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Some Results on Fibrewise Lindelöf and Locally Lindelöf Topological Spaces

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Abstract

In this paper, we define and study new concepts of fibrewise topological spaces over B namely, fibrewise Lindelöf and locally Lindelöf topological spaces, which are generalizations of well-known concepts: Lindelöf topological space (1) "A topological space X is called a Lindelöf space if for every open cover of X has a countable subcover" and locally Lindelöf topological space (1) "A topological space X is called a locally Lindelöf space if for every point x in X, there exist a nbd U of x such that the closure of U in X is Lindelöf space". Either the new concepts are: "A fibrewise topological space X over B is called a fibrewise Lindelöf if the projection function $p : X \to B$ is Lindelöf" and "The fibrewise topological space X over B is called a fibrewise locally Lindelöf if for each point x of X_b, where $b \in B$, there exists a nbd W of b and a nbd $U \subset X_W$ of x such that the closure of U in X_W (i.e., $X_W \cap cl(U)$) is fibrewise Lindelöf space over W". Moreover, we study the relationships between fibrewise Lindelöf (locally Lindelöf) spaces and some fibrewise separation axioms.

Introduction and Preliminaries

Anyone who has worked on the theory of fibrewise topology, and in related areas of mathematics, would become aware of the need for the following information; The fibrewise sets over a given set, called the base set, if the base set is denoted by B then a fibrewise set over B consists of a set X together with a function $p : X \to B$, whichis called the projection, for each point b of B the fibre over b is the subset $X_b = p^{-1}(b)$ of X; fibers may be empty since we do not require p to be surjective, also for each subset B^* of B we regard $X_{B^*} = p^{-1}(B^*)$ as a fibrewise set over B^* with the projection determined by p. In fibrewise topology the term neighborhood (nbd) is used in precisely in the same sense as it is in the ordinary topology. All the above information it can be found in (2).

For a subset A of a topological space X, the closure of A is denoted by cl(A). For other notions or notations which are not defined here, we follow closely Engelking (1).

Basic Definitions

Definition 2.1 (2)

Let X and Y are fibrewise sets over B, with projections $p_X : X \to B$ and $p_Y : Y \to B$, respectively, a function $\varphi : X \to Y$ is said to be fibrewise if $p_Y o \varphi = p_X$, in other words if $\varphi(X_b) \subset Y_b$ for each point b of B.

Notice that a fibrewise function $\varphi : X \to Y$ over B determines by restriction, a fibrewise function $\varphi_{B^*} : X_{B^*} \to Y_{B^*}$ over B^* for each B^* of B.

Definition 2.2 (2)

Suppose that B is a topological space, the fibrewise topology on a fibrewise set X over B, means any topology on X for which the projection p is continuous.

A fibrewise topological space over B is defined to be a fibrewise set over B with a fibrewise topology.

Definition 2.3 (2)

A fibrewise function $\varphi : X \to Y$, where X and Y are fibrewise topological spaces over B, is called:

(a) Continuous if for each point $x \in X_b$, where $b \in B$, the inverse image of each nbd of $\varphi(x)$ is an nbd of x.

(b) Open if for each point $x \in X_b$, where $b \in B$, the direct image of each nbd of x is an nbd of $\varphi(x)$.

Definition 2.4 (2)

A fibrewise topological space X over B is called fibrewise closed if the projection p is closed function.

Definition 2.5 (1)

A topological space X is called a Lindelöf space if for every open cover of X whichas a countable subcover.

Definition 2.6 (1)

A topological space X is called a locally Lindelöf space if for every point x in X, there exist a nbd U of x such that the closure of U in X is Lindelöf space.

Definition 2.7 (1)

For every topological space Y and any subspace X of Y, the function $i_X : X \to Y$ define by $i_X(x)=x$ is called embedded of the subspace X in the space Y.

Observe that i_X is continuous. Since $i_X^{-1}(U)=X \cap U$, where U is open set in Y.

Fibrewise Lindelöf and Locally Lindelöf Topological Spaces.

In this section, we introduce the following new concepts.

Definition 3.1

A function $\varphi : X \to Y$ is called a Lindelöf function, if it is continuous, closed and for each $y \in Y$, $\varphi^{-1}(y)$ is Lindelöf set.

For example, let (IR, τ) where τ is the topology with basis whose members are of the form (a, b) and (a, b)-N, $N=\{1/n \ ; \ n \in Z^+\}$. Define $f : (IR,\tau) \to (IR,\tau)$ by $f(x)=x$, then f is Lindelöf function.

If $\varphi : X \to Y$ is fibrewise and Lindelöf function, then φ is said to be fibrewise Lindelöf function.

Definition 3.2

A fibrewise topological space X over B is called a fibrewise Lindelöf if the projection p is Lindelöf.

For example the topological product $B \times X$ is fibrewise Lindelöf over B, for all Lindelöf space X. For another example, the subset $\{(b, x) \in IR \times IR^n : \|x\| \le b\}$ of $IR \times IR^n$ is fibrewise Lindelöf over IR.

Proposition 3.3 (1)

Let X be a fibrewise topological space over B. Then X is fibrewise closed if for each fibre X_b of X and each nbd U of X_b in X, there exists an nbd W of b such that $X_W \subset U$.

Useful characterizations of fibrewise Lindelöf spaces are given by the following propositions.

Proposition 3.4

The fibrewise topological space X over B is fibrewise Lindelöf if X is fibrewise closed and every fibre of X is Lindelöf.

Proof: ($\Rightarrow$) Let X be a fibrewise Lindelöf space, then the projection $p : X \to B$ is Lindelöf function i.e., p is closed and for each $b \in B$, X_b is Lindelöf. Hence X is fibrewise closed and every fibre of X is Lindelöf.

($\Leftarrow$) Let X be a fibrewise closed and every fibre of X is Lindelöf, then the projection $p : X \to B$ is closed and it is clear that p is continuous, also for each $b \in B$, X_b is Lindelöf. Hence X is fibrewise Lindelöf.

Proposition 3.5

Let X be a fibrewise topological space over B. Then X is fibrewise Lindelöf if for each fibre X_b of X and each covering Γ of X_b by open sets of X there exists an nbd W of b such that a countable subfamily of Γ covers X_W.

Proof: (⇒) Let X be a fibrewise Lindelöf space, then the projection $p : X \to B$ is Lindelöf function, so that X_b is Lindelöf for each $b \in B$. Let Γ be a covering of X_b by open sets of X for each $b \in B$ and let $X_W = \cup X_b$ for each $b \in W$. Since X_b is Lindelöf for each $b \in W \subset B$ and the union of Lindelöf sets is Lindelöf, we have X_W is Lindelöf. Thus, there exists an nbd W of b such that a countable subfamily of Γ covers X_W.

(⇐) Let X be a fibrewise topological space over B, then the projection $p : X \to B$ exists. To show that p is Lindelöf. Now, it is clear that p is continuous and for each $b \in B$, X_b is Lindelöf by takeing $X_b = X_W$. By proposition (3.3), we have p is closed. Thus p is Lindelöf and X is fibrewise Lindelöf.

There are special cases of well-known results of theorems (3.8.8), (3.8.5) and (3.8.7) in (1), as in propositions (3.6)-(3.8) below.

Proposition 3.6

Let $\varphi : X \to Y$ be a Lindelöf fibrewise function, where X and Y are fibrewise topological spaces over B. If Y is fibrewise Lindelöf ,then so is X.

Proof: Suppose that $\varphi : X \to Y$ is Lindelöf fibrewise function and Y is fibrewise Lindelöf space i.e., the projection $p_Y : Y \to B$ is Lindelöf. To show that X is a fibrewise Lindelöf space i.e., the projection $p_X : X \to B$ is Lindelöf. Now, since p_Y and φ are continuous and the composition of continuous functions is continuous so that $p_Y o \varphi = p_X$ is continuous. let F be a closed subset of X_b, where $b \in B$. Since φ is closed, then φ(F) is closed subset of Y_b. Since p_Y is closed, then $p_Y(\varphi(F))$ is closed in B. But $p_Y(\varphi(F)) = (p_Y o \varphi)(F) = p_X(F)$ is closed in B so that p_X is closed. Let $b \in B$, since p_Y is Lindelöf, then Y_b is Lindelöf. Now let $\{U_\alpha : \alpha \in \Lambda\}$ be a family of open sets of X such that $X_b \subset \cup_{\alpha \in \Lambda} U_\alpha$. If $y \in Y_b$, then there exists a countable subset M(y) of Λ such that $\varphi^{-1}(y) \subset \cup_{\alpha \in M(y)} U_\alpha$. Since φ is closed function, so by proposition (3.3) there exists an open set V_y of Y such that $y \in V_y$ and $\varphi^{-1}(V_y) \subset \cup_{\alpha \in M(y)} U_\alpha$. Since Y_b is Lindelöf, there exists a countable subset C of Y_b such that $Y_b \subset \cup_{y \in C} V_y$. Hence $\varphi^{-1}(Y_b) \subset \cup_{y \in C} \varphi^{-1}(V_y) \subset \cup_{y \in C} \cup_{\alpha \in M(y)} U_\alpha$. Thus if $M = \cup_{y \in C} M(y)$, then M is a countable subset of Λ and $\varphi^{-1}(Y_b) \subset \cup_{\alpha \in M} U_\alpha$. Thus $\varphi^{-1}(Y_b) = \varphi^{-1}(p_Y^{-1}(b)) = (p_Y o \varphi)^{-1}(b) = p_X^{-1}(b) = X_b$ and $X_b \subset \cup_{\alpha \in M} U_\alpha$ so that X_b is Lindelöf. Thus p_X is Lindelöf and X is fibrewise Lindelöf.

In particular this holds when φ is a closed fibrewise embedding; thus closed subspaces of fibrewise Lindelöf spaces are fibrewise Lindelöf.

Proposition 3.7

Let X be a fibrewise topological space over B. Suppose that X_j is fibrewise Lindelöf for each member X_j of a finite covering of X. Then X is a fibrewise Lindelöf.

Proof: Let X be a fibrewise topological space over B, then the projection $p : X \to B$ exists. To show that p is Lindelöf. Now, it is clear that p is continuous. Since X_j is fibrewise Lindelöf, then the projection $p_j : X_j \to B$ is closed and for each $b \in B$, $(X_j)_b$ is Lindelöf for each member X_j of a finite covering of X. Let F be a closed subset of X, then $p(F) = \cup p_j(X_j \cap F)$ which is a finite union of closed sets and hence p is closed. Let $b \in B$, then $X_b = \cup (X_j)_b$ which is a finite union of Lindelöf sets and hence X_b is Lindelöf. Thus, p is Lindelöf and X is fibrewise Lindelöf.

Proposition 3.8

Let $\varphi : X \to Y$ be a continuous fibrewise surjection, where X and Y are fibrewise topological spaces over B. If X is fibrewise Lindelöf, then so is Y.

Proof: Suppose that $\varphi : X \to Y$ is a continuous fibrewise surjection and X is a fibrewise Lindelöf i.e., the projection $p_X : X \to B$ is Lindelöf. To show that Y is a fibrewise Lindelöf i.e., the projection $p_Y : Y \to B$ is Lindelöf. Now, it is clear that p_Y is continuous. Let F be a closed subset of Y_b, where $b \in B$. Since φ is continuous fibrewise, then $\varphi^{-1}(F)$ is a closed subset of X_b.

Since p_X is closed, then $p_X(\varphi^{-1}(F))$ is closed in B. But $p_X(\varphi^{-1}(F))=(p_X o \varphi^{-1})(F) = p_Y(F)$ is closed in B, hence p_Y is closed. For any point $b \in B$, we have $Y_b=\varphi(X_b)$ is Lindelöf because X_b is Lindelöf and a continuous image of Lindelöf space is Lindelöf. Thus p_Y is Lindelöf and Y is fibrewise Lindelöf.

Proposition 3.9

Let X be a fibrewise Lindelöf space over B. Then X_{B^*} is a fibrewise Lindelöf space over B^* for each subspace B^* of B.

Proof: Suppose that X is a fibrewise Lindelöf i.e., the projection $p : X \to B$ is Lindelöf. To show that X_{B^*} is a fibrewise Lindelöf space over B^* i.e., the projection $p_{B^*} : X_{B^*} \to B^*$ is Lindelöf. Now, it is clear that p_{B^*} is continuous. Let F be a closed subset of X, then $F \cap X_{B^*}$ is closed in subspace X_{B^*} and $p_{B^*}(F \cap X_{B^*}) = p(F \cap X_{B^*}) = p(F) \cap B^*$ which is closed set in B^*, hence p_{B^*} is closed. Let $b \in B^*$, then $(X_{B^*})_b = X_b \cap X_{B^*}$ which is Lindelöf set in X_{B^*}. Thus p_{B^*} is Lindelöf and X_{B^*} is fibrewise Lindelöf over B^*.

Proposition 3.10

Let X be a fibrewise topological space over B. Suppose that X_{B_j} is a fibrewise Lindelöf over B_j for each member B_j of an open covering of B. Then X is fibrewise Lindelöf over B.

Proof: Suppose that X is a fibrewise topological space over B, then the projection $p : X \to B$ exists. To show that p is Lindelöf. Now, it is clear that p is continuous. Since X_{B_j} is fibrewise Lindelöf over B_j, then the projection $p_{B_j} : X_{B_j} \to B_j$ is Lindelöf for each member B_j of an open covering of B. Let F be a closed subset of X, then we have $p(F) = \cup p_B(X_{B_j} \cap F)$ which is a union of closed sets and hence p is closed. Let $b \in B$ then $X_b = \cup (X_{B_j})_b$ for every $b = \{bj\} \in \cup B_j$. Since $(X_{B_j})_b$ is Lindelöf in X_{B_j} and the union of Lindelöf sets is Lindelöf, we have X_b is Lindelöf. Thus, p is Lindelöf and X is fibrewise Lindelöf over B.

In fact, the last result also holds for locally finite closed coverings, instead of open coverings, and the proof is easy, so is omitted.

The second new concept in this paper is given by the following:

Definition 3.11

The fibrewise topological space X over B is called fibrewise locally Lindelöf if for each point x of X_b, where $b \in B$, there exists an nbd W of b and an nbd $U \subset X_W$ of x such that the closure of U in X_W (i.e., $X_W \cap cl(U)$) is fibrewise Lindelöf over W.

Remark 3.12

Fibrewise Lindelöf spaces are necessarily fibrewise locally Lindelöf by taking W=B and X_W=X. But the conversely is not true for example, let (X, τ_{dis}) where X is infinite set and τ_{dis} is a discrete topology, then X is a fibrewise locally Lindelöf over IR, since for each $x \in X_b$, where $b \in B$, there exists a nbd W of b and a nbd $\{x\} \subset X_W$ of x such that $cl\{x\}=\{x\}$ in X_W is fibrewise Lindelöf over W. But X is not a fibrewise Lindelöf space over IR.

Closed subspaces of fibrewise locally Lindelöf spaces are fibrewise locally Lindelöf spaces. In fact we have

Proposition 3.13

Let $\varphi : X \to Y$ be a closed fibrewise embedding, where X and Y are fibrewise topological spaces over B. If Y is fibrewise locally Lindelöf then so is X.

Proof: Let $x \in X_b$, where $b \in B$. Since Y is fibrewise locally Lindelöf there exist a nbd W of b and a nbd $V \subset Y_W$ of $\varphi(x)$ such that the closure $Y_W \cap cl(V)$ of V in Y_W is fibrewise Lindelöf over W. Then $\varphi^{-1}(V) \subset X_W$ is a nbd of x such that the closure $X_W \cap cl(\varphi^{-1}(V)) = \varphi^{-1}(Y_W \cap cl(V))$ of $\varphi^{-1}(V)$ in X_W is a fibrewise Lindelöf over W. Thus, X is a fibrewise locally Lindelöf.

Fibrewise Lindelöf (Locally Lindelöf) Topological Spaces and Some Fibrewise Separation Axioms.

Now we give a series of results through which we give relationships between fibrewise Lindelöfness (or fibrewise locally Lindelöfness in some cases) and some fibrewise separation axioms which are discussed in (1).

Definition 4.1 (2)

The fibrewise topological space X over B is called fibrewise Hausdorff over B if whenever x_1, $x_2 \in X_b$, where $b \in B$ and $x_1 \neq x_2$, there exists disjoint nbds U_1, U_2 of x_1, x_2 in X.

Definition 4.2 (2)

The fibrewise topological space X over B is called fibrewise regular over B if for each point $x \in X_b$, where $b \in B$, and for each nbd V of x in X, there exists an nbd W of b in B and a nbd U of x in X_W such that the closure of U in X_W (i.e., $X_W \cap cl(U)$) is contained in V.

Definition 4.3 (2)

The fibrewise topological space X over B is called a fibrewise normal if for each point b of B and each pair H, K of disjoint closed sets of X, there exists a nbd W of b and a pair U, V of disjoint nbds of $X_W \cap H$, $X_W \cap K$ in X_W.

Proposition 4.4

Let X be a fibrewise locally Lindelöf and fibrewise regular over B. Then for each point x of X_b, where $b \in B$, and each nbd V of x in X, there exists an nbd U of x in X_W such that the closure $X_W \cap cl(U)$ of U in X_W is fibrewise Lindelöf over W and contained in V.

Proof: Since X is a fibrewise locally Lindelöf there exists a nbd W* of b in B and a nbd U* of x in X_{W^*} such that the closure $X_{W^*} \cap cl(U^*)$ of U* in X_{W^*} is fibrewise Lindelöf over W*. Since X is fibrewise regular there exist a nbd $W \subset W^*$ of b and a nbd U of x in X_W such that the closure $X_W \cap cl(U)$ of U in X_W is contained in $X_W \cap U^* \cap V$. Now $X_W \cap cl(U^*)$ is fibrewise Lindelöf over W, since $X_{W^*} \cap cl(U^*)$ is fibrewise Lindelöf over W*, and $X_W \cap cl(U)$ is closed in $X_W \cap cl(U^*)$. Hence $X_W \cap cl(U)$ is fibrewise Lindelöf over W and contained in V as required.

Proposition 4.5

Let $\varphi : X \rightarrow Y$ be an open continuous fibrewise surjection, where X and Y are fibrewise topological spaces over B. If X is a fibrewise locally Lindelöf and fibrewise regular then so is Y.

Proof: Let y be a point of Y_b, where $b \in B$, and let V be an nbd of y in Y. Pick any point x of $\varphi^{-1}(y)$; Then $\varphi^{-1}(V)$ is an nbd of x in X. Since X is a fibrewise locally Lindelöf there exists an nbd W of b in B and a nbd U of x in X_W such that the closure $X_W \cap cl(U)$ of x in X_W is a fibrewise Lindelöf over W and is contained in $\varphi^{-1}(V)$. Then $\varphi(U)$ is an nbd of y in Y_W, since φ is open, and the closure $Y_W \cap cl(\varphi(U))$ of $\varphi(U)$ in Y_W is a fibrewise Lindelöf over W and contained in V, as required.

Proposition 4.6

Let X be a fibrewise locally Lindelöf and fibrewise regular over B. Let C be a Lindelöf subset of X_b, where $b \in B$, and let V be an nbd of C in X. Then there exists an nbd W of b in B and a nbd U of C in X_W such that the closure $X_W \cap cl(U)$ of U in X_W is a fibrewise Lindelöf over W and contained in V.

Proof: Since X is a fibrewise locally Lindelöf, there exists for each point x of C a nbd W_x of b in B and an nbd U_x of x in X_W such that the closure $X_{W_x} \cap cl(U_x)$ of U_x in X_{W_x} is fibrewise Lindelöf over W_x and contained in V. The family $\{U_x : x \in C\}$ constitutes a covering of the Lindelöf C by open sets of X. Extract a countable subcovering indexed by $x_1, \ldots, x_2, \ldots$, say. Take W to be the intersection $W_{x_1} \cap \ldots \cap W_{x_2} \cap \ldots$, and take U to be the restriction to X_W of the union $U_{x_1} \cup \ldots \cup U_{x_2} \cup \ldots$. Then W is an nbd of b in B and U is a nbd of C in X_W such that the closure $X_W \cap cl(U)$ of U in X_W is a fibrewise Lindelöf over W and contained V, as required.

Proposition 4.7

Let $\varphi : X \rightarrow Y$ be a Lindelöf fibrewise surjection, where X and Y are fibrewise topological spaces over B. If X is fibrewise locally Lindelöf and fibrewise regular then so is Y.

Proof: Let $y \in Y_b$, where $b \in B$, and let V be an nbd of y in Y. Then $\varphi^{-1}(V)$ is a nbd of $\varphi^{-1}(y)$ in X. Suppose that X is fibrewise locally Lindelöf. Since $\varphi^{-1}(y)$ Lindelöf, by proposition (4.6) there exist an nbd W of b in B and a nbd U of $\varphi^{-1}(y)$ in X_W such that the closure $X_W \cap cl(U)$ of U in X_W is fibrewise Lindelöf over W and contained in $\varphi^{-1}(V)$. Since φ is closed there exist a nbd U^* of y in Y_W such that $\varphi^{-1}(U^*) \subset U$. Then the closure $Y_W \cap cl(U^*)$ of U^* in Y_W is contained in $\varphi(X_W \cap cl(U))$ and so is fibrewise Lindelöf over W. Since $Y_W \cap cl(U^*)$ is contained in V this shows that Y is fibrewise locally Lindelöf, as asserted.

Proposition 4.8

Let X be a fibrewise Lindelöf and fibrewise Hausdorff space over B. Then X is fibrewise regular.

Proof: Let $x \in X_b$, where $b \in B$, and let U be an nbd of x in X. Since X is fibrewise Hausdorff there exist for each point $x^* \in X_b$ such that $x^* \notin U$ and an nbd V_{x^*} of x and a nbd $V^*_{x^*}$ of x^* which do not intersect. Now the family of open sets $V^*_{x^*}$, for $x^* \in (X-U)_b$, forms a covering of $(X-U)_b$. Since X-U is closed in X and therefore fibrewise Lindelöf there exist, by proposition (3.5), a nbd W of b in B such that $X_W-(X_W \cap U)$ is covered by a countable subfamily, indexed by $x_1^*, \ldots, x_n^*, \ldots$, say. Now the intersection $V=V_{x_1^*} \cap \ldots \cap V_{x_n^*} \cap \ldots$ is an nbd of x which does not meet the nbd $V^*= V^*_{x_1^*} \cup \ldots \cup V^*_{x_n^*} \cup \ldots$ of $X_W-(X_W \cap U)$. Therefore the closure $X_W \cap cl(V)$ of $X_W \cap V$ in X_W is contained in U as asserted.

We extend this last result to

Proposition 4.9

Let X be a fibrewise locally Lindelöf and fibrewise Hausdorff space over B. Then X is a a fibrewise regular.

Proof: Let $x \in X_b$, where $b \in B$, and let V be an nbd of x in X. Let W be an nbd of b in B and let U be an nbd of x in X_W such that the closure $X_W \cap cl(U)$ of U in X_W is fibrewise Lindelöf over B. Then $X_W \cap cl(U)$ is a fibrewise regular over W, by proposition (4.8), since $X_W \cap cl(U)$ is a fibrewise Hausdorff over W. So there exists an nbd $W^* \subset W$ of b in B and a nbd U^* of x in X_{W^*} such that the closure $X_{W^*} \cap cl(U^*)$ of U^* in X_{W^*} is contained in $U \cap V \subset V$, as required.

Proposition 4.10

Let X be a fibrewise regular space over B and let K be a fibrewise Lindelöf subset of X. Let b be a point of B and let V be a nbd of K_b in X. Then there exist a nbd W of b in B and an nbd U of K_W in X_W such that the closure $X_W \cap cl(U)$ of U in X_W is contained in V.

Proof: We may suppose that K_b is non-empty since otherwise we can take $U=X_W$, where $W=B-p(X-V)$. Since V is an nbd of each point x of K_b there exists, by fibrewise regularity, an nbd W_x of b and a nbd $U_x \subset X_{W_x}$ of x such that the closure $X_{W_x} \cap cl(U_x)$ of U_x in X_{W_x} is contained in V. The family of open sets $\{X_{W_x} \cap U_x : x \in K_b\}$ covers K_b and so there exist a nbd W^* of b and a countable subfamily indexed by $x_1, \ldots, x_n, \ldots$, say, which covers K_W. Then the conditions are satisfied with

$W=W^* \cap W_{x_1} \cap \ldots \cap W_{x_n} \cap \ldots$, $U= U_{x_1} \cup \ldots \cup U_{x_n} \cup \ldots$

Corollary 4.11

Let X be a fibrewise Lindelöf and fibrewise regular space over B. Then X is fibrewise normal.

Proposition 4.12

Let X be a fibrewise regular space over B and let K be a fibrewise Lindelöf subset of X. Let $\{V_j : j=1, \ldots, n, \ldots\}$ be a covering of K_b, where $b \in B$ by open sets of X. Then there exists an nbd W of b and a covering $\{U_j : j=1, \ldots, n, \ldots\}$ of K_W by open sets of X_W such that the closure $X_W \cap cl(U_j)$ of U_j in X_W is contained in V_j for each j.

Proof: Write $V=V_2 \cup \ldots \cup V_n \cup \ldots$, so that X-V is closed in X. Hence $K \cap (X-V)$ is closed in K and so fibrewise Lindelöf. Applying the previous result to the nbd V_1 of $K_b \cap (X-V)_b$ we obtain an nbd W of b and an nbd U of $K_W \cap (X-V)_b$ such that $X_W \cap cl(U) \subset V_1$. Now $K \cap V$ and $K \cap (X-V)$ cover K, hence V and U covers K_W. Thus $U=U_1$ is the first step in the shrinking process. We continue by repeating the argument for $\{U_1, V_2, \ldots, V_n, \ldots\}$, so as to shrink V_2, and so on. Hence the result is obtained.

Proposition 4.13

Let $\varphi : X \rightarrow Y$ be a Lindelöf fibrewise surjection, where X and Y are fibrewise topological spaces over B. If X is fibrewise regular then so is Y.

Proof: Let X be a fibrewise regular. Let y be a point of Y_b, where $b \in B$, and let V be a nbd of y in Y. Then $\varphi^{-1}(V)$ is an nbd of the Lindelöf $\varphi^{-1}(y)$ in X. By proposition (4.10), therefore, there exists an nbd W of b in B and an nbd U of $\varphi^{-1}(y)$ in X_W such that the closure $X_W \cap cl(U)$ of U in X_W is contained in $\varphi^{-1}(V)$. Now since φ_W is closed there exist a nbd V^* of y in Y_W such that $\varphi^{-1}(V) \subset U$, and then the closure $X_W \cap cl(V^*)$ of V^* in X_W is contained in V since $cl(V^*) = cl(\varphi(\varphi^{-1}(V^*))) = \varphi(cl(\varphi^{-1}(V^*))) \subset \varphi(cl(U)) \subset \varphi(\varphi^{-1}(V)) \subset V$.
Thus Y is a fibrewise regular, as asserted.

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بعض النتائج حول الفضاءات التبولوجية الليندلوفية والليندلوفية الموضعية الليفية

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الخلاصة

في هذا البحث عرفنا ودرسنا مفاهيم جديدة من الفضاءات التبولوجية الليفية فوق المجموعة B سميناها، الفضاءات التبولوجية الليندلوفية والليندلوفية الموضعية الليفية. التي تعد تعميمات للمفاهيم المعروفة: الفضاء التبولوجي ليندلوف (1) "الفضاء التبولوجي X يسمى ليندلوف اذا كان كل غطاء" مفتوح لـX يمتلك غطاء جزئي قابل للعد" و الفضاء التبولوجي ليندلوف الموضعي (1) "الفضاء التبولوجي X يسمى ليندلوف موضعي اذا كان لكل نقطة x في X يوجد جوار U لـx بحيث ان علاقة المجموعة U في X تكون فضاء ليندلوف". اما المفاهيم الجديدة هي: الفضاء التبولوجي ليندلوف الليفي "الفضاء التبولوجي الليفي X فوق المجموعة B يسمى ليندلوف ليفي اذا كانت الدالة الاسقاطية $p: X \rightarrow B$ دالة ليندلوف" والفضاء التبولوجي ليندلوف الموضعي الليفي "الفضاء التبولوجي الليفي X فوق المجموعة B يسمى ليندلوف موضعي ليفي اذا كان لكل نقطة x في X_b بحيث $b \in B$ يوجد جوار W لـb وجوار $U \subset X_W$ لـx بحيث ان علاقة المجموعة U في X_W (اي ان $X_W \cap cl(U)$) تكون فضاء تبولوجي ليندلوف ليفي فوق المجموعة W. فضلا" عن ذلك نحن درسنا العلاقة بين الفضاءات التبولوجية الليندلوفية (الليندلوفية الموضعية) ألليفية وبعض بديهيات الفصل ألليفية.

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