On Significance Testimator in Pareto Distribution Via Shrinkage Technique

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Abstract:

In this paper, preliminary test Shrinkage estimator have been considered for estimating the shape parameter α of pareto distribution when the scale parameter equal to the smallest loss and when a prior estimate α_0 of α is available as initial value from the past experiences or from quaintance cases.

The proposed estimator is shown to have a smaller mean squared error in a region around α_0 when comparison with usual and existing estimators.

مُقدر الاختبار المعنوي في توزيع باريتو باستخدام طريقة التقلص

المستخلص:

في هذا البحث،تم دراسة مقدر الاختبار الأولي المقلص لتقدير معلمة الشكل α في توزيع باريتو عند توافر معلومات مسبقة α_0 حول المعلمة الحقيقية α بشكل قيمة ابتدائية المتأتيه من الخبرات السابقة او الحالات المشابهه وعندما تكون معلمة القياس k اقل قيمة ممكنه. بين البحث ان المقدر المقترح ذو متوسط مربعات خطأ قليل وخصوصاً في المنطقة حول α_0 وذلك عند مقارنته مع المقدرات الاعتيادية والموجودة.

Key words: Pareto Distribution, Shrinkage Estimation, Preliminary Test Region, Bias, Mean Squared Error and Relative Efficiency.

1. Introduction

1.1 The Model

The pareto model is very often used as a basis of Excess of loss quotations as it gives a pretty good description of the random behavior of large losses, and often used to model the distribution of income; see [4], [6] and [8].

Consider a random sample x_1, x_2, …, x_n from the pareto distribution with the following p.d.f.

$$f(x;\alpha,\lambda)=\begin{cases}\alpha k^{\alpha} x^{-(\alpha+1)} & \text{for } \alpha, k>0, x\ge k\\ 0 & \text{o.w.}\end{cases}$$

…(1)

In conventional notation, we write x~par(α,k) where α and k are the shape and scale parameter respectively.

Some times we may have a prior guess value α_0 due to past experiences or from a quaintance with similar situation of the parameter α to be estimated. If this value is very close to the true value, the Shrinkage technique is useful to get an improved estimator.

According to [13], such prior estimate may arise for any one of a number of reasons, e.g., we are estimating α and

(i) we believe α_0 is close to the true value of α; or

(ii) We fear that α_0 may be near the true value of α; i.e.; something bad happens if $\alpha = \alpha_0$, and we do not know about it. For such cases, this prior guess value may be utilized to improve the estimation procedure with usual estimator $\hat{\alpha}$ (MLE) via shrinkage estimator $\tilde{\alpha}$ which including Shrinkage weight factor $\psi(\hat{\alpha}), 0 \le \psi(\hat{\alpha}) \le 1$ as follows, see [3]

$$\tilde{\alpha} = \psi(\hat{\alpha})\hat{\alpha} + (1-\psi(\hat{\alpha}))\alpha_0$$
…(2)

The authors in [1], [5], [11], [13] and others suggested Shrunken estimators (2) for the parameters of different distribution when a guess prior value is available

They showed that these estimators perform better than usual estimator in the term of mean square error when the guess value is close to the true value.

Therefore to make sure whether the prior guess value (α_0) is approximately or close to the true value (α) or not, we may test $H_0{:}\alpha = \alpha_0$ vs. $H_A{:}\alpha \neq \alpha_0$ and a preliminary test Shrunken estimator of significance are employed as follows:

$$\tilde{\alpha}_{PT} = \begin{cases} \psi_1(\hat{\alpha})\hat{\alpha} + (1-\psi_1(\hat{\alpha}))\alpha_0 & \text{, if } H_0 \text{ accepted} \\ \psi_2(\hat{\alpha})\hat{\alpha} + (1-\psi_2(\hat{\alpha}))\alpha_0 & \text{, if } H_0 \text{ rejected} \end{cases}$$
…(3)

Using pre-assigned level of significanceΔ, the critical region (R) for such test is pre test region using specific test statistic, where $\psi_i(\hat{\alpha})$, i = 1, 2 are Shrinkage weight factors such that 0 $\leq \psi_i(\hat{\alpha}) \leq 1$ and $\hat{\alpha}$ is the usual estimator (MLE for example).

Preliminary test Shrunken estimators (3) have been considered in different contexts by [12], [9], [7], [10], [1] and [2].

The aim of this paper is to estimate the shape parameter α of pareto distribution using mentioned preliminary test shrunken estimator (3) and study its behavior when we derived the expressions of its Bias, Mean Square Error and Relative Efficiency and study their performance. Numerical results and conclusions for these expressions were made to show the effective of the proposed estimator as well as some comparisons with the usual and existing estimators were made.

1.2 Usual Estimation of the Shape Parameter α

Let $x_1, x_2, \ldots, x_n$ be identically independent pareto distributed random variables.

The maximum likelihood estimator of α when k = min(x_i) [the smallest loss] is defined in [8] as below:

$$\hat{\alpha}_{MLE} = \frac{n}{\sum_{i=1}^{n} \ln\left(\frac{x_i}{\min(x_i)}\right)}$$
…(4)

It follows easily that $\ln\left(\frac{x_i}{\min(x_i)}\right)$ will be exponentially distributed with mean 1 / α. Then

$T = \sum_{i=1}^{n} \ln\left(\frac{x_i}{\min(x_i)}\right)$ will be Gamma distributed with (n) and (1/α) parameters.

Now, when $\hat{\alpha}_{MLE} = \frac{n}{T}$, we get

$$E(\hat{\alpha}_{MLE}) = \frac{n}{n-1}\alpha \quad \text{and} \quad var(\hat{\alpha}_{MLE}) = \frac{n^2\alpha^2}{(n-1)^2(n-2)}$$
…(5)

The maximum likelihood estimator $\hat{\alpha}_{MLE}$ is biased estimator but the following $\hat{\alpha}$ is unbiased estimator

$$\hat{\alpha} = \frac{n-1}{n}\hat{\alpha}_{MLE} = \frac{n-1}{T}$$
…(6)

i.e.; $E(\hat{\alpha}) = \alpha$ and $var(\hat{\alpha}) = \frac{\alpha^2}{n-2}$.

Furthermore,

$$var(\hat{\alpha}) = \frac{\alpha^2}{n-2} < var(\hat{\alpha}_{MLE}).$$

Thus, $\hat{\alpha}$ is a better estimator of α than $\hat{\alpha}_{MLE}$.

Therefore, we used $\hat{\alpha}$ estimator above in preliminary test Shrunken estimator (3) as a usual estimator in the next section.

2. Preliminary Test Shrunken Estimator

Recall the estimator which is defined in (3), to estimate the shape parameter α of pareto distribution as:

$$\tilde{\alpha}_{PT} = \begin{cases} \psi_1(\hat{\alpha})\hat{\alpha} + (1-\psi_1(\hat{\alpha}))\alpha_0 & , \text{if } \hat{\alpha} \in R, \\ \psi_2(\hat{\alpha})\hat{\alpha} + (1-\psi_2(\hat{\alpha}))\alpha_0 & , \text{if } \hat{\alpha} \notin R. \end{cases}$$

where $\hat{\alpha}$ is unbiased estimator defined in (6), $\psi_i(\hat{\alpha})$, i = 1, 2 is a Shrinkage weight factors such that $0 \le \psi_i(\cdot) \le 1$ and it may be a function of $\hat{\alpha}$ or a constant as well as R is the pretest region for testing the hypothesis $H_0:\alpha = \alpha_0$ against $H_A:\alpha \ne \alpha_0$ with level of significance Δ using test statistic $T(\hat{\alpha}/\alpha_0) = \frac{2(n-1)}{\hat{\alpha}}\alpha_0$ See [2].

i.e.; $R = \left[a \le \frac{2(n-1)\alpha_0}{\hat{\alpha}} \le b \right]$

or $R = \left[\frac{2(n-1)\alpha_0}{b}, \frac{2(n-1)\alpha_0}{a} \right]$ …(7)

where a and b are the lower and upper $100(\Delta/2)$ percentile point of chi-square distribution with degree of freedom 2n.

Noted that, we put forward $\psi_1(\hat{\alpha}) = 0$ and $\psi_2(\hat{\alpha}) = k = e^{-\left(\frac{n+10}{n}\right)}$ in $\tilde{\alpha}_{PT}$.

The Expression of the proposed estimator $\tilde{\alpha}_{PT}$ is defined as

$$Bias(\tilde{\alpha}_{PT}/\alpha, R) = E(\tilde{\alpha} - \alpha)$$

$$= \int_R (\alpha_0 - \alpha) f(\hat{\alpha}) d\hat{\alpha} + \int_{\bar{R}} [k(\hat{\alpha} - \alpha_0) + (\alpha_0 - \alpha)] f(\hat{\alpha}) d\hat{\alpha}$$

where $\bar{R}$ is the complement region of R in real space and $f(\hat{\alpha})$ is a p.d.f. of $\hat{\alpha}$ which has been derived as below :

$$f(\hat{\alpha}) = \begin{cases} \dfrac{\left[\frac{(n-1)\alpha}{\hat{\alpha}}\right]^{n+1} e^{-\frac{(n-1)\alpha}{\hat{\alpha}}}}{\Gamma(n)\ (n-1)\alpha} & \text{for } \hat{\alpha} > 0 \\ 0 & \text{o.w.} \end{cases}$$

…(8)

We conclude,

$$Bias(\tilde{\alpha}_{PT}/\alpha, R) = \alpha\{(\zeta-1)J_0(a^*, b^*) + (1-k)(\zeta-1) - (n-1)kJ_1(a^*, b^*) - (1-k)\zeta J_0(a^*, b^*) + J_0(a^*, b^*)\}$$

…(9)

where $J_l(a^*, b^*) = \int_{a^*}^{b^*} y^{-l} \frac{y^{n-1}e^{-y}}{\Gamma(n)} dy, l = 0, 1, 2$, …(10)

also $\zeta = \frac{\alpha_0}{\alpha}, a^* = \zeta^{-1} \cdot a,\ b^* = \zeta^{-1} \cdot b$ and $y = \frac{(n-1)\alpha}{\hat{\alpha}}$ …(11)

we denote to the Bias Ratio as B(·) which is defined as

$$B(\tilde{\alpha}_{PT}) = \frac{Bias(\tilde{\alpha}_{PT}/\alpha, R)}{\alpha} \qquad \ldots(12)$$

The expression of the Mean Squared Error (MSE) of $\tilde{\alpha}_{PT}$ is

$$MSE(\tilde{\alpha}_{PT}/\alpha, R) = E(\tilde{\alpha}-\alpha)^2$$

$$= \alpha^2\{(\zeta-1)^2 J_0(a^*,b^*) + \frac{k^2}{n-2} + (\zeta-1)^2(k-1)^2 - k^2[(n-1)^2 J_2(a^*,b^*) - 2(n-1)\zeta J_1(a^*,b^*) + \zeta^2 J_0(a^*,b^*)] - 2k(\zeta-1)[(n-1)J_1(a^*,b^*) - \zeta J_0(a^*,b^*)] - (\zeta-1)^2 J_0(a^*,b^*)\} \qquad \ldots(13)$$

The expression of Efficiency of $\tilde{\alpha}_{PT}$ relative to the $\hat{\alpha}$ denoted by R.Eff($\tilde{\alpha}_{PT}$/α,R) is defined as below:

$$R.Eff(\tilde{\alpha}/\alpha, R) = \frac{MSE(\hat{\alpha})}{MSE(\tilde{\alpha}_{PT}/\alpha, R)}$$; see [13], [1] and [2] …(14)

3. Conclusions and Numerical Results

The computations of the statistical indecators Relative Efficiency [R.Eff(·)] and Bias Ratio [B(·)] expressions were used for the considered testimators $\tilde{\alpha}$. These computations were performed [using Mat.LAB.program] for the constants Δ = 0.01,0.05,0.1, n = 4,6,8,10,12, 16,20,30 and ζ = 0.25(0.25)2. Some of these computations are displayed in attached table for some samples of these constants. The observation mentioned in the table leads to the following results:

i. The Relative Efficiency [R.Eff(·)] of $\tilde{\alpha}$ are adversely proportional with small value of Δ especially when ζ = 1, i.e. Δ = 0.01 yield highest efficiency

ii. The Relative Efficiency [R.Eff(·)] of $\tilde{\alpha}$ has maximum value when $\alpha=\alpha_0$(ζ=1), for each n, Δ, and decreasing otherwise (ζ≠1). This feature shown the important usefulness of prior knowledge which given higher effects of proposed estimator as well as the important role of shrinkage technique and its philosophy.

iii. Bias ratio [B(·)] of $\tilde{\alpha}$ increases when ζ increases.

iv. Bias ratio [B(·)] of $\tilde{\alpha}$ are reasonably small when $\alpha=\alpha_0$ for each n, Δ, and increases otherwise. This property shown that the proposed estimator $\tilde{\alpha}$ is very close to unbias property especially when $\alpha=\alpha_0$.

v. The Relative efficiency [R.Eff(·)] of $\tilde{\alpha}$ decreases function with increases value of n, for each Δ, ζ. This property shown that the proposed estimator reduce the cost of random sample size, and then overall sample size saved.

vi. The Effective Interval [the value of ζ that makes R.Eff.(·) greater than one] using proposed estimator $\tilde{\alpha}$ is [.75,1.5]. Here the pretest criterion is very important for guarantee that prior information is very closely to the actual value and prevent it faraway from it, which get optimal effect of the considered estimator to obtain high efficiency.

vii. The considered estimator $\tilde{\alpha}$ is better than the classical estimator especially when $\alpha\approx\alpha_0$, which is given the effective of $\tilde{\alpha}$ and important weight of prior knowledge as well as the increment of efficiency may be reach to tens time.

viii. The proposed estimator $\tilde{\alpha}$ has smaller MSE than some existing estimators introduced by authors, see for examples [4] and [8].

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Table (1)

Shown Bias Ratio [B(·)] and Relative Efficiency [R.Eff(·)] of $\tilde{\alpha}$ w.r.t. Δ, n and ζ

			ζ					
Δ	**n**	**R.Eff. Bias**	**0.25**	**0.75**	**1**	**1.25**	**1.75**	**2**
0.01	4	R.Eff(·) B(·)	0.941 (– 0.729)	8.319 (– 0.244)	1.69×10^{3} (3187×10^{-3})	7.85 (0.252)	0.887 (0.751)	0.5 (1)
	8	R.Eff(·) B(·)	0.369 (– 0.671)	3.203 (– 0.224)	138.926 (0.012)	2.535 (0.255)	0.295 (0.751)	0.167 (1)
	16	R.Eff(·) B(·)	0.195 (– 0.602)	1.709 (– 0.198)	40.501 (0.02)	1.077 (0.257)	0.127 (0.75)	0.072 (0.999)
	20	R.Eff(·) B(·)	0.162 (– 0.583)	1.398 (– 0.193)	31.984 (0.019)	0.853 (0.065)	0.099 (0.75)	0.056 (0.998)
0.05	4	R.Eff(·) B(·)	0.944 (– 0.727)	8.568 (– 0.241)	1.393×10^{3} (5894×10^{-3})	7.725 (0.254)	0.885 (0.751)	0.5 (1)
	8	R.Eff(·) B(·)	0.369 (– 0.671)	3.342 (– 0.22)	128.726 (0.016)	2.481 (0.258)	0.295 (0.751)	0.167 (0.997)
	16	R.Eff(·) B(·)	0.195 (– 0.602)	1.68 (– 0.2)	37.736 (0.015)	1.124 (0.251)	0.128 (0.747)	0.072 (0.992)
	20	R.Eff(·) B(·)	0.162 (– 0.583)	1.378 (– 0.194)	27.294 (0.011)	0.927 (0.243)	0.1 (0.745)	0.056 (0.991)