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To cite this article: Balqees K Mahmoud and Yousif Y Yousif 2021 *J. Phys.: Conf. Ser.* **1879** 022103

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Paths and Cycles in Alpha Topological Spaces

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Abstract. This paper presents the concepts of prepaths, paths, and cycles in α -topological spaces and studies them in orderable spaces. Also, many relationships are proved with their equivalences using some properties in topological spaces like compactness and locally connectedness.

1. Introduction

Graph theory is an important part of mathematics, so many sciences are interested in the applications of graphs in multiple aspects [1]. Paths and cycles are some of basic combinatorial properties of graphs which are defined in [2, 3, 4] in addition in general topology in [5, 6, 7]. Cyclically orderable spaces are defined and studied in [8], and some new concepts are presented in [9].

Many years ago, many concepts had been studied in topological spaces, like α -open (α -closed) sets [10] and they have generalized and developed to get another new concept with new relationships to extend the mathematics knowledges [11, 12, 13].

In this paper, we introduce the concepts of prepaths, paths, and cycles with the topological viewpoint in α -topological space. Also, we study their equivalence relationships with convexity and intervals in an orderable space. In addition compactness, order-complete, and locally connectedness.

2. Preliminaries and basic definitions

Through this section, any graph G contains the set of vertices V_G and the set of edges E_G . The non-empty set $X = V_G \cup E_G$ used to define a topology τ on it which satisfied the general topological conditions, and for simplest (X, τ) refers to the topological space that we constructed it. For $A \subseteq X$, the interior and the closure of A in X with respect to τ are denoted by $\text{Int}(A)$ and $\text{Cl}(A)$. A sub set A is said to be an α -open set, if it satisfied $A \subseteq \text{Int}(\text{Cl}(\text{Int}(A)))$ for all $A \subseteq X$, and the complement of A in X is an α -closed set. It is clear that every open set is α -open, but in general the reverse is not true. We obtain an α -topology τ_α by taking all α -open sets, and (X, τ_α) is an α -topological space. Furthermore, the interior of A in this space with respect to τ_α is denoted by $\alpha\text{-Int}(A) = \cup \{ B \mid B \subseteq A, B \text{ is } \alpha\text{-open} \}$, while the closure of A denoted by $\alpha\text{-Cl}(A) = \cap \{ B \mid A \subseteq B, B \text{ is } \alpha\text{-closed} \}$.

We recall some basic definitions, remarks and facts in topology and graph.

Definition 2.1: Let X be a topological space, then:

- 1) X is called connected space if there do not exist two disjoint, nonempty, open sets A and B such that $X = A \cup B$. Otherwise, X is disconnected [7].
- 2) X is called separated space if there are two unordered disjoint subsets of X such that each one is the complement of the other [7].



3) if A is a subset of X , then the intersection of $Cl(A)$ and $Cl(X - A)$ is called the boundary points of A [5].

4) the component of X is the largest connected subspace of X , and an adherent component denoted as $K(x) \setminus \{x\}$ is the component of X contains x without x itself [3].

5) X is said to be S_1 -space if every singleton is either open or closed [3].

6) an α -topological space X is called an α -separated space if there are two unordered disjoint subsets of X , such that each one is the complement of the other [9].

7) any family of X has the finite intersection property if the intersection is non-empty of any finite subfamily [7].

8) if $B \subseteq X$, then the intersection of all α -open sets contained B is called α -surrounding set of B , and denoted by $B^{\alpha\circ}$ [9].

Definition 2.2 [3]: If x is any point in a connected topological space X , and $X \setminus \{x\}$ is not connected, then x is called a cutpoint of X . Otherwise x is a non-cutpoint (that is an endpoint).

Theorem 2.3 [3]: Let X be a topological space, then the next statements are equivalence:

(A) $\forall x, y, z \in X: x < y < z$, there exists a total order \leq on X such that y disconnects x from z .

(B) for each triple of points S in X , there exists a point in S which disconnects the other two.

(C) $\forall x \in X$, every adherent component of x denoted by $K(x) \setminus \{x\}$, has at most one endpoint, also any cutpoint is strongly cutpoint.

Corollary 2.4 [3]: Let X be a connected topological space has at least four points, then X is S_1 -space, and any two non-incident points disconnect the others, but no one do.

Definition 2.5 [3]: Let X be a connected topological space, then :

1) X is a prepath denoted by P if $|X| \leq 2$ with any topology except the indiscrete one, otherwise, X satisfies theorem (2.3) when $|X| \geq 3$.

2) a prepath is called a path if it is locally connected.

3) X is a precycle if it satisfies corollary (2.4).

4) a locally connected precycle is called a cycle.

Facts 2.6 [3,7]:

A) Let A be a connected subset of a connected topological space X , and K a component of $X \setminus A$. Then $X \setminus K$ is connected.

B) Let Y be a subspace of a topological space X , then any component of Y is contained in a component of X .

C) let X be a topological space with exactly two components, then $\{K_1, K_2\}$ is a separation of X if and only if K_1, K_2 are the components of X .

D) If C is a connected subset of a topological space, and A is a subset such that $C \subseteq A \subseteq Cl(C)$, then A is connected.

E) A topological space X is locally connected if and only if each component of each open set is open.

Definition 2.7 [3]: If P is a prepath, then the interval topology on P is the collection of intervals of P which is a base for a this topology on P .

Theorem 2.8 [3]: A prepath is locally connected if and only if the topology concurs with the interval topology.

Definition 2.9 [3]: Let \leq be a binary relation on X with $a \in X$, then $\omega(x) = \{x \in X \mid x > a\}$, $\Omega(x) = \{x \in X \mid x \geq a\}$, $\delta(x) = \{x \in X \mid x < a\}$ and $\Delta(x) = \{x \in X \mid x \leq a\}$.

Definition 2.10 [11]: Let \leq be a total order on X , a subset of X of the form $L \cap U$ is an interval where U and L are taking one of the following forms:

1. X .
2. $\omega(a)$, for some $a \in X$ and $(\delta(b))$, for some $b \in X$.
3. $\Omega(a)$, for some $a \in X$ and $(\Delta(b))$, for some $b \in X$.

Definition 2.11 [3]: Let X be a totally ordered space. A subset S of X is said to be convex if for all $a, b \in S$, and $a < x < b$, then $x \in S$ for all $x \in X$.

Theorem 2.12 [3]: Let S be a subset of a complete totally ordered set, then it is convex if and only if S is an interval.

Definition 2.13 [3]: A convex subset containing exactly two points is called a jump.

Definition 2.14 [3]: A convex subset S of a totally ordered space X is said to be a chain, if S is order isomorphic onto a convex subset of Z .

Corollary 2.15 [3]: Every chain in a totally ordered space is contained in a maximal one.

Corollary 2.16 [3]: Any two maximal chains in a totally ordered set are disjoint.

Definition 2.17 [3]: Let X be any set, $S \subseteq X$ with a binary relation on X , S is said to be strongly upper-bounded (lower-bounded) if S has a maximum (respectively, minimum) M (respectively, m), where M (respectively, m) is strictly bounded from above (respectively, below). Also, S is strongly bounded if it is both strong (upper-bounded and lower-bounded). However, S is partially strongly bounded if it satisfies one of them.

We notice that a chain is finite if it has both minimum and maximum.

Corollary 2.18 [9]: Let X be an α -topological space, and $x, y, z \in X$. If y α -separates x and z , then y α -disconnects x and z .

Corollary 2.19 [9]: Let X be an α -connected topological space, x is an α -cutpoint of X , then only one of the next statements are holding for any α -separation $\{B_1, B_2\}$ of $X \setminus \{x\}$:

1. $\{x\}$ is α -closed, B_i is α -open in X , and $\alpha\text{-Cl}(B_i) = B_i \cup \{x\}$ is α -closed, for $i = 1, 2$
2. $\{x\}$ is α -open, B_i is α -closed in X , and $B_i^{\alpha\circ} = B_i \cup \{x\}$ is α -open, for $i = 1, 2$.

Theorem 2.20 [9]: Let X be an α -connected topological space, x be an α -cutpoint of X , then there exist two nonempty α -closed sets F_1, F_2 in X , such that $F_1 \cup F_2 \cup \{x\} = X$, $\{F_1 \setminus \{x\}, F_2 \setminus \{x\}\}$ is an α -separation of $X \setminus \{x\}$ and only one of the following statements are holding:

1. $\{x\} = (F_1 \cap F_2)$ with $\{x\}$ is an α -closed.
2. $\{F_1, F_2, \{x\}\}$ is a partition of X with $\{x\}$ is an α -open.

Theorem 2.21 [9]: If X is an α -connected topological space, h is an α -hyperedge of X with finite number of α -boundary points, then each α -component of $X \setminus \{h\}$ includes an α -boundary point of h . Specifically, $X \setminus \{h\}$ contains at most two α -connected components, when h is an edge.

Corollary 2.22 [9]: Let X be an α -connected topological space, x is a cutedge, then an α -adherent component $K_\alpha(x) \setminus \{x\}$ contains exactly two α -connected components, each α -component consisting of one α -boundary point of x .

Definition 2.23 [3]: An ordered pair (A, B) is said to be a gap, if A and B have no maximum and no minimum respectively.

Fact 2.24 [3]: Any total order has no gaps if and only if it is order-complete.

Definition 2.25 [3]: Let X be any set, a triplex relation $S \subseteq X^3$ is said to be a cyclic order on X if:

* $(a, b, c) \notin S$ where $a \neq b \neq c \neq a$ if and only if $(c, b, a) \in S$

* if $(a, b, c) \in S$, then $(b, c, a) \in S$

* if $(a, b, c) \in S$ and $(a, c, d) \in S$, then $(a, b, d) \in S$

If S is a cyclic order on X and Y is a subset of X , then there exists a cyclic order $S^* = S \cap Y^3$ inherited on Y by the classes contains all its elements in Y .

Furthermore, the ordered space S^* is called a cyclic subsequence of S , where S is finite.

3. Prepaths in Alpha Topological Spaces

In this section, we introduce some concepts of α -topological spaces and investigate the relationships between them.

The first important theorem can be proven in the same way as in [3], we need it here to define an α -prepath and in the proof of theorem (3.4).

Theorem 3.1: Let X be a topological space, then the next statements are equivalents:

(A) $\forall x, y, z \in X: x < y < z$, there exists a total order \leq on X such that y α -disconnects x from z .

(B) For each triple of points S in X , there exists a point in S which α -disconnects the other two.

(C) $\forall x \in X$, every α -adherent component of x denoted as $K_\alpha(x) \setminus \{x\}$, has at most one endpoint, also any α -cutpoint is strongly α -cutpoint.

(* if X is an α -connected in this theorem, then the 3rd statement means that for all $x \in X$, $X \setminus \{x\}$ has mostly two α -component contains one endpoint in each one.

Definition 3.2: An α -connected topological space X is said to be an α -prepath denoted by P_α if $|X| \leq 2$ with any α -topology except the indiscrete one, otherwise $|X| \geq 3$ and X satisfies the conditions of theorem (3.1).

Definition 3.3: An α -topological space X is said to be α -topologized graph if:

* any singleton is either α -open or α -closed.

* X has at most two α -boundary points, for all $x \in X$.

The next theorem shows that an α -prepath is always an α -topologized graph.

Theorem 3.4: Let X be an α -connected topological space, if X is an α -prepath then:

1. X is an α -topologized graph.

2. Every edge in X is a cutedge or a loop which is an endpoint.

Proof: Suppose X is an α -prepath, then X has many cases. Firstly, X is trivially an α -topologized graph if it is an empty set because it is α -clopen. Secondly, When X is a singleton element, so it has a unique point and it is α -open and α -closed. Thirdly, when X have two points, we can get exactly two α -connected topologies, the first one is the indiscrete α -topology which excepted in definition (3.2) and the second one is an α -topologized graph with exactly one vertex with one loop.

Finally, when X has at least three points, we must prove it is an α -topologized graph by satisfying the conditions of definition (3.3). Now, we discuss being x is an α -cutpoint or an α -endpoint. Suppose that x is an α -cutpoint, it is either α -open or α -closed by corollary (2.19). Furthermore, all α -cutpoints are strong α -cutpoints by theorem (3.1 part C). Thus, when x is an α -open, we have that an α -boundary of

x contains exactly two points by theorem (2.21). So, x is a proper edge, that is a cutedge because x is an α -cutpoint.

Now, suppose that x is an endpoint, so X must have at least one α -cutpoint, and mostly two endpoints because it has at least three points. Let y be any α -cutpoint in X , and A_y, B_y be the α -connected components of $X \setminus \{y\}$. Define C as the set of all α -cutpoints, and $W = \bigcap_{y \in C} \overline{A_y}$, so $x \in A_y$ implies that $x \in W$.

Now, assume that y is another endpoint in W , such that $y \neq x$, and take another point to be an α -cutpoint like w . Since $y \in W$, we have that $y \in \overline{A_w}$, by theorem (2.20) which leads to either $y \in A_w$ or $y = w$. That means x and y lie in the same α -connected component of $X \setminus \{w\}$, that is a contradiction with (*) of theorem (3.1) since the component contains two endpoints together. Also the second choice is impossible, because y is an endpoint and w is an α -cutpoint. Hence, W does not contain any endpoints excepting x .

Suppose that $y_1 \neq y_2$ are two α -cutpoints in W . Clearly, x cannot α -disconnect y_1 and y_2 because it is an endpoint. So, for the three elements $\{x, y_1, y_2\}$, if y_1 α -disconnects x with y_2 , then $y_2 \notin A_{y_1}$ implies $y_2 \notin W$ (otherwise if y_2 α -disconnects x with y_1 , then $y_1 \notin A_{y_2}$ implies $y_1 \notin W$), by using (3.1B) is a contradiction. So, W contains mostly one element besides x , and it is an α -cutpoint. If $W = \{x\}$, then $\{x\}$ is α -closed, since it is the intersection of α -closed sets. Otherwise, when $W = \{x, y\}$ for some α -cutpoint y , $y \in A_y$. Hence $A_y = \{x\}$ is α -open and $\alpha\text{-Cl}(x) = \{x, y\}$ by corollary (2.19), (that is a loop).

Corollary 3.5: Let X be an α -connected topological space, $x, y, z \in X$, then only one of them α -disconnects the others.

Proof: Suppose that y α -disconnects x and z , and assume that $K_\alpha(x)$ define the α -component of $X \setminus \{y\}$ which contains x . So $X \setminus K_\alpha(x)$ is an α -connected set containing y and z without x by (2.6A). So x can not α -disconnect y from z . By the same way, z cannot α -disconnect x from y .

The next theorem is proved in [9], and it is very useful in our proofs.

Theorem 3.6: If X be an α -connected topological space, x is an α -cutpoint of X . If $\{A, B\}$ is an α -separation of $X \setminus \{x\}$, then $A \cup \{x\}$ is an α -connected (so as B).

The next theorem shows that α -disconnected and α -separated are equivalent in α -connected prepaths.

Theorem 3.7: Let X be an α -connected topological space, then the following statements are equivalent:

- (1) $\forall x, y, z \in X$, there exists a total order \leq on X such that, if $x < y < z$, then y is a strong α -cutpoint and α -disconnects x from z .
- (2) $\forall x, y, z \in X$, there exists a total order \leq on X such that, if $x < y < z$, then y α -separates x from z .
- (3) $\forall x, y, z \in X$, there exists a total order \leq on X such that, if $x < y < z$, then y α -disconnects x from z .
- (4) For any three disjoint points, there exists exactly one of them that α -disconnects the others.
- (5) For any three disjoint points, there exists exactly one of them that α -separates the others.
- (6) For any three disjoint points, there exists exactly one strong α -cutpoint of them that α -disconnects the others.

Proof: Firstly, we should prove the equivalent of the first three statements, and later we should prove the last three of them.

(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)

(1) \Rightarrow (2); Let $x < y < z$, then $X \setminus \{y\}$ has exactly two α -connected components, K_x, K_z such that $x \in K_x, z \in K_z$. So we obtain that $\{K_x, K_z\}$ is an α -separation of $X \setminus \{y\}$, and y α -separates x and z by facts (2.6C)

(2) \Rightarrow (3); This prove is coming immediately from corollary (2.18).

(3) \Rightarrow (1); Since y α -disconnects x from z by (3), implies that y is an α -cutpoint, and a strong α -cutpoint by using theorem (3.1).

(6) \Rightarrow (5) \Rightarrow (4) \Rightarrow (6)

(6) \Rightarrow (5); Take any three disjoint points such that one of them is strong α -cutpoint, so it α -disconnects the others. Also it α -separates them.

(5) \Rightarrow (4); This prove is coming immediately from corollary (2.18).

(4) \Rightarrow (6); Since there exists one point that α -disconnects the others, implies that it is an α -cutpoint, and a strong α -cutpoint by using theorem (3.1).

Now to prove the equivalent between the two sets of statements, we shall prove that (3) and (4) are coincides.

(3) \Leftrightarrow (4): This proves is direct from conditions of theorem (3.1).

Corollary 3.8: Let X be an α -connected topological space, and the total order \leq on X is satisfying the 2nd condition of theorem (3.7). If y α -separates x from z , then either $x < y < z$, otherwise $z < y < x$.

Proof: Since y α -separates x from z , then there exists an α -separation $\{H, K\}$ of $X \setminus \{y\}$ such that $x \in H$ and $z \in K$. So $x < y < z$. But, if the result not hold, assume that $y < x < z$, ($x < z < y$). Then x should α -separate y and z (z should α -separate x and y). But by theorem (3.6) $\{y\} \cup K$ ($H \cup \{y\}$) is an α -connected subset containing y and z but not x (containing x and y but not z), this is contradiction.

Definition 3.9: Let P_α be any α -prepath, then every total order defined on P_α is said to be α -compatible with P_α when it is satisfying conditions of theorem (3.7).

Theorem 3.10: Any subset B of an α -prepath P_α is an α -connected if and only if B is an α -prepath. Moreover, any compatible total order of an α -connected subset of P_α is exactly this one inherits from P_α .

Proof: Let B be an α -prepath subset of an α -prepath X , if B is an α -disconnected, then that is contradiction with theorem (3.1), so B is an α -connected.

Conversely, let B is an α -connected subset, and \leq_B is inherited total order by B from a compatible one \leq on P_α . Let $x, y, z \in B$ and $x <_B y <_B z$, implies that $x < y < z$ and y α -disconnects x from z , that is mean, each of x and z belong to different α -components of $X \setminus \{y\}$, so they belong to different α -connected components of $B \setminus \{y\}$. Hence \leq_B on B satisfies the 3rd condition of the theorem (3.7).

Definition 3.11: Let P_α be an α -prepath, a point $x \in P_\alpha$ is internal if it is not extremum with respect to a compatible total order, and terminal otherwise, so the α -cutpoints are the internal points, and the endpoints are the terminal.

Definition 3.12: An α -prepath is said to be α -bounded, if the compatible total orders are α -bounded. That means, it is an α -prepath with two terminal points. Moreover, some compatible total order is α -bounded from above (respectively, below) but not from below (respectively, above), the α -prepath is said to be one-sided.

Definition 3.13: Let X be an α -topological space, $a, b \in X$, a subset P_α of X is said to be an ab- α -prepath if it is a (bounded) α -prepath with two terminal points a minimum point and b maximum point.

Definition 3.14: Let X be an α -topological space, X is said to be an ab- α -prepath connected space, if X contains an α -prepath for every $a, b \in X$.

Theorem 3.15: Let P_α be α -prepath, then a point $x \in P_\alpha$ is an endpoint if and only if it is an extremum with respect to a compatible total order. Furthermore, every α -prepath has at most two endpoints.

Proof: Let $x \in P_\alpha$ be an endpoint, to prove x is extremum. Assume that x is not extremum (not minimum nor maximum) with respect to a compatible total order \leq , then there exist two points a, b such that $a < x < b$, so we obtain that x is an α -cutpoint. That is a contradiction, hence every non α -cutpoint is a maximum or a minimum. Conversely, if x is not an endpoint, it is an α -cutpoint, then there exists at least two points a, b such that x α -separates them. So we get $a < x < b$ ($b < x < a$) by corollary (3.8), implies that x is not maximum nor minimum (not extremum). That is contradicting the hypothesis, hence X is an endpoint.

Let P_α be α -prepath, and has three distinct endpoints a, b, c in X . Then any two of them must be α -disconnected by the third one, which became an α -cutpoint. That is contradiction. Therefore, P_α must has at most two endpoints.

The next corollary shows that if x is an α -cutpoint, then either $\delta(x)$ or $\Delta(x)$ ($\omega(x)$ or $\Omega(x)$) is α -open.

Corollary 3.16: Let P_α be α -prepath, $x \in P_\alpha$ be an internal point with a fixed compatible total order. Then one of the following must satisfy:

- (1) x is an edge, $\Delta(x)$ and $\Omega(x)$ are α -open, $\delta(x)$ and $\omega(x)$ are α -closed, $(\delta(x))^{\alpha^\circ} = \Delta(x)$ and $(\omega(x))^{\alpha^\circ} = \Omega(x)$.
- (2) x is a vertex, $\Delta(x)$ and $\Omega(x)$ are α -closed, $\delta(x)$ and $\omega(x)$ are α -open, $\alpha\text{-Cl}(\delta(x)) = \Delta(x)$ and $\alpha\text{-Cl}(\omega(x)) = \Omega(x)$.

Proof: Assume x is an internal point, and fixed \leq as a compatible order, then x is an α -cutpoint and a strong α -cutpoint by theorem (3.1). Suppose that K_1, K_2 are the two α -components of $P_\alpha \setminus \{x\}$. So we can prove that $\{K_1, K_2\} = \{\delta(x), \omega(x)\}$, by using definition (2.17). But, if not, then there exist two points $a \in K_1, b \in K_2$ such that $a < x < b$. Hence x is α -disconnected a from b . This is contradiction, since K_1, K_2 are two α -connected sets containing a and b respectively but not x .

Definition 3.17: Let P_α be an α -prepath with a fix compatible order, and let $a \in P_\alpha$. The α -open tail from a upwards (respectively, downwards) is the largest α -open subset of P_α contained in $\Omega(a)$ ($\Delta(a)$).

- If a is a terminal point, we have two cases: when a is maximum then the α -open tail from a upwards is P_α , and when a is minimum the α -open tail from a downwards is $\{a\}$ where $\{a\}$ is α -open or equal \emptyset if not.

- If a is an α -cutpoint, the α -open tail from a upwards (downwards) is $\omega(x)$ ($\delta(x)$) or $\Omega(x)$ ($\Delta(x)$) depending on what condition is satisfied from corollary (3.16).

Definition 3.18: Let x_1, x_2 be any two points in an α -topologized hypergraph H , then $\{x_1, x_2\}$ is said to be an edge-vertex incident pair if any one of them is a vertex and the other is an incident edge.

Theorem 3.19: If P_α is an α -prepath, then:

1. $\{x, y\}$ is an edge-vertex incident pair if and only if it is a jump with respect to a compatible total order.
2. any terminal edge exactly has one endvertex, and any internal edge has exactly two endvertices.

Proof: Suppose that e is an edge with an incident vertex v , and $\{e, v\}$ is not a jump. If $e < v$ or ($v < e$), this leads to the existence of a point x such that $e < x < v$ or ($v < x < e$). So we obtain y is an α -cutpoint, by using theorem (3.15), and (x, ∞) is an α -open set containing v but not e , by corollary (3.16) and definition (3.17). That is a contradiction, since e and v are incident.

Conversely, Let \leq be an compatible total order, and $\{x, y\}$ a jump. Assume that $x < y$ or ($y < x$), then $P_\alpha = \Delta(x) \cup \Omega(y)$, where they are disjoint. If x, y are both edges, then both these sets are α -open, but if they are both vertices, then both these sets are α -closed, by corollary (3.16). Hence, $\{\Delta(x), \Omega(x)\}$ is an α -separation of P_α , which contradiction with being P_α is α -connected.

Since every edge is a cutedge or a loop from theorem (3.4). So an internal edge is a cutedge, by theorem (3.15). Therefore, it has exactly two endvertices by corollary (2.22), and since terminal points are not α -cutpoints, then they are loops. Therefore, they are incident with exactly one vertex.

Corollary 3.20: Let P_α be an α -prepath, A be a convex subset of P_α with at least two points. If $a \in A \setminus A^{\alpha^\circ}$, then a is an extremum point for A , but not for P_α .

Proof: Let \leq be a compatible total order for P_α . Assume that there exist two points in A such that a is between them, (that is $\exists x, y \in A$, where $x < a < y$). Then $\langle x, y \rangle$ is α -open contains a , but A is convex, then a is contained in A , so $a \in A^{\alpha^\circ}$, that is a contradiction. Therefore, a is an extremum point of A . To prove that a is not an extremum point of P_α let b be another point in A . By contradiction, let a be a minimum point of P_α , then $[a, b) = (-\infty, b)$ is an α -open subset of A containing a . That is contradiction, hence a cannot be a minimum for P_α . By the same way a can not be a maximum. Therefore, a is not an extremum for P_α .

The following theorem can be proven similarly as in [3], and we need it in the subsequence details of proofs.

Theorem 3.21: A subset of an α -prepath is an α -connected if and only if it is convex.

Corollary 3.22: If P_α is an α -prepath and \leq a compatible total order, then \leq is an order-complete.

Proof: By way of contradiction, we must prove that there are no gaps, let (A, B) be a gap, and A must be convex, since (A, B) is a cut. If A is not α -open, then there exists some point $a \in A \setminus A^{\alpha^\circ}$ by corollary (3.20). So a is an extremum for A , and the same as B , which is contradicting the definition of a gap definition (2.23). Therefore A and B are α -open, then $\{A, B\}$ is an α -separation of the α -connected space P_α , that is contradiction.

Corollary 3.23: Let X be an α -prepath, $M \subseteq X$, then the next statements are equivalent:

- (1) M is α -connected.
- (2) M is an α -prepath.
- (3) M is convex.
- (4) M is an interval.

Proof: We prove (1) \leftrightarrow (2) by using theorem (3.10), and (2) \leftrightarrow (3) by using theorem (3.21), and (3) \leftrightarrow (4) by using theorem (2.12), so to prove (4) \leftrightarrow (1) we use corollary (2.22) and corollary (3.22).

Theorem 3.24: Let X be an α -topological space, $a, b \in X$, then the following statements are equivalent:

- (1) For all $x \in X \setminus \{a, b\}$, x α -disconnects a from b .
- (2) For all $S \subset X$, such that $\{a, b\} \subseteq S$, S is α -disconnected.

Proof: Suppose that (1) satisfies, assume that $S \subset X$ contains $\{a, b\}$. Take $x \in X \setminus S$, and define $W = X \setminus \{x\}$. Let $a \in S_a, b \in S_b$ be the α -connected components of S containing a, b . So $a \in W_a, b \in W_b$ is the α -connected components of W containing a, b . Since $S \subseteq W$, we get that $S_a \subseteq W_a$ and $S_b \subseteq W_b$ by the fact that α -connected components form a partition of the point set of an α -topological space and facts (2.6B). But, W_a and W_b are different, then S_a and S_b are disjoint. So, S is α -disconnected, we get (2).

Now suppose that (2) holds, Assume that $x \in X \setminus \{a, b\}$. So, if $S = X \setminus \{x\} \subset X$ contains a, b , then S is α -disconnected. Assume that a, b are in the same α -connected component K of $X \setminus \{x\}$. So K is a subset of $X \setminus \{x\}$ containing a, b , and hence K is α -connected, that is a contradiction. So a and b must contain in a different α -connected components of $X \setminus \{x\}$.

4. Paths and Cycles in Alpha Topologica Spaces

In this section, we introduce some basic definitions and facts on α -topological spaces which we need it to define our new concepts. Moreover, we study some important relationships in α -space.

Definition 4.1: Let P_α be an α -prepath, we can define the interval α -topology on P_α by taking a subbase contains X and all open intervals $(-\infty, x)$, (x, ∞) on P_α for some vertex x and we can obtain (the base, the topology, and the α -topology) of P_α . We should refer to that the α -open sets in the interval α -topology are α -open sets in general topology on P_α .

Corollary 4.2: An α -topological space X is a locally α -connected if and only if each component of each α -open set in X is α -open.

Proof: Let K be a component of α -open set U in a locally α -connected space X , $x \in K$. So there is an α -open connected set V such that $x \in V \subset U$, and $V \subset K$, therefor K is α -open.

Conversely, assume that the components of every α -open set is α -open in X . Let U be any α -open neighborhood containing x in X , then there is an α -component K containing x of U which is an α -open connected neighborhood in U . That is, $x \in K \subset U$. Hence, X is locally α -connected.

Theorem 4.3: An α -prepath is locally α -connected if and only if the α -topology concurs with the interval α -topology.

Proof: Let P_α be an α -prepath, and let U be an α -open set in P_α containing x , assume that P_α is locally α -connected. We have to show that U contains an α -open interval containing x . By corollary (3.23), we obtain that $K_U(x)$ is an interval, because it is an α -connected of P_α . Since P_α is locally α -connected and U is α -open, the components $K(U)(x)$ is α -open by corollary (4.2).

Conversely, assume that P_α has the interval α -topology and $U \subseteq P_\alpha$ is α -open. Then the union of every α -open intervals is α -open, means, $\bigcup_{i \in I} J_i$ where the J_i are α -open intervals. Now, if $x \in K$, where K is any α -component of U , is an α -open interval. So we get $x \in J_i$ for some i . But J_i is an α -connected subset of U , therefore $J_i \subseteq K$. Hence, P_α is locally α -connected.

Definition 4.4: Let X be an α -connected topological space, an α -prepath is called an α -path if it is locally α -connected.

If we have an α -prepath say ab , we can express it by ab - α -prepath, and it is an ab - α -path if it is locally α -connected. If there exists an α -topology on X such that X is an α -prepath(α -path) with a total order \leq is compatible with X , then \leq on X is an α -prepath(α -path) compatible.

Theorem 4.5: Let X be an α -connected topological space with the interval α -topology τ_α^* , and X is an α -prepath, then X is an α -path with compatible orders, and it has the same α -connected subsets.

Proof: Let τ_α^* be the interval α -topology on X , then any α -separation in τ_α^* should be an α -separation in τ_α . Since $\tau_\alpha^* \subseteq \tau_\alpha$. Hence, X is α -connected with respect to τ_α^* .

Assume that \leq is a compatible total order on X with respect to τ_α . Let x, y, z be any three points on X such that $x < y < z$. If y is a vertex, then we have an α -separation $\{(-\infty, y), (y, \infty)\}$ of $X \setminus \{y\}$, and y α -separates x from z . But, if y is an edge (not external), then it has a previous vertex a and a latter vertex b . So we have an α -separation $\{(-\infty, b) \setminus \{y\}, (a, \infty) \setminus \{y\}\}$ of $X \setminus \{y\}$. Hence, X is an α -prepath compatible with the same total order \leq with respect to τ_α^* . That is implying that the intervals with respect to the orders compatible with τ_α^* are exactly the intervals with respect to the original α -topology τ_α . Hence, X is locally α -connected with respect to τ_α^* , by (4.3), that is, (X, τ_α^*) is an α -path.

The next definition was introduced in [13].

Definition 4.6: An α -topological space X is said to be α -compact if every α -open cover has a finite subcover.

Corollary 4.7: If X is an α -topological space, then the following statements are equivalent:

- (1) X is α -compact.
- (2) If every family of α -closed sets has the finite intersection property, then it has a non-empty intersection.

Proof: (1) \rightarrow (2) Let X be an α -compact space, by contradiction, let $\{F_i \mid i \in I\}$ be a collection of α -closed sets such that $\bigcap_{i \in I} F_i = \emptyset$. Then we have an α -open cover $\{X - F_i \mid i \in I\}$. Since X is α -compact, then there exists a finite subcover $\{X - F_i\}_{i=1}^n$, and $\bigcap_{i=1}^n F_i = \emptyset$. That is contradicting with the finite intersection property, hence $\bigcap_{i \in I} F_i \neq \emptyset$.

(2) \rightarrow (1) Let $\{U_i \mid i \in I\}$ be an α -open cover of X such that $X = \bigcup \{U_i \mid i \in I\}$ and does not have a finite subcover, then $X - \{U_i \mid i \in I\} = \{F_i \mid i \in I\}$ is a nonempty α -closed sets. Hence $\bigcap_{i=1}^n F_i \neq \emptyset$ when $\bigcap_{i \in I} F_i \neq \emptyset$ because of the finite intersection property by (2). So $X - \bigcap \{F_i \mid i \in I\} = \bigcup \{U_i \mid i \in I\}$, moreover $X - \bigcap_{i=1}^n F_i = \bigcup_{i=1}^n U_i$ which is a finite subcover of $\{U_i \mid i \in I\}$. Hence, X is an α -compact.

The next theorem is very important because it shows the relation between compactness and locally connectedness in α -topological spaces. Therefore, we take it here without proof, so we can write its proof in another place.

Theorem 4.8: If P_α is an α -bounded prepath, then P_α is α -compact if and only if it is locally α -connected. From the above theorem, we can conclude the next corollary.

Corollary 4.9: Let P_α be an α -bounded prepath, then the next statements are equivalent:

- (1) P_α is α -compact.
- (2) P_α is locally α -connected.
- (3) The α -topology of P_α concurs with the interval α -topology.

Proof: we can see the equivalence the above statements directly by using theorem (4.8) and theorem (4.3).

Definition 4.10: An α -topological space X is said to be α - S_1 space if every singleton is either α -open or α -closed.

We introduced the next theorem to use it when we define precycle and cycle.

Theorem 4.11: If X is an α -connected topological space with at least four points, then the following statements are equivalent:

1. For every four points, no one α -disconnects the remaining, however, some two α -disconnect, and are α -disconnected by the others.
2. There exists a cyclic order on X such that for every four points a, b, c, d , we have that a, c α -separate(α -disconnect) b and d if and only if $\langle w, x, y, z \rangle$ is a cyclic subsequence for some choice of $w, y \in \{a, c\}$ and $x, z \in \{b, d\}$.
3. X is α - S_1 , and for every finite subset A , the complement of A and A have the same number of α -connected components.
4. X is α - S_1 and every two non-incident points α -disconnects, but no one point do.

Proof: (1) \rightarrow (2) Let S be a cyclic order on X , and a, b, c, d , be disjoint points with a, c α -separate(α -disconnect) b and d . Now suppose that $Y \subseteq X$, then Y inherits a cyclic order $S^* = S \cap Y^4$ such that all quadruple with entries in Y . Since S is finite, then the obtaining ordered space is a cyclic subsequence of S . By (1), since a, c α -separate(α -disconnect) b and d , that means they are belong to different component. So, there is four points w, x, y, z in Y such that $\{w, y\} = \{a, c\} \cap Y$, and $\{x, z\} = \{b, d\} \cap$

Y, since Y is a subspace of X. That leads to $w, y \in \{a, c\}$ and $x, z \in \{b, d\}$, because they belong to the intersection.

(2) \rightarrow (3) from (2) we have a cyclic subsequence ordered space such that $w, y \in \{a, c\}$ and $x, z \in \{b, d\}$ which makes an α -separation any four points into two disjoint finite sets. Each one of them is being the complement of the other by definition (2.1). Hence, any one of these sets have the same number of α -connected components by facts (2.6A).

(3) \rightarrow (4) X is α - S_1 space, and A is a finite subset that has the same number of α -component with its complement. So there is an α -separation of X into two α -connected subsets by facts (2.6C). If A contains only one element, t then it is an α -open(α -closed) connected which can not α -disconnect the other points in X. However, if it contains two elements, it is a jump. So, it is an edge-vertex incident pair by theorem (3.19). Therefore, the two points can not α -separate the two others points. Hence, they are non-incident to α -separate the others.

(4) \rightarrow (1) we have X is α - S_1 space, for every four points on X, any two points which non-incident are α -disconnect the other two, so they are α -disconnected. Since X is α - S_1 and α -connected, then there has not any point that α -disconnect the other three points in X.

Definition 4.12: An α -connected topological space X is said to be an α -precycle, if $|X| = 2$, and X is an α -topologized graph contains one vertex and one edge (a loop) only. Otherwise, If $|X| \geq 4$, and X satisfies the statements of theorem (4.11).

Definition 4.13: An α -precycle is called an α -cycle if it is locally α -connected.

The following simple example shows the idea of precycles and cycles by using the above definitions.

Example 4.14: Let $X = V_G \cup E_G = \{v, e\}$ be a set consists of one vertex and one edge (loop), with $\tau_X = \{\emptyset, X, \{v\}\}$, then (X, τ) is a topological space such that the elements of τ are open sets and their complements are closed sets. So the α -topology of X is $\tau_{\alpha X} = \{\emptyset, X, \{v\}\}$ which contains an α -open sets and their complements are α -closed set. We can see that X is an α -connected space, therefore it is an α -precycle by definition (4.12). Moreover, it is an α -cycle since it is locally α -connected by definition (4.13).

5. Conclusion

Through this paper, alpha-prepaths, alpha-paths, and alpha-cycles have been formulated and some relationships have been discussed. Moreover, several theorems and corollaries have been proved to continuum the future searches and works to define a new concepts which are related with them like cycle spaces.

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