On Semi-Essential Submodules

Muna A. Ahmed
math.200600986@yahoo.com
Maysaa R. Abbas
maysaa.alsaher@yahoo.com
Department of Mathematics, college of Science for Women,University of Baghdad, Iraq-Baghdad.

Received in : 20 October 2014 , Accepted in :5 January 2015

Abstract

Let R be a commutative ring with identity and let M be a unitary left R-module. The purpose of this paper is to investigate some new results (up to our knowledge) on the concept of semi-essential submodules which introduced by Ali S. Mijbass and Nada K. Abdullah, and we make simple changes to the definition relate with the zero submodule, so we say that a submodule N of an R-module M is called semi-essential, if whenever $N \cap P = (0)$, then $P = (0)$ for each prime submodule P of M. Various properties of semi-essential submodules are considered.

Keywords: Essential submodules, Semi-essential submodules, Uniform modules, Semi-uniform modules, Fully prime modules and Fully essential modules.

his paper is a part of a thesis submitted by the second author and supervised by the first

1. Introduction

Throughout this paper, R represents a commutative ring with identity and M is a unitary left R-module. Assume that all R-modules under study contain prime submodules. It is well known that a submodule N of M is called essential, if whenever $N \cap L = (0)$, then $L = (0)$ for each submodule L of M [7] and [9].

Ali and Nada in [1] introduced the concept of semi-essential submodules as a generalization of the class of essential submodules, where they say that a nonzero submodule N of M is called semi-essential, if $N \cap P \neq (0)$ for each nonzero prime R- submodule P of M [1], where a submodule P of M is called prime, if whenever $rm \in P$ for $r \in R$ and $m \in M$, then either $m \in P$ or $r \in (P \dot{:}_R M)$ [12]. In this paper we rewrite the definition of the semi-essential submodules which introduced [1] in another formula, in fact we didn't find any reasonable reason to exclude the zero submodule from the definition of semi-essential submodules. Also we give some new results (up to our knowledge) about this concept, and illustrate that by some remarks and examples. We start by the formula of the definition of the semi-essential submodules.

Definition (1.1): A submodule N of an R-module M is called semi-essential if whenever $N \cap P = (0)$, then $P = (0)$ for every prime submodule P of M.

We see it is necessary to put some simple remarks about the class of semi-essential submodules which not mentioned in [1].

Remarks (1.2):

1. Consider the Z-module $M = Z_8 \oplus Z_2$. In this module there are eleven submodules which are $<(\bar{0}, \bar{0})>$, $<(\bar{1}, \bar{0})>$, $<(\bar{0}, \bar{1})>$, $<(\bar{1}, \bar{1})>$, $<(\bar{2}, \bar{0})>$, $<(\bar{2}, \bar{1})>$, $<(\bar{4}, \bar{0})>$, $<(\bar{4}, \bar{1})>$, $<(\bar{0}, \bar{1}), (\bar{4}, \bar{0})>$, $<(\bar{2}, \bar{0}), (\bar{4}, \bar{1})>$, and M. The semi-essential submodules of M are $<(\bar{1}, \bar{1})>$, $<(\bar{1}, \bar{0})>$, $<(\bar{2}, \bar{0})>$, $<(\bar{2}, \bar{1})>$, $<(\bar{4}, \bar{0})>$, $<(\bar{0}, \bar{1}), (\bar{4}, \bar{0})>$, $<(\bar{2}, \bar{0}), (\bar{4}, \bar{1})>$ and M. In fact each one of them intersects with each nonzero prime submodule of M is nonzero, where the prime submodules of M are $<(\bar{2}, \bar{0}), (\bar{4}, \bar{1})>$, $<(\bar{1}, \bar{1})>$, $<(\bar{1}, \bar{0})>$, and $<(\bar{2}, \bar{0})>$.
2. When a submodule N of an R-module M is nonzero in the Def (1.1), then N is a semi-essential submodule if $N \cap P \neq (0)$ for each prime submodule P of M, and this is the same definition which is said by Ali and Nada in [1].
3. Every module is a semi-essential submodule of itself.
4. For the concept of the essential submodules, (0) is an essential submodule of an R-module M if and only if $M = (0)$, but (0) may be semi-essential submodule in a nonzero module. In fact $(0) \leq_{sem} M$ if and only if M has only one prime submodule which is (0), for example $(\bar{0})$ is a semi-essential submodule of the Z-module, Z_2, while (0) is not semi-essential submodule of Z.
5. The sum of two semi-essential submodules is also semi-essential submodule.

Proof (5): Let M be an R-module and let L and K be two essential submodules of M. Note that $L \leq L+K$, since $L \leq_{sem} M$, so by [1], $L+K \leq_{sem} M$.

6. Let M be an R-module, and let $N \leq M$. Then for each R-module M' and for each homomorphism $f: M \rightarrow M'$ with $\ker f \cap N \neq (0)$, implies that $N \leq_{sem} M$.

Proof (6): Let P be a nonzero prime submodule of M, and let π: $M \to \frac{M}{P}$ be the natural epimorphism. By assumption ker $\pi \cap N \neq (0)$. But ker π = P, then $P \cap N \neq (0)$, hence $N \leq_{sem}$ M.

Proposition (1.3): Let f: M → M' be an isomorphism. If $N \leq_{sem} M$, then $f(N) \leq_{sem} M'$.

Proof: Let P be a nonzero prime submodule of M'. Since f is an epimorphism, then $f^{-1}(P)$ is a prime submodule of M [12, Prop. 3.8, P.10]. But $N \leq_{sem} M$, then $N \cap f^{-1}(P) \neq (0)$, On the other hand f is a monomorphism thus $f(N) \cap P \neq (0)$, and we are done.

In [1], Ali and Nada gave an example verified, that the class of semi-essential submodules was didn't satisfy the transitive property for nonzero submodules. But in this work, we show that the example which they gave it in [1] is not true, and we prove that the class of semi-essential submodules satisfies the transitive property. In fact Ali and Nada said that $(\bar{4}) \leq_{sem} (\bar{2})$ and $(\bar{2}) \leq_{sem} Z_{12}$, but $(\bar{4}) \not\leq_{sem} Z_{12}$. In fact $(\bar{4}) \not\leq_{sem} (\bar{2})$ since $(\bar{6})$ is a prime submodule of $(\bar{2})$ and $(\bar{4}) \cap (\bar{6}) = (\bar{0})$. However, in the following proposition we give the proof of the transitive property for nonzero semi-essential submodules. Before that we need the following Lemma which appeared in [3, Prop (1.7), p.11].

Lemma (1.4): Let C be an R-module, if P is a prime submodule of C and B is a submodule of C, such that $B \not\leq P$, then $P \cap B$ is a prime submodule in B.

Proposition (1.5): Let A, B, C be R-modules such that $A \leq B \leq C$. Suppose that A is a nonzero submodules of M. If $A \leq_{sem} B$ and $B \leq_{sem} C$ then $A \leq_{sem} C$.

Proof: Let P be a prime submodule of C such that $A \cap P = (0)$. Note that $(0) = A \cap P = (A \cap P) \cap B = A \cap (P \cap B)$. But P is a prime submodule of C, so we have two cases. If $B \leq P$ then $(0) = A \cap (P \cap B) = A \cap B$, hence $A \cap B = (0)$, but $A \leq B$, so $A \cap B = A$, which is implies that A = (0). But this is a contradiction with our assumption. Thus $B \not\leq P$, and by Lemma (1.4), $P \cap B$ is a prime submodule of B. But $A \leq_{sem} B$, therefore $P \cap B = (0)$, and since $B \leq_{sem} C$, then P = (0), that is $A \leq_{sem} C$.

Remark (1.6): The condition $A \neq (0)$ in Prop (1.5) is necessary. In fact in the Z-module Z_{12}, $(\bar{0})$ is a semi-essential submodule of $\{\bar{0}, \bar{6}\}$ and $\{\bar{0}, \bar{6}\}$ is a semi-essential submodule of Z_{12}, but $(\bar{0})$ not semi-essential in Z_{12}.

The converse of Prop (1.5) is not true in general, as the following example shows.

Example (1.7): Consider the Z-module, Z_{36}, the submodule $(\overline{18})$ is a semi-essential submodule of Z_{36}. But $(\overline{18})$ is not semi-essential submodule of $(\bar{2})$.

2. Other results on semi-essential submodules

In this section, we introduce other properties of semi-essential submodules. Recall that an R-module M is called fully prime, if every proper submodule of M is a prime submodule [5], and a nonzero R-module is called fully essential, if every nonzero semi-essential submodule of M is an essential submodule of M [11].

Tamadher in [8, Lemma 3.7], proved that if A and B are prime submodules of an R-module M and $A \leq B$, then A is a prime submodule in B. In fact B need not be necessary prime submodule in M. We use this statement to prove the following proposition, which is forming a generalization of the result which was given in [11, Lemma (1.4)].

Proposition (2.1): Let M be a fully prime R-module, and let $(0) \neq N \leq M$. Then $N \leq_{sem} L$ if and only if $N \leq_e L$ for every submodule L of M.

Proof ⇒): Assume that N is a semi essential submodule of L, and let A be a submodule of L such that $N \cap A = (0)$. Since M is a fully prime module then both of N and A are prime submodules of M, and by [8, Lemma 3.7] A is a prime submodule of L. But N is a semi-essential submodule of L, therefore A = (0), that is N is an essential submodule of L.

⇐): It is clear.

Corollary (2.2): Every fully prime module is a fully essential module.

Recall that a nonzero R-module M is called semi-uniform if every nonzero R-submodule of M is semi-essential. A ring R is called semi-uniform if R is a semi-uniform R-module, [1].

Proposition (2.3): Let M be an R-module. Then M is uniform if and only if M is semi-uniform and fully essential.

Proof ⇒): It is clear.

⇐): Let N be a nonzero submodule of M, since M is a semi-uniform module, then $N \leq_{sem} M$. But M is a fully essential module, then $N \leq_e M$.

Corollary (2.4): Let M be a fully prime module, then a module M is uniform if and if M is a semi-uniform module.

The following proposition appeared in [11], it deals with the direct sum of semi-essential submodules, and we give the proof for completeness.

Proposition (2.5): Let $M = M_1 \oplus M_2$ be a fully prime R-module where M_1 and M_2 are submodules of M, and let $(0) \neq K_1 \leq M_1$ and $(0) \neq K_2 \leq M_2$. Then $K_1 \oplus K_2$ is a semi-essential submodule of $M_1 \oplus M_2$ if and only if K_1 is a semi-essential submodule of M_1 and K_2 is a semi-essential submodule of M_2.

Proof ⇒): Since M is a fully prime module, then by [11, Lemma (1.14)] $K_1 \oplus K_2$ is an essential submodule of $M_1 \oplus M_2$, and by [2], K_1 is an essential submodule of M_1 and K_2 is an essential submodule of M_2. But every essential submodule is a semi-essential, so we are done.

⇐): It follows similarly.

In the following proposition, we give another result for the direct sum of semi-essential submodules.

Proposition (2.6): Let $M = M_1 \oplus M_2$ be an R-module where M_1 and M_2 are submodules of M, and let $K_1 \leq M_1$ and $K_2 \leq M_2$. If $K_1 \oplus K_2$ is a semi-essential submodule of $M_1 \oplus M_2$, then K_1 is a semi-essential submodule of M_1, provided that every prime submodule of M_1 is a prime submodule of M.

Proof: Let P_1 be a prime submodule of M_1 such that $K_1 \cap P_1 = (0)$. We can easily prove that $(K_1 \oplus K_2) \cap P_1 = (0)$. By assumption P_1 is a prime submodule of M and $K_1 \oplus K_2 \leq_{sem} M$, Thus $P_1 = (0)$.

Recall that the prime radical of an R-module M is denoted by rad(M), and it is the intersection of all prime modules of M [10].

Proposition (2.7): Let M be an R-module and let $(0) \neq N \leq M$. If N' is a semi relative complement of N in M, and $N' \leq rad(M)$, then $N \oplus N' \leq_{sem} M$.

Proof: Consider the natural epimorphism $\pi: M \to \frac{M}{N'}$. Since N' is a semi relative complement of N in M, so by [1], $\frac{N \oplus N'}{N'} \leq_{sem} \frac{M}{N'}$. But $\ker \pi = N'$ and $N' \leq rad(M)$, then by [1], $\pi^{-1}(\frac{N \oplus N'}{N'}) \leq_{sem} M$. Hence $N \oplus N' \leq_{sem} M$.

Ali and Nada in [1] showed by an example that the intersection of two semi-essential submodules need not be semi-essential submodule, and they satisfied that under certain condition, see [1]. In this work we give a deferent condition.

Proposition (2.8): Let M be an R-module and let N_1 and N_2 be semi-essential submodules of M such that $N_1 \cap N_2 \neq (0)$ and all prime submodules of N_1 are prime submodules of M, then $N_1 \cap N_2 \leq_{sem} M$.

Proof: Let P be a prime submodule of M such that $(N_1 \cap N_2) \cap P = (0)$. This implies that $N_2 \cap (N_1 \cap P) = (0)$. If $N_1 \leq P$, then we have a contradiction with the assumption, thus $N_1 \nleq P$. This implies that $N_1 \cap P$ is a prime submodule of N_1 [3, Prop (1-7, P. 11)]. Since $N_2 \leq_{sem} M$

and by our assumption $N_1 \cap P$ is a prime submodule of M, then $N_1 \cap P = (0)$. But $N_1 \leq_{sem} M$, therefore $P = (0)$, hence $N_1 \cap N_2 \leq_{sem} M$.

Note that the condition "all prime submodules of N_1 are prime submodules of M" which we used in Prop (2.8) can be applied also for N_2.

Proposition (2.9): Let M be an R-module, and let N_1 and N_2 are semi-essential submodules of M such that $N_2 \cap P$ is a prime submodules of M for all prime submodule P of M, then $N_1 \cap N_2 \leq_{sem} M$.

Proof: Let P be a prime submodule of M such that $(N_1 \cap N_2) \cap P = (0)$. This implies that $N_1 \cap (N_2 \cap P) = (0)$. But $N_2 \cap P$ is a prime submodule of M and since $N_1 \leq_{sem} M$, then $N_2 \cap P = (0)$. Moreover, since $N_2 \leq_{sem} M$, thus $P = (0)$, and hence $N_1 \cap N_2 \leq_{sem} M$.

Recall that, an R-module M is called multiplication, if for each submodule N of M, there exists an ideal I of R such that $N = IM$ [4].

Proposition (2.10): Let M be a faithful and multiplication module such that M satisfies the condition (*), and let I, J be ideals of R. If $IM \leq_{sem} JM$, then $I \leq_{sem} J$.

Condition (*): For any two ideals L and K of R, if L is a prime ideal of K, then LM is a prime submodule of KM.

Proof: Let P be a prime ideal of J such that $I \cap P = (0)$, then $(I \cap P)M = (0)M$. Since M is a faithful and multiplication, therefore $IM \cap PM = (0)$ [6, Th (1.7)]. By condition (*), PM is a prime submodule of JM. But $IM \leq_{sem} JM$, then $PM = (0)$. Since M is a faithful module so $P = (0)$, thus $I \leq_{sem} J$.

The converse of Prop (2.10) is true without using the condition (*), but we need other condition as the following proposition shows.

Proposition (2.11): Let M be a finitely generated, faithful and multiplication R-module. If $I \leq_{sem} J$ then $IM \leq_{sem} JM$ for every ideals I and J of R.

Proof: Let P be a prime submodule of JM such that $IM \cap P = (0)$. Since M is a multiplication module, then $P = EM$ for some prime ideal E of R [6, Cor (2.11)]. So $IM \cap EM = (0)$, this implies that $(I \cap E) M = (0)$. Since M is a faithful module, then $I \cap E = (0)$. Since $EM \leq JM$ and M is a finitely generated, faithful and multiplication module so by [6, Th (3.1)] $E \leq J$. But E is a prime ideal of R, then E is a prime ideal of J [8, Lemma 3.7]. Since I is a semi-essential ideal of J, then $E = (0)$, and hence $P = (0)$. That is $IM \leq_{sem} JM$.

From Prop (2.10) and Prop (2.11) we have the following theorem.

Theorem (2.12): Let M be a finitely generated, faithful and multiplication module such that M satisfies the condition (*). Then $I \leq_{sem} J$ if and only if $IM \leq_{sem} JM$ for every two ideals I and J of R.

It is well known that If a ring R has only one maximal ideal I, then I is an essential ideal of R if and only if $I \neq (0)$. In the following proposition we generalize this statement in one direction to essential (hence semi-essential) submodules.

Proposition (2.13): Let M be a nonzero multiplication R-module with only one maximal submodule N, if $N \neq (0)$. Then N is an essential (hence semi-essential) submodule of M.

Proof: Let P be a submodule of M with $P \cap N = (0)$. If $P = M$, then $M \cap N = (0)$, hence $N = (0)$ which is a contradiction. Thus P is a proper submodule of M, and since M is a nonzero multiplication module, so by [6, Th (2.5)], P contained in some maximal submodule of M. But M has only one maximal submodule which is N. Thus $P \subseteq N$, this implies that $P = (0)$, that is N is an essential (hence semi-essential) submodule of M.

Proposition (2.14): Let M be a finitely generated R-module with only one nonzero maximal submodule N, then N is an essential (hence semi-essential) submodule of M.

Proof: In similar way, and by using [13, Prop (1.6), P. 7] instead of [6, Th (2.5)].

We end this work by the following theorem which gives the hereditary of fully essential property between R-module, M and the ring R.

Theorem (2.15): Let M be a nonzero faithful and multiplication R-module. Then M is a fully essential module if and only if R is a fully essential ring.

Proof ⇒): Assume that M is a fully essential module, and let I be a nonzero semi-essential ideal of R, then IM is a submodule of M say N. This implies that N is a semi-essential submodule of M [1]. Since $I \neq (0)$ and M is a faithful module, then $N \neq (0)$. But M is a fully essential module, thus N is an essential submodule of M. Since M is a faithful and multiplication module, therefore I is an essential ideal of R [6, Th (2.13)], that is R is a fully essential ring.

⇐): Suppose that R is a fully essential ring and let $(0) \neq N \leq_{sem} M$. Since M is a multiplication module, then N = IM for some semi-essential ideal I of R. By assumption I is an essential ideal of R. But M is a faithful and multiplication module then N is an essential submodule of M [6, Th (2.13)]. That is M is a fully essential module.

References

[1] Ali. S. Mijbass and Nada. K. Abdullah, (2009), Semi-essential submodule and semi-uniform modules. J. of Kirkuk University-Scientific studies; 4 (1), 48-58.

[2] Anderson, F.W. and Fuller, K.R, (1992), Rings and categories of modules, Springer-Verlag, New York., Academic Press Inc. London.

[3] Athab, E. A., (1996), Prime and semi prime submodules M. SC. Thesis, university of Baghdad

[4] Barnard, A., (1981), Multiplication modules. J. Algebra, 71: 174-178.

[5] Behboodi, M., Karamzadeh, O. A. S. and Koohy, H., (2004), Modules whose certain submodule are prime, Vietnam J. of Mathematics, 32(3): 303-317.

[6] El-Bast, Z. A. and Smith, P. F., (1988), Multiplication modules, Comm. In Algebra, 16: 755-779.

[7] Goodearl, K. R., (1972), Ring theory, Marcel Dekker, New York.

[8] Ibrahiem, T. A., (2011), Prime extending module and S-prime module, J. of Al-Nahrain Univ., 14(4),166-170.

[9] Kasch, F., (1982), Modules and rings. London: Academic Press.

[10] Larsen, M. D. and McCarthy, P. J.,(1971), Multiplicative theory of ideals, Acad. press, New York and London.

[11] Ahmed, M. A. and Dakheel, Sh. O., S-maximal submodules, J. of Baghdad for Science, Preprint.

[12] Saymeh, S. A., (1979), On prime R-submodules, Univ. Ndc. Tucuma'n Rev. Ser. A29, 129-136.

[13] Sharp**,** D.W. and Vamos, P., (1972), Injective modules, Cambridge Univ. Press.

حول المقاسات الجزئية شبه الجوهرية

منى عباس أحمد
ميساء رياض عباس
قسم الرياضيات – كلية العلوم للبنات– جامعة بغداد

أستلم البحث في : 20 تشرين الاول 2014 قبل البحث في :5 كانون الثاني 2015

الخلاصة

لتكن R حلقة ابدالية ذات عنصر محايد, وليكن M مقاساً أحادياً أيسر على R. هدفنا في هذا البحث هو التقصي عن بعض النتائج الجديدة (على حد علمنا) حول المقاسات الجزئية شبه الجوهرية التي قدمها الباحثان علي سبع وندى الدبان، إذ يقال للمقاس الجزئي N من M بأنه شبه جوهري، إذا كان $N \cap P \neq 0$ لكل مقاس جزئي أولي غير صفري N من M. لقد قمنا بإجراء تعديل يسير لهذا التعريف ليشمل المقاس الصفري، كما قدمنا العديد من القضايا والخواص الجديدة لهذا النوع من المقاسات الجزئية.

الكلمات المفتاحية: المقاسات الجزئية الجوهرية، المقاسات الجزئية شبه الجوهرية، المقاسات المنتظمة، المقاسات شبه المنتظمة، المقاسات الأولية المتكاملة, المقاسات الجوهرية المتكاملة.