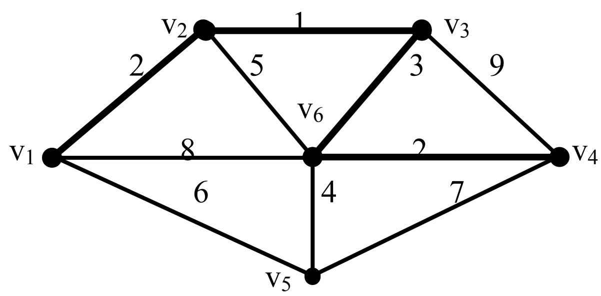


figure 1.1. A (v_1, v_4)-path of minimum weight

Weight graphs occur frequently in application of graph theory. In the friendship graph, for example, weights might indicate intensity of friendship; in the communications graph, they could represent the construction or maintenance costs of the various communication links.

If H is a subgraph of a weighted graph, the weight w(H) of H is the sum of the weights $\sum_{e\in w(H)} w(e)$ an its edge. Many optimization problems amount to finding, in a weighted graph, a subgragh of a certain type with minimum (or maximum) weight. One such is the shortest path problem: given a railway network connecting various towns, determine a shortest route between two specified towns in the network.

Here one must find, in a weighted graph, a path of minimum weight connecting two specified vertices a and b; the weights represent distances by rail between directly-linked towns, and are therefore non-negative. The path indicated in the graph of figure(1) is a (v_1, v_4)-path of minimum weight.

We now present a pre-topology and algorithm for solving the shortest path problem. For clarity of exposition, we shall refer to the weight of a path in a weighted graph as its length; similarly the minimum weight of a (v, u)-path will be called the distance between v and u and denoted by d(v, u). These definitions coincide with the usual notions of length and distance.

It clearly suffices to deal with the shortest path problem for the simple graphs; so we shall assume here that G is simple. We shall also assume that all the weights are positive. This, again, is not a serious restriction because, if the weight of an edge is zero, then its ends can be identified. We adopt the convention that $w(v,u)=\infty$ if $vu\not\in E$.

Dijkstra's Algorithm for shortest path

Given a connected graph G = (V(G), E(G)) with vertices 1, … ,n and edges (i, j) having weighted $w_{ij} > 0$, this algorithm determines the weights of shortest paths from vertex 1 to the vertices 2, … ,n.

INPUT: Number of vertices n, edges (i, j), and weights w_{ij}

OUTPUT: Weights W_j of shortest paths $1 \to j$, j = 2, 3, … ,n

1. Initial step
   Vertex 1 get PW : $W_1 = 0$
   Vertex j (= 2, … ,n) gets TW : $W^*_j = w_{1j}$ ($= \infty$ if there is no edge (i, j) in G).
   Set $\mathcal{PL} = \{1\}$, $\mathcal{TL} = \{2, 3, \dots ,n\}$.
2. Fixing a permanent labeled
   Find a k in $\mathcal{TL}$ for which W^*_k is minimum, set $W_k = W^*_k$. Take the smallest k if there are several. Delete k from $\mathcal{TL}$ and in dude it in $\mathcal{PL}$.
   If $\mathcal{TL} = \phi$ (that is, $\mathcal{TL}$ is empty) then

OUTPUT $W_2, \dots, W_n$. Stop.
Else continue (that is, go to step 3).
3. Updating temporary labels
For all j in $\mathcal{TL}$, set $W^*_j = \min_k \{W^*_j, W_k + w_{kj}\}$ (that is, take the smallest of W^*_j and $W_k + w_{kj}$ as your new W^*_j).
Go to step 2.
End Dijkstra

2 j-pre-Topology Generated by The Short Path Problems; j = 1,2

In the following H^i denotes the subgraph $H^i = (V(H^i), E(H^i))$ which is represent the subgraph in step i for the shortest path problem of a graph $G = (V(G), E(G))$, H^e denotes the subgraph $H^e = (V(H^e), E(H^e))$ which is represent the subgraph in end step for the shortest path problem, and τ^i will be used for a family of subgrahs of the power set of $P(V(H^i))$ obtained by closure operators Cl(K) where $K \subseteq H^i$.

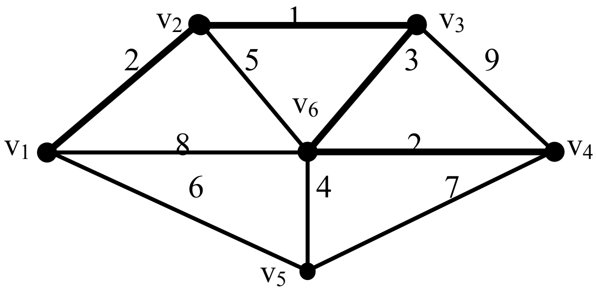
Definition 2.1. Let $G = (V(G), E(G))$ be a connected graph, a 1-pre-topology on a subgraph H^i of a shortest path problem in step i of a graph G is a family $1\text{-}\tau^i = \{1\text{-}Cl(K) : K \subseteq H^i\} \subseteq P(V(H^i))$ where $1\text{-}Cl(K) = \{k \in K : (k, h)\text{-shortest path}, h \in H^i\}$ together V(G). The elements of $1\text{-}\tau^i$ are called 1-open subgraphs of the 1-pre-topology in step i.

Definition 2.2. Let $G = (V(G), E(G))$ be a connected graph, a 2-pre-topology on a subgraph H^i of a shortest path problem in step i of a graph G is a family $2\text{-}\tau^i = \{2\text{-}Cl(K) : K \subseteq H^i\} \subseteq P(V(H^i))$ where $2\text{-}Cl(K) = V(K) \cup \{k \in K : (k, h)\text{-shortest path}, h \in H^i\}$ together V(G). The elements of $2\text{-}\tau^i$ are called 2-open subgraphs of the 2-pre-topology in step i.

This notations is close to the classical notation of topology. pre-topologies differ from topologies in that they do not require the open subgraphs to be stable with respect to finite intersection. Furthermore, there exists the maximal open subgraph denoted as $\max(j\text{-}\tau^i)$ in $j\text{-}\tau^i : j = 1, 2$.

The complement of j-open subgraph is called j-closed subgraph. The j-interior of subgraph K is $j\text{-}Int(V(K)) = \cup\{V(O) : O \text{ is j-open subgraph}, V(O) \subseteq V(K)\}$, and the j-closure of subgraph K is $j\text{-}Cl(V(K)) = \cap\{V(F) : F \text{ is j-closed subgraph}, V(K) \subseteq V(F)\}$ where j = 1, 2.

Example 2.1. Consider the graph in figure 1.1.



Here one must find, in a weighted graph, a path of minimum weight connecting two specified vertices a and d.

		v_1	v_2	v_3	v_4	v_5	v_6	$Max(1-\tau^i)$
Step.0		--	--	--	--	--	--	ϕ
Step.1	v_1	--	**2**	∞	∞	6	8	$\{v_1, v_2\}$
Step.2	v_2	--	--	**1**	∞	∞	5	$\{v_1, v_2, v_3\}$
Step.3	v_3	--	--	--	9	∞	**3**	$\{v_1,v_2,v_3,v_6\}$
Step.4	v_6	--	--	--	**2**	4	--	$\{v_1,v_2,v_3,v_6,v_4\}$

If $H^0 = \{v_1\}$, then $1\text{-}\tau^0 = \{V(G), \phi\}$

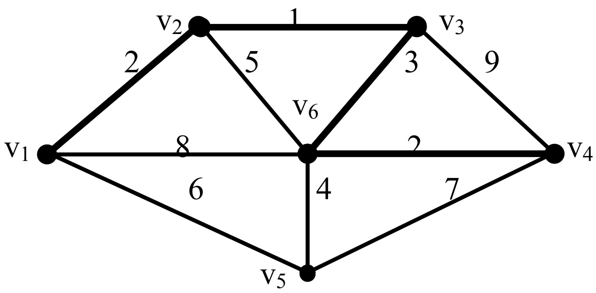
If $H^1 = \{v_1, v_2\}$, then $1\text{-}\tau^1 = \{V(G), \phi, \{v_1, v_2\}\}$

If $H^2 = \{v_1, v_2, v_3\}$, then $1\text{-}\tau^2 = \{V(G),\phi,\{v_1,v_2\}, \{v_2, v_3\},\{v_1, v_2, v_3\}\}$,

If $H^3 = \{v_1, v_2, v_3, v_6\}$, then $1\text{-}\tau^3 = \{V(G), \phi, \{v_1, v_2\}, \{v_2, v_3\},\{v_3, v_6\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_6\}, \{v_2, v_3, v_6\}, \{v_1, v_2, v_3, v_6\}\}$

If $H^4 =\{v_1,v_2,v_3,v_6,v_4\}$, then $1\text{-}\tau^4 = \{V(G), \phi,\{v_1,v_2\},\{v_2,v_3\},\{v_3,v_6\},\{v_6,v_4\}, \{v_1,v_2,v_3\},\{v_1,v_2,v_3,v_6\},\{v_1,v_2,v_6,v_4\},\{v_1,v_2,v_4\}, \{v_2,v_3,v_6\},\{v_2,v_3,v_6,v_4\},\{v_2,v_3,v_6\},\{v_3,v_6,v_4\}, \{v_1,v_2,v_3,v_6,v_4\},\{v_1,v_2,v_3,v_4\}\}$.

Example 2.2. Consider the graph in figure 1.1.



Here one must find, in a weighted graph, a path of minimum weight connecting two specified vertices a and d.

		v_1	v_2	v_3	v_4	v_5	v_6	Max(2-τ^i)
Step.0		--	--	--	--	--	--	$\{v_1\}$
Step.1	v_1	--	**2**	∞	∞	6	8	$\{v_1, v_2\}$
Step.2	v_2	--	--	**1**	∞	∞	5	$\{v_1, v_2, v_3\}$
Step.3	v_3	--	--	--	9	∞	**3**	$\{v_1, v_2, v_3, v_6\}$
Step.4	v_6	--	--	--	**2**	4	--	$\{v_1,v_2,v_3,v_6,v_4\}$

If $H^0 = \{v_1\}$, then $2\text{-}\tau^0 = \{V(G), \phi, \{v_1\}\}$

If $H^1 = \{v_1, v_2\}$, then $2\text{-}\tau^1 = \{V(G), \phi, \{v_1, v_2\}, \{v_2\}\}$

If $H^2 = \{v_1, v_2, v_3\}$, then $2\text{-}\tau^2 = \{V(G),\phi,\{v_1,v_2\},\{v_2,v_3\},\{v_3\},\{v_1,v_2,v_3\}\}$

If $H^3 = \{v_1, v_2, v_3, v_6\}$, then $2\text{-}\tau^3 = \{V(G),\phi,\{v_1, v_2\}, \{v_2, v_3\},\{v_3, v_6\}, \{v_6\}, \{v_1,v_2 ,v_3\}, \{v_1, v_2, v_3, v_6\}, \{v_2, v_3, v_6\}, \{v_1, v_2, v_6\}\}$

If $H^4 =\{v_1,v_2,v_3,v_6,v_4\}$, then $2\text{-}\tau^4 =\{V(G),\phi,\{v_1,v_2\},\{v_2,v_3\},\{v_3,v_6\},\{v_6,v_4\}, \{v_4\},\{v_1, v_2,v_3\},\{v_1,v_2,v_3,v_6\},\{v_1,v_2,v_6,v_4\},\{v_1,v_2,v_4\}, \{v_2,v_3,v_6\}, \{v_2,v_3, v_6,v_4\},\{v_2,v_3, v_6\},\{v_3,v_6,v_4\}, \{v_1,v_2,v_3,v_6,v_4\},\{v_1,v_2,v_3,v_4\}\}$.

Remark 2.1. From above example we have
$1\text{-}\tau^0 \subseteq 1\text{-}\tau^1 \subseteq \ldots \subseteq 1\text{-}\tau^e$

Proposition 2.1. Let G = (V(G), E(G)) be a connected graph, and $H^i = (V(H^i), E(H^i))$, $H^{i+1} = (V(H^{i+1}), E(H^{i+1}))$ are subgraphs in G represent the steps i, i+1in the shortest path problem such that $H^i \subseteq H^{i+1}$, then $1\text{-}\tau^i \subseteq 1\text{-}\tau^{i+1}$.
Proof. Clear.

Definition 2.3. An 2-pre-topology $2\text{-}\tau^1$ is called 2-pre-subtopology of a 2-pre-topology $2\text{-}\tau^2$ if for each 2-open subgraph $O_1 \in 2\text{-}\tau^2$, there exists 2-open subgraph $O \in 2\text{-}\tau^1$ such that $O \subseteq O_1$. and denoted by $2\text{-}\tau^1 \subseteq_s 2\text{-}\tau^2$.

Remark 2.2. From above definition and example 2.2 we have
$2\text{-}\tau^0 \subseteq_s 2\text{-}\tau^1 \subseteq_s \ldots \subseteq_s 2\text{-}\tau^e$.
Proposition 2.2. Let G=(V(G), E(G)) be a connected graph, and $H^i=(V(H^i), E(H^i))$, $H^{i+1} = (V(H^{i+1}), E(H^{i+1}))$ are subgraphs in G represent the steps i, i+1in the shortest path problem such that $H^i \subseteq H^{i+1}$, then $1\text{-}\tau^i \subseteq_s 1\text{-}\tau^{i+1}$.
Proof. Clear.

Definition 2.5. Let G = (V(G), E(G)) be a connected graph and $H^i = (V(H^i), E(H^i))$ be a subgraph represented the shortest path problem in step i. The 1-dependency (resp. 2-dependency) of a vertices of H^i on G with respect to 1-pre-topology (resp. 2-pre-topology) is denoted by 1-$\mathfrak{D}^i(V(H^i))$ (resp. 2-$\mathfrak{D}^i(V(H^i))$) and defined as follows:

$$1\text{-}\mathfrak{D}^i(V(H^i)) = \frac{|1-\mathrm{Int}(V(H^i))|}{|V(G)|} \quad (\text{resp. } 2\text{-}\mathfrak{D}^i(V(H^i)) = \frac{|2-\mathrm{Int}(V(H^i))|}{|V(G)|}\ )$$

Example 2.5. From examples 2.1 and 2.2, we have:
1-$\mathfrak{D}^0(V(H^0))$ = 0/6 = 0, 1-$\mathfrak{D}^1(V(H^1))$ = 2/6 = 1/3, 1-$\mathfrak{D}^2(V(H^2))$ = 3/6 =1/2, 1-$\mathfrak{D}^3(V(H^3))$ = 4/6 = 2/3, 1-$\mathfrak{D}^4(V(H^4))$ = 5/6.
2-$\mathfrak{D}^0(V(H^0))$ = 1/6, 2-$\mathfrak{D}^1(V(H^1))$ = 2/6, 2-$\mathfrak{D}^2(V(H^2))$ = 3/6 = 1/2, 2-$\mathfrak{D}^3(V(H^3))$ = 4/6 = 2/3, 2-$\mathfrak{D}^4(V(H^4))$ = 5/6.

Remark 2.3.
(a) 1-$\mathfrak{D}^0(V(H^0)) \leq$ 1-$\mathfrak{D}^1(V(H^1)) \leq \ldots \leq$ 1-$\mathfrak{D}^e(V(H^e))$
(b) 2-$\mathfrak{D}^0(V(H^0)) \leq$ 2-$\mathfrak{D}^1(V(H^1)) \leq \ldots \leq$ 2-$\mathfrak{D}^e(V(H^e))$

Proposition 2.3. Let G = (V(G), E(G)) be a connected graph, and H^i, H^{i+1} be a subgraphs in G represent of the shortest path problem in steps i, i+1, then
(a) 1-$\mathfrak{D}^i(V(H^i)) \leq$ 1-$\mathfrak{D}^{i+1}(V(H^{i+1}))$ for every i = 0, 1, … , e-1.
(b) 2-$\mathfrak{D}^i(V(H^i)) \leq$ 2-$\mathfrak{D}^{i+1}(V(H^{i+1})) \leq$ for every i = 0, 1, … , e-1.

Proof. The proofs of the two facts are similar; so, we will only proof the fact (a). Let H^i, H^{i+1} be a subgraphs in G represent the shortest path problem of steps i, i+1: i = 0, 1, … , e-1. Since $V(H^i) \subseteq V(H^{i+1})$ for every i = 0, 1, … , e-1, implies 1-Int$(V(H^i)) \subseteq$ 1-Int$(V(H^{i+1}))$, then |1-Int$(V(H^i))| \leq$ |1-Int$(V(H^{i+1}))$| and

$$\frac{|1-\mathrm{Int}(V(H^i))|}{|V(G)|} \leq \frac{|1-\mathrm{Int}(V(H^{i+1}))|}{|V(G)|}.$$

Hence 1-$\mathfrak{D}^i(V(H^i)) \leq$ 1-$\mathfrak{D}^{i+1}(V(H^{i+1}))$ for every i = 0, 1, … , e-1.

Proposition 2.4. Let G = (V(G), E(G)) be a connected graph, and H^e be a subgraph in G represent of the shortest path problem in end step such that $V(H^e)$ = V(G), then j-$\mathfrak{D}^e(V(H^e))$ = 1.

Proof. Clear.

Example 2.6. In examples 2.1 and 2.2. If we take the shortest path, P = $v_1v_2v_3v_6v_4$, then.
1-$\mathfrak{D}^0(V(P))$ = 0/6 = 0, 1-$\mathfrak{D}^1(V(P))$ = 2/6 = 1/3, 1-$\mathfrak{D}^2(V(P))$ = 3/6 =1/2, 1-$\mathfrak{D}^3(V(P))$ = 4/6 = 2/3, 1-$\mathfrak{D}^4(V(P))$ = 5/6.

2-$\mathfrak{D}^0$(V(P)) = 1/6, 2-$\mathfrak{D}^1$(V(P)) = 2/6, 2-$\mathfrak{D}^2$(V(P)) = 3/6 = 1/2, 2-$\mathfrak{D}^3$(V(P)) = 4/6 = 2/3, 2-$\mathfrak{D}^4$(V(P)) = 5/6.

Corollary 2.1. Let G = (V(G), E(G)) be a connected graph, and P be a shortest path of end step on G, then.

(a) 1-$\mathfrak{D}^0$(V(P)) ≤ 1-$\mathfrak{D}^1$(V(P)) ≤ … ≤ 1-$\mathfrak{D}^e$(V(P)).

(b) 2-$\mathfrak{D}^0$(V(P)) ≤ 2-$\mathfrak{D}^1$(V(P)) ≤ … ≤ 2-$\mathfrak{D}^e$(V(P)).

Example 2.7. In examples 2.1 and 2.2. If the vertices of the shortest path (subgraphs) in step i are, V(H^1) = v_1v_2, V(H^2) = $v_2v_3v_6$, and V(H^1) ∪ V(H^2) = $v_1v_2v_3v_6$, then.

1-$\mathfrak{D}^3$(V(H^1)) = 2/6 = 1/3, 1-$\mathfrak{D}^3$(V(H^2)) = 3/6 = 2/3, 1-$\mathfrak{D}^3$(V(H^1) ∪ V(H^2)) = 4/6 =2/3, and

2-$\mathfrak{D}^3$(V(H^1)) = 2/6 = 1/3, 2-$\mathfrak{D}^3$(V(H^2)) = 3/6 = 2/3, 2-$\mathfrak{D}^3$(V(H^1) ∪ V(H^2)) = 4/6 =2/3,

Notes that, j-$\mathfrak{D}^3$(V(H^1) ∪ V(H^2)) ≤ j-$\mathfrak{D}^3$(V(H^1)) + j-$\mathfrak{D}^3$(V(H^2)) ; j = 1, 2.

Also,

If the vertices of the shortest path (subgraphs) in step i are, V(H^1) = $v_1v_2v_3$, V(H^2) = $v_2v_3v_6$, and V(H^1) ∩ V(H^2) = v_2v_3, then.

1-$\mathfrak{D}^3$(V(H^1)) = 3/6 = 1/3, 1-$\mathfrak{D}^3$(V(H^2)) = 3/6 = 2/3, 1-$\mathfrak{D}^3$(V(H^1) ∩ V(H^2)) = 2/6 =1/3, and

2-$\mathfrak{D}^3$(V(H^1)) = 3/6 = 1/2, 2-$\mathfrak{D}^3$(V(H^2)) = 3/6 = 2/3, 2-$\mathfrak{D}^3$(V(H^1) ∩ V(H^2)) = 2/6 =1/3,

Notes that, j-$\mathfrak{D}^3$(V(H^1) ∩ V(H^2)) ≤ j-$\mathfrak{D}^3$(V(H^1)) + j-$\mathfrak{D}^3$(V(H^2)) ; j = 1, 2.

Remark 2.3. Let G = (V(G), E(G)) be a connected graph, H^h, H^k be two subgraphs represented the shortest path problem of steps h, k respectively, then.

(a) j-$\mathfrak{D}^i$(V(H^h) ∪ V(H^k)) ≤ j-$\mathfrak{D}^i$(V(H^h)) + j-$\mathfrak{D}^i$(V(H^k)) ; j = 1, 2 , i = 0, 1, ... , e-1.

(b) j-$\mathfrak{D}^i$(V(H^h) ∩ V(H^k)) ≤ j-$\mathfrak{D}^i$(V(H^h)) + j-$\mathfrak{D}^i$(V(H^k)) ; j = 1, 2 , i = 0, 1, ... , e-1.

If we take two paths has the same initial vertex and the other vertices of first path inside the step i and the other vertices of second path outside step i, to illustrate this idea take the following example:

Example 2.8. In examples 2.1 and 2.2.

In particular take 1-τ^3 and paths P = v_1v_2, P* = $v_1v_6v_2$, then.

1-$\mathfrak{D}^1$(V(P)) = 2/6 = 1/3, and 1-$\mathfrak{D}^1$(V(P*)) = 2/6 = 1/3.

Also. In 1-τ^2 and paths P = $v_1v_2v_3$, P* = $v_1v_6v_3$, then.

$1\text{-}\mathfrak{D}^2(V(P)) = 3/6 = 1/2$, and $1\text{-}\mathfrak{D}^2(V(P^*)) = 0$.

Also. In $1\text{-}\tau^3$ and paths $P = v_1v_2v_3v_6$, $P^* = v_1v_6$ or $v_1v_5v_6$, then.

$1\text{-}\mathfrak{D}^3(V(P)) = 4/6 = 2/3$, and $1\text{-}\mathfrak{D}^3(V(P^*)) = 0$.

Also. In $1\text{-}\tau^4$ and paths $P = v_1v_2v_3v_6v_4$, $P^* = v_1v_5v_6$, then.

$1\text{-}\mathfrak{D}^4(V(P)) = 5/6 = 1/3$, and $1\text{-}\mathfrak{D}^4(V(P^*)) = 0$.

Remark 2.4. If we take two paths has the same initial vertex and the other vertices of first path P inside the step i and the other vertices of second path P* outside step i, we have $j\text{-}\mathfrak{D}^i(V(P^*)) \leq j\text{-}\mathfrak{D}^i(V(P))$; $j = 1, 2$, $i = 0, 1, \ldots , e-1$.

3 Restriction and Separators in pre-Topology Generated by The Short Path Problems

We will denote by τ for $1\text{-}\tau$ or $2\text{-}\tau$, and τ^e to a pre-topology on H^e.

Definition 3.1. (Restricting). Let τ^e be j-pre-topology on H^e. Restricting τ^e on H^i means considering a j-pre-topology on H^i which is naturally related to τ^e, and it denoted by $\tau^e|H^i$. By definition, $\tau^e|H^i$ consists of sets of the form $U \cap H^i$, where $U \in \tau^e$. In other words, open sets in H^i are the traces on Z of open sets in G. It is clear that $\tau^e|H^i$ is an pre-topology.

Definition 3.2. (Separators). A subgraph $H^i \subseteq H^e$ is a separators for τ^e if $U \cap H^i$ is open subgraph for every open subgraph U; that is , $U \in \tau^e$ implies $U \cap H^i \in \tau^e$. An equivalent formulation of the condition is $\tau^e|H^i \subseteq \tau^e$.

Example 3.1.

(a) The empty subgraph and H^e are trivial separators.

(b) An open singleton subgraph is a separator.

(c) The maximal open subgraph $\max(\tau^e)$ is a separator.

(d) If H^i does not intersect $\max(\tau^e)$, then H^i is a separator.

(e) If τ^e is a topology, the set of separators for τ^e coincides with τ^e.

Lemma 3.1. Let τ^e be a pre-topology on H^e.

(a) Suppose that $H^i \subseteq H^e$ is a separator for τ^e. Then, for any $H^k \subseteq H^e$, the subgraph $H^i \cap H^k$ is a separator for $\tau^e|_{H^i}$.

(b) Suppose that $H^i \subseteq H^e$ is a separator for τ^e and $H^k \subseteq H^e$ is a separator for $\tau^e|_{H^i}$. Then H^k is a separator for τ^e.

Proof. (a) Let O be an open subgraph in H^k (with respect to $\tau^e|_{H^k}$). We have to check that $O \cap (H^i \cap H^k)$ is in $\tau^e|_{H^k}$. By definition, $O = U \cap H^k$ for some $U \in \tau^e$. Therefore $O \cap (H^i \cap H^k) = (U \cap H^i) \cap H^k$. Since H^k is a separator for τ^e, the set $U \cap H^i$ is in τ^e. Hence $(O \cap H^i) \cap H^k$ is in $\tau^e|_{H^k}$.

(b) It holds that $\tau^e|_{H^k} = (\tau^e|_{H^i})|_{H^k} \subseteq \tau^e|_{H^i} \subseteq \tau^e$.

The following proposition tell us that the set of separators always is a topology.

Proposition 3.1.

(a) Intersection of a finite family of separators is a separator.

(b) Union of any family of separators is a separator.

Proof. (a) It suffices to consider the case of two separators subgraphs H^1 and H^2. Let $U \in \tau^e$. Since H^1 is a separator, $U \cap H^1 \in \tau^e$. Since H^2 is a separator, $U \cap (H^1 \cap H^2) = (U \cap H^1) \cap H^2 \in \tau^e$.

(b) Let $\{H^i\}$ be a family of separators, and $H = \cup_i H^i$. Let $U \in \tau^e$. Since H^i is a separators, $U \cap H^i \in \tau^e$. As the union of open subgraphs, $U \cap (\cup_i H^i) = \cup_i (U \cap H^i)$ is an open subgraph.

The set of $Sep(\tau^e)$ of separators for τ thus is a topology on H^e. In general, this topology is incompatible with τ^e. (though, if the subgraph H^e is open, each separator is open and $Sep(\tau^e) \subseteq \tau^e$). Separators allow decomposing τ^e in to finer parts. An appreciation for the decomposing issue can be gained by considering the following simple case; later on we shall discuss a more general setting.

Decomposition: Suppose that τ^e is a pre-topology on H^e, $H^i \subseteq H^e$, and $Y = H^e \setminus H^i$. In general, the restriction $\tau^e|_{H^i}$ and $\tau^e|_Y$ do not determine the pre-topology τ^e. To reconstruct τ^e we consider, for every open subgraph $O \in \tau^e|_{H^i}$, $\tau^O = \{ U \subseteq Y : O \cup U \in \tau^e\}$, in P(Y). (Note that $\tau^O \subseteq \tau^e|_Y$).

Lemma 3.2. Let H^i be a separators for τ^e. Then:

(a) For every $O \in \tau^e|_{H^i}$, τ^O is a pre-topology on H^i.

(b) τ^O is isotonic in $O \in \tau^e|_{H^i}$.

Proof. (a) It holds that $\phi \in \tau^O$, that is, $O = O \cup \phi$ is open with respect to τ^e. We now use the fact that H^i is a separator for τ^e; since $O \in \tau^e|_{H^i}$, we have $O \in \tau^e$. The proof for τ^O being stable with respect to arbitrary union is straightforward (and does not require H^i to be a separator).

(b) Suppose $U \in \tau^O$ and $O \subseteq W \in \tau^e|_{H^i}$. We have to show that $U \in \tau^W$. Since $U \in \tau^O$, $O \cup U$ is open subgraph in H^e. Since $W \in \tau^e|_{H^i}$ and H^i is a separator for τ^e, W is open subgraph in H^e as well. Hence, their union $(O \cup U) \cup W = (O \cup W) \cup U = W \cup U$ is open subgraph in H^e, and $U \in \tau^W$.

These data (the pre-topology $\tau^e|_{H^i}$ on H^i and the isotone family (τ^O, $O \in \tau^e|_{H^i}$) of pre-topologies on $H^e \setminus H^i$) allow us to restore the initial pre-topology τ^e. Namely $\tau^e = \{O \cup U \subseteq Y$, where $O \in \tau^e|_{H^i}$ and $U \in \tau^O\}$.

Indeed, $O \cup U$ belongs to τ^e (by definition of τ^O). Conversely, if $W \in \tau^e$, then $W = (W \cap H^i) \cup (W \setminus H^i)$, $W \cap H^i \in \tau^e|_{H^i}$, and $W \setminus H^i \in \tau^{W \cap H^i}$.

Synthesis: Conversely, suppose we have the following data:

(a) A pre-topology τ^i on H^i, and

(b) For every $O \in \tau^i$, a pre-topology τ^O on $H^e \setminus H^i$ such that the correspondence $O \rightarrow \tau^O$ is isotone.

Give these data we define the following collection τ^e of subgraph of H^e,

$\tau^e = \{O \cup U$, where $O \in \tau^i$ and $U \in \tau^O\}$.

Proposition 3.2.

(a) τ^e is a pre-topology on H^e,

(b) It holds that $\tau^e|_{H^i} = \tau^i$,

(c) H^i is a separator for τ^e,

(d) For every $O \in \tau^e|_{H^i}$, $\tau^O = \{U \in H^e \setminus H^i$, such that $O \cup U \in \tau^e\}$.

Proof. (a) Let $\{O_i \cup U_i\}$ be a family of subgraph in τ^e. Since τ^i is stable with respect to union, $O = \cup_i O_i$ is in τ^i. Even U_i is in τ^{Oi}; since $O_i \subseteq O$, $\tau^{Oi} \subseteq \tau^O$ and $U_i \in \tau^O$. As τ^O is stable with respect to unions, $\cup_i U_i \in \tau^O$, and consequently, $O \cup (\cup_i U_i) \in \tau^e$.

(b) The inclusion $\tau^e|_{H^i} \subseteq \tau^i$ is obvious; the inverse inclusion follows from $\phi \in \tau^O$. This prove (b), and the inclusion $\tau^i \subseteq \tau^e$ implies (c).

(d) This statement is obvious.

This assembly construction thus is the reverse to the previously considered decomposition. One can say that τ^e is the semi-direct sum of the pre-topology τ^e and the family $\{\tau^O : O \in \tau^i\}$. If both H^i and $Y = H^e \setminus H^i$ are separators, τ^e is the direct sum of the pre-topology $\tau^e|_{H^i}$ and $\tau^e|_Y$.

Antimatroids are special pre-topologies. In terms of closed sets (i.e., complements of open sets) they are known as the so-called convex geometries; see Theorem 1.3 in chapter 3 of Korte, Lovasz, and Schrader (1991) [5]. The following definition more appropriate for the purposes of this note as it directly refers to open subgraphs.

Definition 3.3. An Antimatroid on a subgraph H^e is a pre-topology τ^e on H^e possessing the following property: for every non-empty open subgraph $U \in \tau^e$, a vertex $v \in U$ exists such that the subgraph $U \setminus \{v\}$ is open as well.

In the following we will write $U \setminus v$ instead of $U \setminus \{v\}$. Antimatroids possess a property formally stronger than the property given in the definition.

Lemma 3.3. Let τ^e be an antimatroids on H^e, and let $O \underset{\neq}{\subset} U$ be two distinct open subgraphs. Then a vertex $v \in U \setminus O$ exists such that $U \setminus v$ is open.

Proof. We shall use an induction on the size of the subgraph $U \setminus O$. If $U \setminus O$ consists of a single vertex, the assertion of the lemma is obviously true. For the general case, let v be an arbitrary vertex of $U \setminus O$. Let $U_v \subseteq U$ be a minimal open subgraph containing the vertex v.

Note that the subgraph $U_v \setminus v$ is open. This is obvious if $U_v = \{v\}$. Otherwise, there exists $u \in U_v$ such that $U_v \setminus u$ is open subgraph. If $u \neq v$, then $U_v \setminus u$ is a proper subgraph of U_v containing the vertex v, This contradicts the minimality of U_v. Hence $u = v$.

Consider now two cases. If $U_v \cup O = U$, then $U \setminus v = (U_v \setminus v) \cup O$, as the union of the open subgraphs $U_v \setminus v$ and O, is open subgraph. If $O^* = U_v \cup O$ is strictly smaller than U, we obtain the pair of open subgraphs $O^* \subseteq U$ with strictly smaller difference $U \setminus O^*$. By the induction assumption, there exists a vertex $v^* \in U \setminus O^*$ such that $U \setminus v^*$ is open subgraph.

4 Rate of Liaison in pre-topology between Two Subgraphs

In this section, we will study the rate of liaison in pre-topology between two subgraphs. It is formally shown that the new distance measure is a metric

Definition 4.1. Let G be a non-empty graph, and H^1, H^2 are two non-empty subgraphs of G, the rate of liaison between H^1, H^2 in G is denote by $\Re(H^1, H^2)$ and defined as:

$$\Re(H^1, H^2) = \frac{|H^1 \cap H^2|}{\max(|H^1|, |H^2|)}$$

Where | G | denote the number of vertices of G.

Example 4.1. From example 2.2. we have
$\Re(H^1, H^1) = 2/2 = 1$, $\Re(H^1, H^2) = 2/3$, $\Re(H^1, H^3) = 2/4 = 1/2$, $\Re(H^1, H^4) = 2/5$,
$\Re(H^2, H^2) = 3/3 = 1$, $\Re(H^2, H^3) = 3/4$, $\Re(H^2, H^4) = 3/5$,
$\Re(H^3, H^3) = 4/4 = 1$, $\Re(H^3, H^4) = 4/5$,
$\Re(H^4, H^4) = 5/5 = 1$.

Remark 4.1.
(a) $\Re(H^i, H^i) = 1$ for all subgraph H^i of G.
(b) $\Re(H^i, H^j) = 0$ if $H^i \cap H^j = \phi$.
(c) $\Re(H^i, H^k) \leq \Re(H^1, H^j)$ for all $H^i \subseteq H^j \subseteq H^k$.
(d) $\Re(H^i, H^k) \leq \Re(H^j, H^k)$ for all $H^i \subseteq H^j \subseteq H^k$.

Theorem 4.1. Let H^1, H^2, and H^3 be any graphs. The following properties holds true.
(a) $0 \leq \Re(H^1, H^2) \leq 1$,
(b) $\Re(H^1, H^2) = 1$ if and only if $H^1$and H^2 are isomorphic to each other,
(c) $\Re(H^1, H^2) = \Re(H^2, H^1)$,
(d) $\Re(H^1, H^3) \leq \Re(H^1, H^2) + \Re(H^2, H^3)$.

Proof. (a) Since $H^1 \cap H^2 \subseteq H^1$ and $H^1 \cap H^2 \subseteq H^2$, then $|H^1 \cap H^2| \leq |H^1|$ and $|H^1 \cap H^2| \leq |H^2|$, so $|H^1 \cap H^2| \leq \max(|H^1|, |H^2|)$, and

$$0 \leq \frac{|H^1 \cap H^2|}{\max(|H^1|, |H^2|)} \leq 1. \text{ Hence } 0 \leq \Re(H^1, H^2) \leq 1.$$

(b) $\Re(H^1, H^2) = 1$ if and only if $\frac{|H^1 \cap H^2|}{\max(|H^1|, |H^2|)} = 1$

if and only if $|H^1 \cap H^2| = \max(|H^1|, |H^2|)$
if and only if $|H^1| = |H^2|$
if and only if H^1 and H^2 are isomorphic to each other.

(c) $\Re(H^1, H^2) = \dfrac{|H^1 \cap H^2|}{\max(|H^1|, |H^2|)} = \dfrac{|H^2 \cap H^1|}{\max(|H^2|, |H^1|)} = \Re(H^2, H^1)$.

(d) In the following proof of the triangle we distinguish two case:

Case 1. The graphs $H^1 \cap H^2$ and $H^2 \cap H^3$ are disjoint, or speaking more strictly, the Intersection of $H^1 \cap H^2$ and $H^2 \cap H^3$ is empty. (Figer.1)

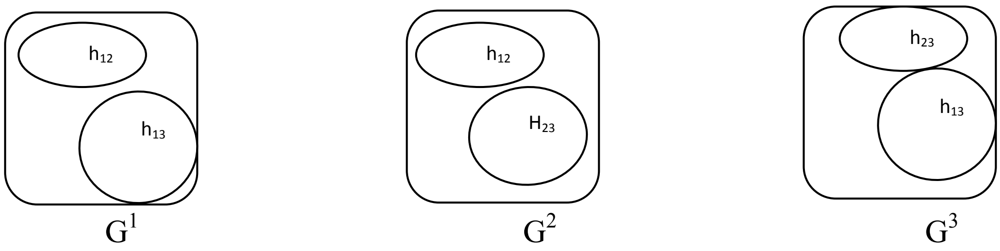


Figure 4.1. Illustration of disjoint and overlapping intersection subgraphs: the intersection subgraphs $H^1 \cap H^2 = h_{12}$, $H^1 \cap H^3 = h_{13}$, $H^2 \cap H^3 = h_{23}$ are disjoint.

Let $I_{12} = |H^1 \cap H^2|$, $I_{23} = |H^2 \cap H^3|$, and $I_{13} = |H^1 \cap H^3|$. Then the following relation holds true.

$I_{12} + I_{23} \le |H^2|$. -------- (1)

Property (d) is equivalent to the following inequality:

$$\frac{I_{12}}{\max(|H^1|, |H^2|)} + \frac{I_{23}}{\max(|H^2|, |H^3|)} \ge \frac{I_{13}}{\max(|H^1|, |H^3|)} \text{ -------- (2)}$$

We will show that the left-hand side of this inequality is always smaller or equal to (1). Which is equivalent to:

$\max(|H^1|, |H^2|) - \max(|H^2|, |H^3|) \ge I_{12} \max(|H^2|, |H^3|) + I_{23} \max(|H^1|, |H^2|)$ ---(3)

We proceed by a simple case analysis.

Case 1.a: $|H^1| \ge |H^2| \ge |H^3|$. Here Eq.(3) is equivalent to

$|H^1||H^2| \ge I_{12}|H^2| + I_{23}|H^1|$ -------- (4)

From Eq.(1) we conclude that

$|H^1||H^2| \ge I_{12}|H^1| + I_{23}|H^1| \ge I_{12}|H^2| + I_{23}|H^1|$.

Case 1.b: $|H^1| \ge |H^3| \ge |H^2|$. Here Eq.(3) become

$|H^1||H^3| \ge I_{12}|H^3| + I_{23}|H^1|$ -------- (5)

Using Eq.(1) again we conclude

$|H^1||H^3| \ge |H^1||H^2| \ge I_{12}|H^1| + I_{23}|H^1| \ge I_{12}|H^3| + I_{23}|H^1|$.

The remaining four cases $|H^2| \ge |H^1| \ge |H^3|$, $|H^2| \ge |H^3| \ge |H^1|$, $|H^3| \ge |H^1| \ge |H^2|$, and $|H^3| \ge |H^2| \ge |H^1|$ can be shown similarly.

Case 2. Here we assume that the intersection of $H^1 \cap H^2$ and $H^2 \cap H^3$ is not empty (Figer.2)

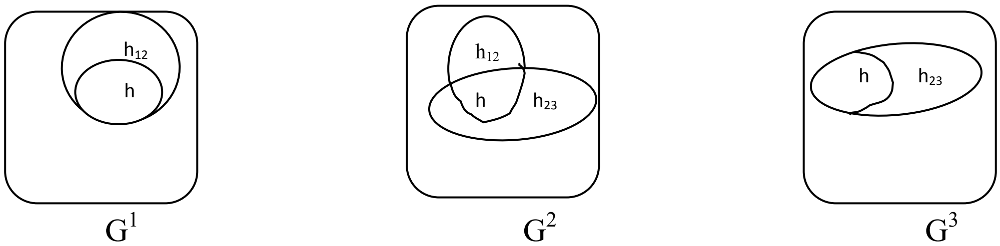


Figure 4.2. Illustration of disjoint and overlapping intersection subgraphs: $H^1 \cap H^2$ and $H^2 \cap H^3$ share an intersection subgraph h i.e., $h = (H^1 \cap H^2) \cap (H^2 \cap H^3)$.

Let $I = |(H^1 \cap H^2) \cap (H^2 \cap H^3)| > 0$. It follows that $|H^1 \cap H^3| \ge I$.

Furthermore it follows that

$I_{12} + I_{23} - I \le |H^2|,\ I \le I_{12},\ I \le I_{23}$ -------- (6)

We will show that

$$\frac{I_{12}}{\max(|H^1|,|H^2|)} + \frac{I_{23}}{\max(|H^2|,|H^3|)} \ge \frac{I}{\max(|H^1|,|H^3|)} \text{ -------- (7)}$$

Which implies property (d). Obviously inequality (7) is equivalent to

$I_{12} \max(|H^2|, |H^3|) \max(|H^1|, |H^3|) + I_{23} \max(|H^1|, |H^2|) \max(|H^1|, |H^3|)$

$\ge I \max(|H^1|, |H^2|) \max(|H^2|, |H^3|)$ -------- (8)

Again we proceed by case analysis.

Case 2.a: $|H^1| \ge |H^2| \ge |H^3|$. Here Eq.(8) is equivalent to

$I_{12} |H^2| |H^1| + I_{23} |H^1| |H^1| \ge I |H^1| |H^2|$

Which can be simplified to

$I_{12} |H^2| + I_{23} |H^1| \ge I |H^2|$ -------- (9)

From Eq.(6) if follows that

$I_{12} + I_{23} \ge 2 I \ge I$

$I_{12} |H^2| + I_{23} |H^1| \ge I_{12} |H^2| + I_{23} |H^2| \ge I |H^2|$

From which we get Eq.(9)

Case 2.b: $|H^1| \ge |H^3| \ge |H^2|$. Here Eq.(8) become.

$I_{12} |H^3| |H^1| + I_{23} |H^1| |H^1| \ge I |H^1| |H^3|$

Which can be simplified to

$I_{12} |H^3| + I_{23} |H^1| \ge I |H^3|$ -------- (10)

We proceed by analogously to case 2.a.

$I_{12} |H^3| + I_{23} |H^1| \ge I_{12} |H^3| + I_{23} |H^3| \ge I |H^3|$ -------- (11)

The remaining cases can be shown similarly.

Remark 4.1. From theorem above it follows in particular that our proposed rate of liaison is a metric.

5 Discussion and conclusion

We have shown that the graph distance measure of Definition 4.1 is in fact a metric. As discussed earlier it is often difficult to form a metric from edit distance measures. Therefore in applications where the properties of a metric are important, the intersection subgraph metric could be used.

One application where this is important is information retrieval from images and video databases Chang et al., 1987 [2], Lee and Hsu, 1992[6], Shearer et al., 1997 [7]. This area relies heavily on browsing to locate required database elements. Thus it is necessary for the distance measure chosen to be ‘‘well behaved’’ to allow sensible navigation of the database. The use of a metric, such as that proposed, for the distance measure ensures that the behavior of the similarity retrieval will be consistent and comprehensible, aiding the user in their search task.

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Received: October, 2011