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Some Generalizations of Continuity Functions

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الخلاصة

في هذا البحث عرفنا ودرسنا تعميمات جديدة من الدوال المستمرة سميناها الدوال المستمرة الضعيفة (المغلقة، القوية) من النمط-ω. واهم الخواص التي درست هي: (أ) أذا كان $f : X \to Y$ دالة مستمرة ضعيفة (مغلقة، قوية) من النمط-ω . فان لآي مجموعه $A \subset X$ و آي $B \subset Y$ الدوال المقصورة $f|_A : A \to Y$ و $f_B : f^{-1}(B) \to B$ تكون دوال مستمرة ضعيفة (مغلقة، قوية) من النمط-ω. (ب) المقارنة بين مختلف أشكال تعميمات الدوال المستمرة. (ج) العلاقة بين تركيب مختلف أشكال تعميمات الدوال المستمرة. اضافة لذلك وسعنا التعميمات اعلاه وسميناها الدوال المستمرة الضعيفة (المغلقة، القوية) من النمط-ω على الاغلب. كذلك اعطينا وبرهنا العديد من النتائج المتعلقة بها.

Abstract

In this paper we define and study new generalizations of continuous functions namely, ω-weakly (resp., ω-closure, ω-strongly) continuous and the main properties are studies: (a) If $f : X \to Y$ is ω-weakly (resp., ω-closure, ω-strongly) continuous, then for any $A \subset X$ and any $B \subset Y$ the restrictions $f|_A : A \to Y$ and $f_B : f^{-1}(B) \to B$ are ω-weakly (resp., ω-closure, ω-strongly) continuous. (b) Comparison between diferent forms of generalizations of continuous functions. (c) Relationship between compositions of deferent forms of generalizations of continuous functions. Moreover, we expanded the above generalizations and namely almost ω-weakly (resp., ω-closure, ω-strongly) continuous functions and we state and prove several results concerning it.

1. Introduction and Notations.

Continuity functions are a fairly old concept studied by many mathematicians and first considered by M. Frechet (1) in 1910. In this paper we introduce some new generalizations of continuous functions and expanded these generalizations.

ω, denotes the cardinal number of integers. For a subset A of a spaces X, the closure of A denoted by cl(A). For other notions or notations not defined here we follow closely N. Bourbaki (2).

2. Basic Definitions.

Definition 2.1 (3, 4, and 5)

A function $f : X \to Y$ is called weakly (resp., closure, strongly) continuous at a point $x \in X$ if given any open set V containing f(x) in Y, there exists an open set U containing x in X such that $f(U) \subseteq cl(V)$ (resp., $f(cl(U)) \subseteq cl(V)$, $f(cl(U)) \subseteq V$).

If this condition is satisfied at each point $x \in X$, then f is said to be weakly (resp., closure, strongly) continuous.

Definition 2.2 (2)

A point x of a space X is called a condensation point of the set $A \subseteq X$ if every nbd of the point x contains an uncountable subset of this set.

Definition 2.3 (6)

A subset of a space X is called ω-closed if it contains all its condensation points. The complement of a ω-closed set is called ω-open set. Also the ω-closed of a set A is the intersection of all ω-closed sets which contains A, and denoted by $cl^{\omega}A$. i.e., $cl^{\omega}A = \cap\{F: F \text{ is } \omega\text{-closed and } A \subseteq F\}$, then A is ω-closed iff $A = cl^{\omega}A$.

Observe that A is ω-open iff for every $x \in A$ there is an open nbd U of x such that U-A is countable.

3. Basic Results.

The first new concepts in this paper are given now.

Definition 3.1

A function $f : X \to Y$ is called ω-weakly (resp., ω-closure, ω-strongly) continuous, if for each point $x \in X$ and every open set V of f(x) in Y, there exists an open set U containing x in X such that $f(U) \subseteq cl^{\omega}(V)$ (resp., $f(cl^{\omega}(U)) \subseteq cl^{\omega}(V)$, $f(cl^{\omega}(U)) \subseteq V$).

Definition 3.2

A space X is called ω-Urysohn if for every $x \neq y \in X$, there exists an open set U containing x and an open set V containing y such that $cl^{\omega}(U) \cap cl^{\omega}(V) = \phi$.

Clearly $cl(A) \subseteq cl^{\omega}A$, but not equal as it is shown in the next example.

Example 3.3

Let (IR, τ_{cof}) be the cofinite topology on IR, then every finite subset of IR is closed, but the ω-closure of every non empty set is IR.

It is well-known that if $f : X \rightarrow Y$ is continuous, then for any $A \subset X$ and any $B \subset Y$ the restrictions $f|_A : A \rightarrow Y$ and $f_B : f^{-1}(B) \rightarrow B$ are continuous, this is still the case in ω-weakly (resp., ω-closure, ω-strongly) continuous, as it is shown in the next theorem.

Theorem 3.4

If $f : X \rightarrow Y$ is ω-weakly (resp., ω-closure, ω-strongly) continuous, then for any $A \subset X$ and any $B \subset Y$ the restrictions $f|_A : A \rightarrow Y$ and $f_B : f^{-1}(B) \rightarrow B$ are ω-weakly (resp., ω-closure, ω-strongly) continuous.

Proof: Let $x \in X$ and let V be any open set containing f(x) in Y. Since $A \subset X$, then $x \in X$, since f is ω-weakly (resp., ω-closure, ω-strongly) continuous, there is an open set U containing x in X such that $f(U) \subseteq cl^{\omega}(V)$ (resp., $f(cl^{\omega}(U)) \subseteq cl^{\omega}(V)$, $f(cl^{\omega}(U)) \subseteq V$). Also $A \cap U$ is an open set containing x in A such that $A \cap U \subseteq U$ and $cl^{\omega}(A \cap U) \subseteq cl^{\omega}(U)$, so that $f(A \cap U) \subseteq f(U)$ and $f(cl^{\omega}(A \cap U)) \subseteq f(cl^{\omega}(U))$. Therefore, there is an open set $A \cap U$ containing x in A such hat $f(A \cap U) \subseteq cl^{\omega}(V)$ (resp., $f(cl^{\omega}(A \cap U)) \subseteq cl^{\omega}(V)$, $f(cl^{\omega}(A \cap U)) \subseteq V$). Thus $f|_A$ is ω-weakly (resp., ω-closure, ω-strongly) continuous.

The proof of $f_B : f^{-1}(B) \rightarrow B$ is ω-weakly (resp., ω-closure, ω-strongly) continuous similar to the proof $f|_A : A \rightarrow Y$, so it is omitted.

Also it is will-known that if $f : X \rightarrow Y$ is continuous, then $f_{f(X)} : X \rightarrow f(X)$ is continuous. This is not the case in ω-weakly (ω-closure) continuous even over a ω-Urysohn space as it is shown in the next example, but it is true for ω-strongly continuous as it is shown in theorem (3.6).

Example 3.5

Let P be the upper half of plane and L be the x-axis. Let $X = P \cup L$. If τ_{hdis} is the half disc topology on X and τ_r be the relative topology that X inherits by virtue of being a subspace of IR^2. The identity function $f : (X, \tau_r) \rightarrow (Y, \tau_{hdis})$ is ω-weakly (ω-closure) continuous but not continuous. And $f : (L, \tau_r) \rightarrow (X, \tau_{hdis})$ is ω-weakly (ω-closure) continuous, but $f : (L, \tau_r) \rightarrow (L, \tau_{hdis})$ is not ω-weakly (ω-closure) continuous.

Theorem 3.6

Let $f : X \to Y$ be ω-strongly continuous, then $f_{f(X)} : X \to f(X)$ is ω-strongly continuous.

Proof: Let $x \in X$ and let V be any open set containing f(x) in f(X), also in Y because $f(X) \subseteq Y$. Since f is ω-strongly continuous, there is an open set U containing x in X such that $f(cl^{\omega}(U)) \subseteq V$, hence $f_{f(X)}$ is ω-strongly continuous.

Now we will compare between diferent forms of generalizations continuity.

Theorem 3.7

Let $f : X \to Y$ be a ω-strongly continuous. Then f is continuous.

Proof: Let $x \in X$ and let V be any open set containing f(x) in Y. Since f is ω-strongly continuous, there is an open set U containing x in X such that $f(cl^{\omega}(U)) \subseteq V$. Since $U \subseteq cl^{\omega}(U)$. Then $f(U) \subseteq f(cl^{\omega}(U))$, therefore $f(U) \subseteq V$. Hence f is continuous.

The converse of the above theorem is not true, as it is shown in the next example.

Example 3.8

Let (IR, τ) where τ is the topology with basis whose members are of the form (a, b) and (a, b)-N, $N=\{1/n ; n \in Z^{+}\}$. Let $f : (IR,\tau) \to (IR,\tau)$, $f(x)=x$, then f is continuous but not ω-strongly continuous.

Theorem 3.9

Let $f : X \to Y$ be a continuous, then f is ω-closure continuous.

Proof: Let $x \in X$ and let V be any open set containing f(x) in Y. Since f is continuous, there is an open set U containing x in X such that $f(U) \subseteq V$. Hence $cl^{\omega}f(U) \subseteq cl^{\omega}(V)$. To show, $f(cl^{\omega}(U)) \subseteq cl^{\omega}(f(U))$, if $y \notin cl^{\omega}(f(U))$ there is nbd V_1 of y such that $V_1 \cap f(U)$ countable, also $f^{-1}(V_1)$ is a nbd for some $x \in f^{-1}(y)$ such that $f^{-1}(V_1) \cap U$ countable, then $x \notin cl^{\omega}(U)$ and $f(x)=y \notin f(cl^{\omega}(U)$. Therefore $f(cl^{\omega}U) \subseteq cl^{\omega}(V)$. Hence f is ω-closure continuous.

The converse of the above theorem is not true, as it is shown in the next example.

Example 3.10

Let X=[0, 1] with topology τ_{cof} consisting of the empty set together with all sets whose complements are finite, let Y=[0, 1] with topology τ_{coco} consisting of the empty set together with all sets whose complements are countable. Let $f : (X,\tau_{cof}) \to (Y,\tau_{coco})$ be the identity function, then f is ω-closure continuous since for every nonemty open set U in Y, $cl^{\omega}U=Y$. It is clear that for every $x \in X$, f is not continuous at x. Hence f is not continuous.

Theorem 3.11

Let $f : X \to Y$ be ω-closure continuous, then f is ω-weakly continuous.

Proof: Let $x \in X$ and let V be an open set containing f(x) in Y. Since f is ω-closure continuous, there is an open set U containing x in X such that $f(cl^{\omega}(U)) \subseteq cl^{\omega}(V)$, since $U \subseteq cl^{\omega}(U)$, then $f(U) \subseteq f(cl^{\omega}(U))$, therefore $f(U) \subseteq cl^{\omega}(V)$. Hence f is ω-weakly continuous.

The converse of the above theorem is not true, as it is shown in the next example.

Example 3.12

Let X=(1, 5) with topology $\tau_X=\{\phi, (3, 4), (3, 5), (1, 4), X\}$ and let Y=(-5, -1) with topology $\tau_Y=\{\phi, (-4, -3), (-2, -1), (-5, -3), (-4, -3)\cup(-2, -1), (-5, -3)\cup(-2, -1), (-4, -1), Y\}$. Define $g : (X, \tau_X) \to (Y, \tau_Y)$, by g(x)=-x. Then g is ω-weakly continuous but not ω-closure continuous.

Therefore, ω-strongly continuous ⇒ continuous ⇒ ω-closure continuous ⇒ ω-weakly continuous, but not conversely.

It is well-known that the composition of continuous function is continuous. Similar results hold for ω-closure and ω-strongly continuous but it is not true for ω-weakly continuous.

Theorem 3.13

Let $f : X \to Y$ be ω-strongly continuous and let $g : Y \to Z$ be ω-strongly continuous. Then $gof : X \to Z$ is ω-strongly continuous.

Proof: Let $x \in X$ and let W open set containing (gof)(x) in Z, since g is ω-strongly continuous, there is an open set V containing f(x) in Y such that $g(cl^{\omega}(V)) \subseteq W$. Since f is ω-strongly continuous, there exists an open set U of x in X such that $f(cl^{\omega}(U)) \subseteq V$, since $V \subseteq cl^{\omega}(V)$, then $f(cl^{\omega}(U)) \subseteq cl^{\omega}(V)$, so $g(f(cl^{\omega}(U))) \subseteq g(cl^{\omega}(V))$ and $(gof)(cl^{\omega}(U)) \subseteq g(cl^{\omega}(V))$. Therefore, there is an open set U containing x in X such that $(gof)(cl^{\omega}(U)) \subseteq W$ and gof is ω-strongly continuous.

The proofs of next theorems are similar to that proof of theorem (3.13) and thus will be omitted.

Theorem 3.14

Let $f : X \to Y$ be ω-closure continuous and let $g : Y \to Z$ be ω-closure continuous. Then $gof : X \to Z$ is ω-closure continuous.

Theorem 3.15

Let $f : X \to Y$ be ω-closure continuous and let $g : Y \to Z$ be ω-strongly continuous. Then $gof : X \to Z$ is ω-strongly continuous.

Theorem 3.16

Let $f : X \to Y$ be ω-weakly continuous and let $g : Y \to Z$ be ω-strongly continuous. Then $gof : X \to Z$ is continuous.

Theorem 3.17

Let $f : X \to Y$ be ω-weakly continuous and let $g : Y \to Z$ be ω-closure continuous. Then $gof : X \to Z$ is ω-weakly continuous.

Theorem 3.18

Let $f : X \to Y$ be continuous and let $g : Y \to Z$ be ω-weakly continuous. Then $gof : X \to Z$ is ω-weakly continuous.

The next example shows that the continuity of f in last theorem can not be weakened into ω-closure continuous, and it also shows that the composition of ω-weakly continuous is not to be ω-weakly continuous.

Example 3.19

In example (3.12) it is show that g is ω-weakly continuous but not ω-closure continuous. Define $f : (IR, \tau_u) \to (X, \tau_X)$, where τ_u is the usual topology on IR by $f(x=\text{rational}) = \frac{5}{2} + \frac{1}{\pi}\tan^{-1} x$, $f(x=\text{irrational}) = \frac{9}{2} + \frac{1}{\pi}\tan^{-1} x$. Then f is ω-closure continuous but not continuous, and gof is not ω-weakly continuous.

4. Main Results.

The second new concepts in this paper are given now.

Definition 4.1

A point x of a space X is called almost condensation point of the set $A \subseteq X$ iff $cl^{\omega}(U) \cap A \neq \phi$ for every open set U containing x. The set of all almost condensation points of A is called almost ω-closure of A and denoted by $al^{\omega}(A)$. A subset A of a space X is called almost ω-closed iff $A = al^{\omega}(A)$. The complement of almost ω-closed set is called almost ω-open. Similarly, the almost ω-interior of a set A in X and denoted by $int^{\omega}(A)$ is $\{x \in X : cl^{\omega}(U) \subseteq A$ for some open set U containing $x\}$ i.e., $al^{\omega}(U) \subseteq A$ for some open set U containing x. A subset A of a space X is called almost ω-open iff $A = int^{\omega}(A)$. Clearly every almost ω-closed (almost ω-open) is closed (open).

Definition 4.2

A function $f : X \to Y$ is called almost ω-weakly (resp., ω-closure, ω-strongly) continuous, if for each point $x \in X$ and every open set V of f(x) in Y, there exists an open set U containing x in X such that $f(U) \subseteq al^{\omega}(V)$ (resp., $f(al^{\omega}(U)) \subseteq al^{\omega}(V)$, $f(al^{\omega}(U)) \subseteq V$).

Clearly $cl(A) \subseteq al^{\omega}(A)$

By analogue of definition closure compact in (7) we will generalization this definition as follows.

Definition 4.3

A space X is called ω-closure compact if for every open cover of X, there exists a finite subcollection whose ω-closures cover X.

Theorem 4.4

An almost ω-closed subset of ω-closure compact space is ω-closure compact.

Proof: Let A be almost ω-closed subset of ω-closure compact space X and let $\mathcal{A}$ be an open cover of A. Since X\A is almost ω-open, then for each $x\in X\setminus A$ there exists an open set U_x such that $cl^{\omega}(U_x)\subseteq X\setminus A$. Thus $\mathcal{B}=\mathcal{A}\cup\{U_x : x\in X\setminus A\}$ is an open cover of X. Since X is ω-closure compact, there exists a finite subcollection $\mathcal{C}$ of $\mathcal{B}$ whose ω-closures cover X. Hence $\mathcal{C}\cap\mathcal{A}$ is a finite subcollection of $\mathcal{A}$ whose ω-closures cover A, proving that A is ω-closure compact.

Corollary 4.5

Every clopen subset of a ω-closure compact space is ω-closure compact.

Theorem 4.6

Let $f : X\to Y$. Then the following conditions are equivalent:

(a) $f(al^{\omega}(A))\subseteq cl(f(A))$, for every $A\subset X$.

(b) The inverse image of every closed set is almost ω-closed.

(c) The inverse image of every open set is almost ω-open.

(d) f if almost ω-strongly continuous.

Proof: (a)⇒(b) Let B be a closed subset of Y and let $A=f^{-1}(B)$. Let $x\in al^{\omega}(A)$, then $f(x)\in f(al^{\omega}(A))\subseteq cl(f(A))\subseteq cl(B)=B$, therefore $x\in f^{-1}(B)=A$. Thus $al^{\omega}(A)=A$.

(b)⇒(c) Let O be an open subset of Y and thus Y\O is closed, then $f^{-1}(Y\setminus O)=X\setminus f^{-1}(O)$ is almost ω-closed and thus $f^{-1}(O)$ is almost ω-open.

(c)⇒(d) Let $x\in X$ and let V be an open set of f(x) in Y. By hypothesis, it follows that $f^{-1}(V)$ is almost ω-open and thus there exists an open set U of x such that $al^{\omega}(U)\subseteq f^{-1}(V)$. Thus $f(al^{\omega}(U))\subseteq V$, proving that f is almost ω-strongly continuous.

(d)⇒(a) Let $f : X\to Y$ be almost ω-strongly continuous and let $x\in al^{\omega}(A)$. Let V be an open set containing f(x). By almost ω-strongly continuous of f there exists an open set U containing x such that $f(al^{\omega}(U))\subseteq V$. Therefore $al^{\omega}(U)$ meets A and thus V meets f(A). Hence $f(x)\in cl(A)$.

Corollary 4.7

Let $f : X\to Y$ be almost ω-strongly continuous where Y is T_1-space. Then f has almost ω-closure point inverses.

The hypothesis in the above corollary that Y is a T_1-space can't be weakened into T_0-space as shown in the next example.

Example 4.8

Let (IR, τ) where τ is the lower limit topology and (IR, τ_r) where τ_r is the right ray topology. Define f : (IR, τ)$\rightarrow$(IR, τ_r) as follows f(x)=0, for all x<0, f(x)=1, for all x$\geq$0. Then f is almost ω-strongly continuous, and {0} is compact but $f^{-1}(0)=(-\infty, 0)$ is not even closed.

Theorem 4.9

Let f : X$\rightarrow$Y be almost ω-closure continuous. Then the following holds:

(a) $f(al^{\omega}(A))\subseteq al^{\omega}(f(A))$, for every $A\subset X$.

(b) The inverse image of every almost ω-closed set is almost ω-closed.

(c) The inverse image of every almost ω-open set is almost ω-open.

Proof: (a) Let f : X$\rightarrow$Y be almost ω-closure continuous and let $x\in al^{\omega}(A)$. Let V be an open set containing f(x) in Y. By almost ω-closure continuous of f there exists an open set U containing x such that $f(al^{\omega}(U))\subseteq al^{\omega}(V)$. Therefore, $al^{\omega}(U)$ meets A and thus $al^{\omega}(V)$ meets f(A). Hence $f(x)\in al^{\omega}(f(A))$.

(b) Let B be almost ω-closed set of Y and let $A=f^{-1}(B)$. Let $x\in f^{-1}(B)$, by part (a), $f(x)\in f(al^{\omega}(A))\subseteq al^{\omega}(f(A))\subseteq al^{\omega}(B)=B$. Therefore $x\in f^{-1}(B)=A$. Thus $al^{\omega}(A)=A$.

(c) Let O be almost ω-open subset of Y and thus Y\O is almost ω-closed. Then $f^{-1}(Y\backslash O)=X\backslash f^{-1}(O)$ is almost ω-closed and thus $f^{-1}(O)$ is almost ω-open.

Corollary 4.10

Let f : X$\rightarrow$Y be almost ω-closure continuous where Y is a Urysohn space. Then f has almost ω-closure point inverses.

The hypothesis in the above corollary that Y is a Urysohn space can't be weakened into T_1-space as shown in the next example.

Example 4.11

Let (IR, τ_{cof}) where τ_{cof} is the cofinite topology. Let f : (IR, τ_{cof})$\rightarrow$ (IR, τ_{cof}) be the identity function. Then f is almost ω-closure continuous, but $f^{-1}(\{0\})$ is not almost ω-closed.

Theorem 4.12

Let f : X$\rightarrow$Y be almost ω-weakly continuous. Then the following holds:

(a) $f(cl(A))\subseteq al^{\omega}(f(A))$, for every $A\subset X$.

(b) The inverse image of every almost ω-closed set is closed.

(c) The inverse image of every almost ω-open set is open.

Proof: (a) Let $f : X \to Y$ be almost ω-weakly continuous and let $x \in cl(A)$. Let V be an open set containing f(x) in Y. By almost ω-weakly continuous of f there exists an open set U containing x such that $f(U) \subseteq al^{\omega}(V)$. Therefore, U meets A and thus $al^{\omega}(V)$ meets f(A). Hence $f(x) \in al^{\omega}(f(A))$.

(b) Let B be almost ω-closed set of Y and let $A=f^{-1}(B)$. Let $x \in cl(A)$, by part (a), $f(x) \in f(cl(A)) \subseteq al^{\omega}(f(A)) \subseteq al^{\omega}(B)=B$. Therefore $x \in f^{-1}(B)=A$. Thus $cl(A)=A$.

(c) Let O be almost ω-open subset of Y and thus Y\O is almost ω-closed. Then $f^{-1}(Y \setminus O)=X \setminus f^{-1}(O)$ is closed and thus $f^{-1}(O)$ is open.

Corollary 4.13

Let $f : X \to Y$ be almost ω- weakly continuous where Y is a Urysohn space. Then f has closed point inverses.

The hypothesis in the above corollary that Y is a Urysohn space can't be weakened into T_1-space as shown in the next example.

Example 4.14

Let (IR, τ_u) where τ_u is the usual topology and (IR, τ_{coco}) where τ_{coco} is the cocountable topology. Define f : (IR, τ_u)→(IR, τ_{coco}) as follows f(x=rational)=0, f(x=irrational)=1. Then f is almost ω-weakly continuous, and {0} is compact but $f^{-1}(\{0\})$ is neither closed nor ω-closure compact.

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