# Using Sensitivity Analysis in Linear Programming with Practical Physical Applications

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**Abstract**

Linear programming currently occupies a prominent position in various fields and has wide applications, as its importance lies in being a means of studying the behavior of a large number of systems as well. It is also the simplest and easiest type of models that can be created to address industrial, commercial, military and other dilemmas. Through which to obtain the optimal quantitative value. In this research, we dealt with the post optimality solution, or what is known as sensitivity analysis, using the principle of shadow prices. The scientific solution to any problem is not a complete solution once the optimal solution is reached. Any change in the values of the model constants or what is known as the inputs of the model that will change the problem of linear programming and will affect the optimal solution, and therefore we need a method that helps us to stand on the impact of changing these constants on the optimal solution that has been reached. General concepts about the binary model and some related theories have also been addressed. By analyzing the sensitivity, we relied on real data for a company that transports crude oil and its derivatives. The mathematical model was formulated for it and the optimal solution was reached using the software. Ready-made sop WINQSB and then calculate the shadow price values for the binding constraints, in addition to what

**Keyword:** Linear programming, dual model, shadow prices, sensitivity analysis, fuzzy numbers .

**المستخلص:**

تحتل البرمجة الخطية في الوقت الحاضر مركزاً مرموقاً في مجالات مختلفة ولها تطبيقات واسعة , اذ تكمن اهميتها بكونها وسيلة لدراسة سلوك عدد كبير من الانظمة كذلك فانها تعد ابسط واسهل انواع النماذج التي يمكن انشاؤها لمعالجة معضلات صناعية وتجارية وعسكرية واخرى ,فهي مجموعة من الاساليب الفنية التي يمكن بواسطتها الحصول على المقدار الكمي الامثل.في هذا البحث تعاملنا مع الحل ما بعد الامثلية post optimality او ما يعرف بتحليل الحساسية Sensitivity Analysis باستخدام مبدء اسعار الظل Shadow Prices ,ان الحل العلمي لاي مشكلة لايكون حلاً كاملاً بمجرد الوصول الى الحل الامثل , ان اي تغيير في قيم ثوابت النموذج او ما يعرف بمدخلات النموذج الذي سيغير من مشكلة البرمجة الخطية وسيؤثر على الحل الامثل وعليه نحن بحاجة الى اسلوب يساعدنا في الوقوف على اثر تغير هذه الثوابت على الحل الامثل الذ تم التوصل اليه , كذلك تم تناول مفاهيم عامة عن النموذج الثنائي وبعض النظريات المتعلقة بتحليل الحساسية واعتمدنا على بيانات حقيقية لشركة تنقل النفط الخام ومشتقاته وقد تم صياغة النموذج الرياضي لها والتوصل للحل الامثل باستخدام برنامج الحاسوب الجاهز winqsb ومن ثم حساب قيم اسعار الظل للقيود المستنفدةbinding constraints , اضافة لما ذكر اعلاه فقد استعرضنا في هذا البحث نموذج البرمجة الخطية في ظل البيئة الضبابية واستخدم فيه اسلوب جديد يعتمد على الاعداد الاولية prime numbers في حل معالم النموذج الضبابية.

**الكلمات المفتاحية:** البرمجة الخطية ,النموذج الثنائي,اسعار الظل, تحليل الحساسية,الاعداد الضبابية .

## 1. Introduction

The fact that linear programming is currently generally recognized as a helpful technique in operations research, management science, and other sciences is one of its benefits. Numerous businesses employ this kind of modeling to address a variety of real-world issues, including those relating to allocation issues, production mix issues, transportation issues, and challenges with planning production. The research problem's choice variables have linear relationships, which is the Linear Programming Problem (LPP) [1] the decision variables can be constrained to a certain solution area by various constraints, and the linear objective function (for example) is determined by maximizing profits or minimizing costs. The binary or corresponding model, which has a helpful economic explanation and is frequently utilized in economic theory, is one of the intriguing aspects of linear programming. Theoretically significant, it also touches on the topic of sensitivity analysis in linear programming, and it is generally known that in linear programming, the best values of binary model variables are seen as the shadow prices (border values) of the coefficients on the right side of the constraints. On the basis of optimization, Simplex has been well developed because it involves less computational work and many studies and books, including [2], have used this technique.

The validity of shadow pricing and how they could be determined with or without pre-made programs were reviewed in 2000 by James K. [3]. He also discussed some of the well-known outcomes in this area. Furthermore, in 2005 Jan Staller [4] presented a study in which he demonstrated how to alter the ideal solution when the right side changed. He explained the origin of the issue, how his proposed solution used the pivoting algorithm method and the relationship with the interior-point method's post-optimization outcomes to establish the best change vector as a change in the amounts of resources that were accessible, [5]

## 2. Dual Modeling

The target function of the model or the value of the optimal profits are both equal to the value of the available economic resources, valued at shadow prices. Whether it seeks to maximize or reduce a particular function is related to the idea of the binary model that supports the original model. In the event that the binary model contains fewer constraints and variables than what is present in the original linear model of the problem, the binary model offers management a wealth of information that aids in decision-making, lowers calculations, and saves time and money [5] .

Since the primary objective of the original linear model was to maximize marginal profits, this marginal profit is the return on variable costs, or the difference between the selling price of goods and their variable expenses. And outputs (price of goods), and hence their profits. In light of this, since the original model seeks to maximize profits and is concerned with selecting the best production assortment (outputs) in light of constrained economic resources or inputs, and thus profits cannot be maximized without these resources. The binary model seeks to reduce these readily available resources that aid in generating profits and to determine the shadow prices of these resources under fundamental constraints that demand that the cost of the resources required for each product at its shadow prices be greater than the marginal profit for each product. The binary model therefore aims to identify and lower the shadow pricing of the readily available economic resources.

The binary model is built from the standard formula for the inequalities of the linear programming model, as shown in the table below in the form of matrices:

| Original modeling | Dual modeling |
|---|---|
| Maximize $z_p = c$ x<br>s.t<br>$A$x≤b<br>x ≥0 | Minimize $z_D = b^T \pi$<br>s.t<br>$A^T\ b \geq c^T$<br>Y ≥0 |

Such that

A: comparable to the technical coefficients matrix for a linear model with m rows and n columns c,

X: vectors of unknown variables with dimension n.

Y: vectors of unknown variables for the duality problem.

C: vector with n coefficients in binary modeling.

B: vector with m arbitrary values.

There are numerous connections between the outcomes of the initial model and the binary problem, and these connections are crucial for understanding the outcomes. As x represents any value alternative to the initial model, Paul A. Jensen & Jonathan F. Bard in 2002 [6] went as far as to regard these relationships as theorems with their proofs. And that is any viable answer for the binary model, x*, as well as, if any, the best answers for each of the two models discussed above.

**Theorem 1(weak duality)** [7]

If the acceptable solution is represented by x, $Z_{p(x)}$ is the value of the objective function that is (Maimize) for the original model, the acceptable solution is represented by, and z D (x) is the value of the objective function that is (Minimize) for the binary model, then z p (x) $\leq z_{D\,(x)}$ is the value of the objective function that is (Minimize) for the binary model, then

1- $z_p$ (x) $\leq z_D$(x). Hypothesis Ax $<$ b accepts the prototype's solution as correct.
2- The result of multiplying both sides by $\pi$ of the previous equation by is $Ax\pi \ \leq b\,\pi$.
3- The assumption $A\,\pi \ \geq \ c$ accepts the answer for the relevant model.
4- We obtain "Ax" by multiplying both sides of the previous equation by $\pi Ax \geq cx$.
5- Steps 2 and 4's quotients are added to obtain cx≤Ax$\pi$ ≤b$\pi$ or $Z_P$ (x)≤ $Z_D$ (x).

The above hypothesis leads to a number of practical relationships.

- The value of $Z_p$ (x) for any x is the lower bound of $Z_D$ $(\pi^*)$.
- The value of $Z_D(x)$for any $\pi$ is the upper bound of $Z_P(\pi^*)$.

If there are acceptable solutions for x and the original problem is unbounded, then the solution is no feasible for the binary problem $\pi$.

If there are acceptable solutions to $\pi$ and the binary form is unbounded, then the solution is no feasible for the original problem $x$.

**Theorem 2(Sufficient Optimality Criterion)** [7]

Assume that the binary model's objective function is $z_D$(x) and the prototype's objective function is $z_p$(x). If $x$ a pair of feasible solutions for the prototype and the binary model achieves $z_p(\hat{x}\ )$ =$z_D$ $(\hat{\pi})$ , then x is the prototype's optimal solution and is the binary model's optimal solution

1. Determine the optimization of the initial solution: $z_p$ $(\hat{x}\ ) \leq z_p(x\,*)$.

2. The accepted solution of the binary form for $z_p, z_p$ $(\hat{x}\ ) \leq z_p$ (π*).

3. Determine the optimization of the binary solution: $z_D$ (π* )≤$z_D$ $(\hat{\pi})$.

4. Adding up the above steps: $z_p(\hat{x}\ ) \leq z_p$(x*) ≤ $z_D$ (π*) ≤ $z_D$ $(\hat{\pi})$.

5. Suppose objective functions: $z_p(\hat{x}) = z_D(\hat{\pi})$.

6. Combining steps 4 and 5: $z_p(\hat{x}) = z_p(x^*) = z_D(\pi^*) = z_D(\hat{\pi})$.

So $\hat{x}, \hat{\pi}$ is optimal.

**We come out with some conclusions from the previous theorem:**

- If the two objective functions are equal, then both solutions that are acceptable for both the initial and binary models are optimal.

- If x* is an optimal solution for the prototype, there is a finite optimal solution for the binary model with the objective function $z_p(x*)$

- If π* is an optimal solution for the binary model, there is a finite optimal solution for the prototype with a target function $z_D(\pi^*)$.

**Theorem 4: (Strong Duality)** [7]

The values of the goal functions for the two models are equivalent if either the initial or binary model has a workable optimal solution. To put it another way, allow the basic model to have an ideal outcome x*= ($x_1$*,$x_2$*…$x_n$*), and the binary model has an optimal solution also π*= ($\pi_1$*,$\pi_2$*…$\pi_n$*), then $\sum_{i=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_j x_j^*$

**Theorem 5 :( Shadow Prices)** [7]

The standard approach of resolving the linear programming model the shadow prices and opportunity cost, which are defined as the lost profits for the best alternative that comes after the chosen alternative or as an expected theoretical value for the alternatives abandoned as a result of choosing a specific alternative, are two examples of information that the simplex method provides that the graphic method does not. Three different sorts of these fees include:

1. The cost of acquiring a unit of production resource from a source outside the production institution is one example of an external opportunity cost.

2. The internal opportunity cost, which is the rate of return the institution can get in exchange for helping to pay its fixed costs and earn a profit.

3. The total opportunity cost, which comprises the internal opportunity cost (the return obtained by the facility) and the external opportunity cost (the cost of getting the manufacturing

resource), is of obvious benefit in justifying the choice to add a new product to the production mix.

The idea of a resource's shadow price can be summed up as the rise or fall in the value of the objective function as a result of an improvement or deficiency increasing the quantity of that resource that is readily available by one unit will result in higher marginal profitability.

1. The indirect method (by-product method): The values that appear under the Slack Variables in the best possible simplex table for the (original) LPP [8] can be used to determine the shadow pricing.

2. Binary model approach: Shadow prices are produced by converting the initial issue of the LPP into a binary model, which primarily seeks to ascertain the shadow price of the available economic resources.

The rate of change in the objective function as a result of a change in the value of the resource ($bi$), sometimes referred to as the right side of the constraint, is the shadow price of a constraint, let it be $I(i)$. When the objective function in the initial model is identical to the objective function in the binary model, the value of the objective function in the best solution is represented as follows [2].

Since Z*=v*=$b^T$* and the initial solution is described as a non-degenerate one, this connection can be utilized to display the cost associated with $b_i$ at the ideal value (*). This idea states that the partial derivative of the function Z with respect to $b_i$, which is expressed as ∂Z/ (∂$b_i$)=$\pi$*, can be used to determine the value of the change in Z as a result of a change in the resource ($b_i$).

The price connected to the right-hand side of the constraint, also known as the shadow price, can be inferred from the definition above, which is credited to Paul Samuelson in 1965 [9]

## 3. Economic Interpretation of Shadow Prices

The simplex method's goal is to identify the fundamentally workable solution that employs the most economical technique. The whole cost, i.e., is represented by the binary model's objective function.

$$v=\pi^T b = \sum_{i=1}^{m} \pi_i b_i \quad ..... (1)$$

Since * indicates the implicit income to cover the direct costs (shadow price) for each resource in light of the initial basic variables, $I\ b_i$ is the compensation for the direct costs. Where $\pi_i b_i$ is the factorial of the simplex associated with the basic variables. $\sum_{j=1}^{n} c_j x_j$

By studying each row $(j)$ of the binary problem that corresponds to column $(j)$ of the initial problem and applying the aforementioned interpretation of the objective function and variables of the binary model, $v$, we can determine that each unit $j$ of activity (or product) in the initial problem $a_{ij}$ consumes a unit of resource $i$

The indirect implicit cost of the resource mix used or produced by one unit of activity $j$, as determined by the binary problem's use of shadow pricing (the constraint's left side), is known as:

$$\sum_{i=1}^{m} a_{ij}\pi_i \qquad \ldots. (2)$$

Since the direct cost per unit of activity in the binary problem is derived from constraint $c_j$the constraint j in the binary form

$$\sum_{i=1}^{m} a_{ij}\pi_i \leq c_j \quad \ldots. (3)$$

Can be thought of as including implicit indirect costs. For the materials consumed by activity $c_j$and must not exceed the direct expenses $c_j$ if these costs are exactly less than $c_j$, he is not compensated for participating in the activity $c_j$

$$\sum_{i=1}^{m} a_{ij}\pi_i > c_j \quad \ldots. (4)$$

On the other hand, the constraints in the binary form associated with the non-basic variables $x_N$, may fulfill the acceptable or unacceptable solution, and this means $\bar{C}$=$C_N$-$N^T$ π , as it can be $\bar{C}_j \geq 0$ then the constraint in the binary form is acceptable or $\bar{C}_j$< 0, then the restriction is not acceptable. From an economic point of view, $\bar{C}_j$< 0 means that activity j uses resources more economically than any other activity out of the sum of activities

## 4. Sensitivity Analysis

Sensitivity analysis is a technique that assesses the impact of changes in the decision model's inputs on its outputs. Through this technique, it is possible to examine changes in the values of the model constants and determine how much these constants can vary before the previously specified optimal solution becomes suboptimal. The greater the degree of sensitivity of the decision to change, the more these constants can vary before this happens. When one of the model's constants must be carefully estimated in order to avoid deviating from the ideal later, this takes additional time and effort. Post-optimization analysis is another name for sensitivity analysis [10]. The majority of real-world issues involve data that is not known for sure, for instance, the cost of raw materials may vary after the model is solved or the costs utilized may just be an educated guess as to what will be in the future [11]. The right-hand side of the constraint may change as a result of a rise or fall in the amount of resources on the market or transactions that may alter as a result of modifications to product specifications. Sensitivity analysis is crucial for a number of reasons [12]:

1. Considering the modifications to the model's parameters, the stability of the optimal solution might not be acceptable.

2. It is possible to modify the values of the constraints and the coefficients of the objective function to some extent at some costs; in this case, we are interested in the consequences of doing so as well as the associated costs.

3. Since the values of uncontrolled transactions may be approximations, it's critical to understand how much they change over time in order to maintain the best solution or produce more accurate estimates

**A. Insert (Add) A New Variable [15]:**

Sensitivity analysis is helpful in assessing whether it is feasible to add a new variable to the solution that has already been reached (without this variable) and whether or not doing so would have an impact on the optimization of the original solution [13].

The linear programming model is as follows: After determining the best solution, x=x*, let's add a new variable, $x_{n+1}$, with cost coefficient $c_{n+1}$ and technical coefficients $A_{n+1}$ about following:

$$\left.\begin{aligned} Minimize\ z = c^T x + c_{n+1} x_{n+1} \\ s.t \\ Ax + A_{n+1} x_{n+1} = b \end{aligned}\right\} \quad \ldots.\ (5)$$

$$x_i \geq 0 \,, x_{n+1} \geq 0$$

The accepted solution does not change when a new variable is added to the model since it becomes a non-essential variable and its value is zero at the lower bounds. However, the solution must be optimal because the cost of lowering$\bar{c}_{n+1}$ to correspond to the new variable $x_{n+1}$ is high. It might be negative, and in order to be sure the best answer was reached, we compute

$$\bar{c}_{n+1} = c_{n+1} - A_{n+1}^{T}\, \pi \quad ..... (6)$$

The previous solution, which had the variable $x_{n+1}$ equal to 0, is the best one if $\bar{c}_{n+1}$ _$c_{n+1} - A_{n+1}^{T} \geq 0$ , but if $\bar{c}_{n+1}$< 0, the solution can be improved by making the $x_{n+1}$variable basic.

The most typical scenario is when the new variable $x_{n+1}$ has lower and upper limits, but neither of them must be zero $l_{n+1} \leq x_{n+1} \leq u_{n+1}$. The maximum value is ∞.

**B .Entering (Adding) A New Entry** [13]**:**

If we assume that after finding the best solution, the need to add a new entry might arise due to a change in the environment of the producing institution, which led to a change in the specifications of some of the resources available and changed that product, it is obvious that the added restriction will not widen the solution space. On the other hand, the accepted one may be subtracted from it. As a result, the goal function that was achieved before the new constraint was introduced either stays the same or begins to deteriorate. Let's say the constraint that needs to be added has the form:

$$\left.\begin{array}{l} A_{m+1}x = b_{m+1}, \\ A_{m+1}x \leq b_{m+1}\ , \\ A_{m+1}x \geq b_{m+1} \\ \text{s.t } b_{m+1} \geq 0. \end{array}\right\} \quad ... (7)$$

Any of the aforementioned restrictions can be expressed in the following way by imposing various limits on the variable $x_{n+1}$:

$$A_{m+1}x + x_{n+1} \ = b_{m+1}, \quad .... (8)$$

If "$b_{m+1}$" then $0 \leq x_{n+1} \leq 0$ , if "$b_{m+1}$" $>$ then $0 \leq x_{n+1} \leq \infty$ and $0 \geq b_{m+1}$

Then $-\infty \leq x_{n+1} \leq 0$ When a new constraint is added, the cost $c_{n+1}$ associated with the variable $x_{n+1}$ is zero, but what if the solution x*=x is not acceptable? If so, the model is solved and enhanced [14]

**C. Change in the Right Side of the Constraints:**

It is important and necessary to be able to study the impact of changes that occur on the right side of the constraints, particularly those that determine what resources are available. Assume that we have the best solution for the linear programming model in the standard form and that the right side of the constraint (q) is given by the parameter (λ) and with a given value $b_q$: [15]

$$b(\lambda) = b + (\lambda - b_q)e_q \quad \text{..... (9)}$$

Where $e_q$ represents the column (q) in the neutral matrix. The updated basic solution continues to be the best option if the value λ of the parameter does not render the basic solution inacceptable.

he range of parameter values λ is on the right side of the constraint q, which keeps the basic solution acceptable and optimal, by finding the values of $x_{B\,(\lambda)}$ where

$$Bx_B(\lambda) = b(\lambda) \quad \text{.... (10)}$$

And $x_B(\lambda) \geq 0$ we get

$$x_B(\lambda) = B^{-1}b + (\lambda - b_q)B^{-1}e_q = x_B^* + (\lambda - b_q)B^{-1}e_q$$

Since $x_B$* represents the optimal solution when λ = $b_q$, and to reach the accepted solution it is required that $x_B$ (λ) ≥0, and the field of the parameter λ is calculated according to the formula below:

$$b_q + \max_{\beta_{iq}>0} \frac{-(x_B^*)_i}{\beta_{iq}} \leq \lambda \leq b_q + \min_{\beta_{iq}<0} \frac{-(x_B^*)_i}{\beta_{iq}} \quad \text{..... (11)}$$

Where $B_{iq}$ represents the element $(i, q)$ of the matrix $B^{-}$

## 5. Fuzzy Linear Programming

The concept of fuzzy logic is one of the forms of logic, as it is used in some expert systems and artificial intelligence applications. The Azerbaijani scientist Lutfi Zadeh invented this method in 1965 [9], where he developed it to use it as a better way to process data, but his theory did not receive attention until 1974, when it was used in Steam engine regulation, then evolved to be used in many scientific, engineering and other applications.

**Definition 1**: The pair set $\tilde{A}$, also known as the function $\tilde{A}=\{(\mu_{\tilde{A}}(x),x)\}$ of belonging to the fuzzy set X, $\mu_{\tilde{A}}(x):X\rightarrow[1,0]$ is a subset of the universal set. The value of the affiliation function of the element is called $\mu_{\tilde{A}}(x)$ the degree of affiliation [15].

-$\tilde{A}^c$ The complement of the fuzzy function $\tilde{A}$,- which is symbolized by it, and it is a fuzzy group that can be written $\mu_{\tilde{A}^c}(x)=1-\mu_{\tilde{A}}(x), \forall x\in X$ .

-Intersection of two fuzzy sets $\tilde{A}$ $B$, to define $\tilde{C}$ write as:

$$\mu_{\tilde{C}}(x)=\mu_{\tilde{A}\cap\tilde{B}}=\min(\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x)), \forall x\in X \quad \text{..... (12)}$$

- Union of two fuzzy sets $\tilde{A}$ $B$, s.t define $\tilde{C}$ write as :

$$\mu_{\tilde{C}}(x)=\mu_{\tilde{A}\cup\tilde{B}}=\max(\mu_{\tilde{A}}(x),\mu_{\tilde{B}}(x)), \forall x\in X \quad \text{..... (13)}$$

**Definition 2:** The fuzzy set $\tilde{A}$ is convex if the condition is set $x,y\in X,\lambda\in[0,1]$, [15].

$$\mu_{\tilde{A}}(\lambda x+(1-\lambda)y)\geq\min(\mu_{\tilde{A}}(x),\mu_{\tilde{A}}(y)) \quad \text{.....(14)}$$

**Definition 3**: fuzzy number is pair of functions $(u(r),v(r))$, $r\in[1,0]$ satisfy the following condition, [15].

1. $u(r)$ Is a definite decreasing function from the left lies within the interval [0, 1]?
2. $v(r)$ Is a definite decreasing function from the right lies within the interval [0, 1]?
3. $v(r)$ < $u(r)$ $r\in[1,0]$.

**Definition 4**: suppose that $\tilde{A}$ fuzzy number in trigonometric form (a, b, c) the affiliation function can be calculated $\mu_{\tilde{A}}(x)$ as [15],

$$\mu_{\tilde{A}}(x)=\frac{x-a}{b-a},\ x\in[a,b] \quad , \quad \mu_{\tilde{A}}(x)=\frac{c-x}{c-b}, x\in[b,c], \quad \mu_{\tilde{A}}(x)=0, x\notin[a,c] \quad \text{....(15)}$$

Fuzzy number in trigonometric form as function:

$$v(r)=\frac{c-rc}{c-b}, r\in[b/c,1],\ r\in[a/c,b/c]\ u(r)=\frac{cr-a}{b-a}, \quad \text{.... (16)}$$

## 6. Using the Prime Numbers in the Era to Create the Fuzzy Number $[k_1, k_2]$ [16]

**Definition 5:** Let it be the prime number $P_j(a) \geq 0, a \geq 0, j \in Z$ ∈ [a,∞) when $j \geq 0$ or [0, a) j<0, $P_j(a) \geq 0$ prime number from prime numbers set a > 0, [15].

1- The following list of a prime number's crucial characteristics is an overview:

$$P_0(0) = 0, P_1(0) = 0,, 1 = P_1(0)\ P_{-1}(1), = 0. \qquad \ldots.. (17)$$

2- $P_0(a) = a$ if a > 0 prime number, $P_0(a)$ not found if a > 0 non-prime number

3. $P_j(a) \leq P_k(a)$ if $j \leq k$, $P_j(a) < P_k(a)$, for all $j \in Z$, $j < k$ ,, $k \in Z$ .

$P_j(a) = P_j(a+1) = ... = P_j(a+l)$ 4- $j = 0,1,2,...$ for all ,$1 \leq l < P_{j+1}(a) - P_j(a)$ , $a \geq 0$ .

5- If $a \geq 0$ prime number. $P_j(a) = P_1(P_{j-1}(a)) == P_1(P_1(P_{j-2}(a))) = P_2(P_{j-2}(a)) = ... = P_{j-1}(P_1(a))$

**Definition 6:** let fuzzy number $\tilde{n}$ called tripe fuzzy number (k, n, l), $k \leq n \leq l, k, n, l \in Z$

Such that, [15].

$$l = \begin{cases} P_1(n), n \geq 0, \\ -P_{-1}(-n), n < 0, \end{cases} \quad k = \begin{cases} P_{-1}(n), n \geq 0, \\ -P_1(-n), n < 0, \end{cases} \qquad ... (18)$$

$P_1(\cdot), P_{-1}(\cdot)$ The preceding and subsequent (initial) number of the $n < 0, -n, n \geq 0$

According to the above formula with the use of the linear affiliation function:

$$\mu_{\tilde{n}}(x) = \frac{x-k}{n-k}, x \in [k, n], \mu_{\tilde{n}}(x) = \frac{l-x}{l-n}, x \in [n, l], \mu_{\tilde{n}}(x) = 0, x \notin [k, l] \quad ... (19)$$

Using the definition of fuzzy integers, the traditional arithmetic operations (addition, subtraction, multiplication and division) for any two fuzzy integers $\tilde{m}, \tilde{n}$ . It is given as fuzzy trigonometric numbers $(k_n, n, l_n), (k_m, m, l_m)$ both straight:

1. $\tilde{n} + \tilde{m} = (k_+, n+m, l_+)$, $k_+ = \begin{cases} P_{-1}(n+m), n+m \geq 0, \\ -P_1(-n-m), n+m < 0, \end{cases}$ $l_+ = \begin{cases} P_1(n+m), n+m \geq 0, \\ -P_{-1}(-n-m), n+m < 0, \end{cases}$

2 . $\tilde{n} - \tilde{m} = (k_-, n-m, l_-)$, $k_- = \begin{cases} P_{-1}(n-m), n-m \geq 0, \\ -P_1(-n+m), n-m < 0, \end{cases}$ $l_- = \begin{cases} P_1(n-m), n-m \geq 0, \\ -P_{-1}(-n+m), n-m < 0, \end{cases}$

3. $\tilde{n} * \tilde{m} = (k_*, n*m, l_*)$, $k_* = \begin{cases} P_{-1}(n*m), n*m \geq 0, \\ -P_1(-n*m), n*m < 0, \end{cases}$ $l_* = \begin{cases} P_1(n*m), n*m \geq 0, \\ -P_{-1}(-n*m), n*m < 0, \end{cases}$

4. $\tilde{n}\ /\ \tilde{m}\ =\ \left(k_{div}, n/m, l_{div}\right),\ \ m \neq 0,\quad k_{div} = \begin{cases} P_{-1}(n/m), n/m \geq 0, \\ -P_1(-n/m), n/m < 0, \end{cases} \quad l_{div} = \begin{cases} P_1(n/m), n/m \geq 0, \\ -P_{-1}(-n/m), n/m < 0, \end{cases}$

5. $\tilde{n}\ \%\ \tilde{m}\ =\ \left(k_{\text{mod}}, n\%m, l_{\text{mod}}\right),\ n \geq 0,\ m > 0,$

$$k_{\text{mod}} = \begin{cases} P_{-1}(n\%m), n\%m \geq 0, \\ -P_1(-n\%m), n\%m < 0, \end{cases} \quad l_{\text{mod}} = \begin{cases} P_1(n\%m), n\%m \geq 0, \\ -P_{-1}(-n\%m), n\%m < 0, \end{cases}$$

It is necessary to draw attention to one of the important details, including calculating the prime numbers related to any number $a \geq 0$ , at the same time calculating the prime numbers $P_1(a)$ and $P_{-1}(a)$, that the representation of any fuzzy $\tilde{k}$ integer depends on K and is characterized by the features of the belonging function, so the representation period is unknown (fuzzy values ) . The fuzzy trigonometric numbers $\tilde{k} = \{(k_1, k, k_2)\}$ with belonging functions can be considered as a group of triangles; the prime numbers $k_1, k_2$ are calculated according to the following formula the company's production data: $k_1 = P_{-1}(k), k_2 = P_1(k), .$

## 7. Case Study in the Field of Oil

The information below was taken from Bhumik [17], a significant integrated oil company that imports the remaining oil to meet its needs and imports the majority of its oil. Bhumik also has a vast distribution network that is used to transport oil to refineries and then oil products from refineries to distribution centers, which are located in a variety of places. In addition, RJ Vanderbei [18], the mathematical and physical practical applications of sensitivity analysis but the geometric dimension are presented. **In Table 1**, these facilities are listed (1).

The management of the company has decided to enhance its output by constructing an additional refinery along with an increase in its crude oil imports in order to increase market share from its primary products. The choice of where to put the new refinery is crucial. On how the distribution system is running including decisions regarding the amount of crude oil that may be transported from each source to each refinery, as well as the amount of the finished product that can be transported from each refinery to the distribution centers. As a result, the three primary criteria used by management to determine where to locate the new refinery are:

1.The cost of transporting crude oil from the sources to all the new refinerie*s* .

2. The cost of transporting the final product from the refineries to the distribution centers.

3.The costs of operating the new refinery, including labor costs, taxes, guarantees, the effect of financial incentives provided by the state or the city, as well as the capital costs, because they will be the same in any available location. After verification, the concerned working group found that there are three locations (Los Angeles, Galveston, and Missouri), has major advantages shown in **Table (2)**, and **Table (3)** Shows:

| **Table (1)**: The Current Locations of the Company's Facilities | |
|---|---|
| **Facility type** | **Locations** |
| Oil Fields | **1.Texas/ 2.California/ 3.Alaska** |
| Refineries | **1. Near New Orland/ 2.Near South Carolina/ 3.Near Seattle/Washington** |
| Distribution Centers | **1. Pittsburgh, Pennsylvania / 2. Atlanta, Georgia / 3. Kansas City, Missouri 4. San Francisco, California** |

| **Table (2):** Available Locations for the New Refinery, with the Advantages Of Each Location | |
|---|---|
| **Main advantages** | **Available sites** |
| 1. Close to California Oilfields/ 2. Easy access to Alaska Oilfields/ 3. Fairly close to San Francisco Distribution Center | **Near Los Angeles, California** |
| 1. Near Texas oil fields/ 2. Easy access to imported oil/ 3. Close to the company's headquarters | **Near Galveston** |
| 1. Low operating costs/ 2. Central location for distribution sites/ 3. Easy access to crude oil by the Mississippi River | **Near Missouri** |

| **Table (3):** Shows the Company's Production Data | | | |
|---|---|---|---|
| **Filtered** | **The annual needs of the refinery of crude oil (barrels / million)** | **oil fields** | **Annual production of crude oil per oil field (barrels/million)** |
| 1. New Orlando<br>2. South Carolina<br>3. Seattle/Washington<br>4. New | **100**<br>**60**<br>**80**<br>**120** | 1.Texas<br>2. California<br>3. Alaska | **80**<br>**60**<br>**100** |
| Total | **360** | **The total needs of the importer 360-240 = 120** | |

To do the necessary analysis on the data, the work team must gather a lot of it. The administration desires that all refineries function at maximum output. The work team's responsibility is to determine the annual crude oil requirements for each refinery because the amounts of crude oil produced or the team came to the conclusion that, with the exception of transportation charges, the costs of production or purchase are unrelated to the choice of the refinery's new location because the customer will remain the same regardless of the location. On the other hand, the expenses associated with moving crude oil from the sources to the refineries are crucial is shown in **Table 4**. The cost of shipping crude oil from refineries to distribution centers is shown in **Table 5** for all three planned refineries, and the bottom row of the table lists the quantity of the finished product that is required by each distribution center. In addition to the previously mentioned, the third and final major aspect of the data focuses on operating costs for each refinery in each of the three sites that will be established in it. Cost estimation necessitates field visits by numerous team members to gather detailed information about local labor costs, and tasks. The land assignment and other matters are enumerated in **Table (6):**

**Table (4):** The Cost of Shipping Crude Oil from Sources to Refineries

| **Cites** | Shipping cost (in million dollars / per million barrels) from the fields to the refineries, including the proposed sites | | | | | |
|---|---|---|---|---|---|---|
| | **NBO Orlando** | **Charleston** | **Seattle** | **Los Angeles** | **Galveston** | **Missouri** |
| **Texas** | 2 | 4 | 5 | 3 | 1 | 7 |
| **California** | 5 | 5 | 3 | 1 | 3 | 4 |
| **Alaska** | 5 | 7 | 3 | 4 | 5 | 7 |
| **importer** | 2 | 3 | 5 | 4 | 3 | 4 |

**Table (5):** Costs of Shipping Each Item (In Million Dollars for Million Barrels) From Refineries to Distribution Centers

| | | **Pittsburgh** | **Atlanta** | **Kansas** | **San Francisco** |
|---|---|---|---|---|---|
| **Refineries** | **New Orlando** | 6.5 | 5.5 | 6 | 8 |
| | **Charleston** | 7 | 5 | 4 | 7 |
| | **Seattle Los** | 7 | 8 | 4 | 3 |
| **Suggested Refineries** | **Angeles** | 8 | 6 | 3 | 2 |
| | **Galveston** | 5 | 4 | 3 | 6 |
| | **Missouri** | 4 | 3 | 1 | 5 |
| **Needs** | | 100 | 80 | 80 | 100 |

| Table (6): Refinery Operating Costs | |
|---|---|
| **Location** | **Annual operating costs (in millions of dollars)** |
| **Los Angeles** | **620** |
| **Galveston** | **570** |
| **Missouri** | **530** |

## 8. Constructing (Formulating) the Problem's Mathematical Model:

Before developing the mathematical model for the aforementioned issue, it is necessary for us to establish a general understanding of either- Or restrictions, which follow the generic $K$ out of $N$ formula. This can be stated as [5]:

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i - My_i \quad for\ all \quad i = 1,2,\dots,m\ , \quad \sum_{i=1}^{N} y_i = N - K \quad \text{.....} (20)$$

Where $M \neq 0, y_i \epsilon\ [0, 1]$ because the second condition connected to $y_i$ guarantees that the K of the original problem constraints will remain unaltered and the rest of the constraints will be removed.

**Define Decision Variables:**

$x_{ijk}$: The quantities transported of crude oil from field i to the j refinery, and the quantities transferred from the j refinery to the distribution centers k (annually).

$C_{ijk}$ : Transporting goods from refinery j to distribution centers in addition to the shipping charges from field I to refinery j.

$\delta_{ijk}$ : Refinery operational cost suggested $F_j$, binary variable.

$$i = 1,2,3,4\ , \qquad j = 1,2,3, A\ OR\ B\ OR\ C\ , \qquad k = 1,2,3,4$$

Objective function

$$Min\ Z = \sum_{i=1}^{4}\sum_{j=1}^{3}\sum_{k=1}^{4} c_{ijk}x_{ijk} + \sum_{i=1}^{4}\sum_{j\in\{A,B,C\}}\sum_{k=1}^{4} c_{ijk}\delta_{ijk}x_{ijk} + F_j\delta_j \quad \text{....} (21)$$

$$\delta_{ijk=} \begin{cases} 1 \ \text{ if } \ x_{ijk} > 0\ , j \in \{A,B,C\}\ , i = \overline{1,m}\ , k = \overline{1,p} \\ \\ 0 \ \text{ if } \ \nexists\ (x_{iAk} > 0\ , x_{iBk} > 0\ , x_{iCk} > 0), \end{cases} \quad \text{......} (22)$$

$j \in \{A, B, C\}, i = \overline{1, m}, k = \overline{1, p}$

$$\delta_j = \begin{cases} 1 & \text{if } \delta_{ijk=} 1 \\ 0 & \text{Otherwise} \end{cases} \quad \ldots\ldots. (23)$$

## 9. Restrictions on Imported and Crude Oil Fields

$$\sum_{j=1}^{3}\sum_{k=1}^{4} x_{1jk} + \sum_{k=1}^{4} x_{1Ak} + My_1 \geq 80 \quad , \sum_{j=1}^{3}\sum_{k=1}^{4} x_{2jk} + \sum_{ik=1}^{4} x_{2Ak} + My_1 \geq 60, .. (24)$$

$$\sum_{j=1}^{3}\sum_{k=1}^{4} x_{1jk} + \sum_{k=1}^{4} x_{1Bk} + My_2 \geq 80 \quad , \quad \sum_{j=1}^{3}\sum_{k=1}^{4} x_{2jk} + \sum_{k=1}^{4} x_{2Bk} + My_2 \geq 60 \quad \ldots (25)$$

$$\sum_{j=1}^{3}\sum_{k=1}^{4} x_{1jk} + \sum_{k=1}^{4} x_{1Ck} + My_3 \geq 80 \quad , \quad \sum_{j=1}^{3}\sum_{k=1}^{4} x_{2jk} + \sum_{k=1}^{4} x_{2Ck} + My_3 \geq 60 \quad \ldots. (26)$$

$$\sum_{j=1}^{3}\sum_{k=1}^{4} x_{3jk} + \sum_{k=1}^{4} x_{3Ak} + My_1 \geq 100 \quad , \quad \sum_{j=1}^{3}\sum_{k=1}^{4} x_{4jk} + \sum_{k=1}^{4} x_{4Ak} + My_1 \geq 120$$

$$\sum_{j=1}^{3}\sum_{k=1}^{4} x_{4jk} + \sum_{k=1}^{4} x_{4Bk} + My_2 \geq 120, \sum_{j=1}^{3}\sum_{k=1}^{4} x_{3jk} + \sum_{k=1}^{4} x_{3Bk} + My_2 \geq 100, \quad ..$$

(27)

$$\sum_{j=1}^{3}\sum_{k=1}^{4} x_{3jk} + \sum_{k=1}^{4} x_{3Ck} + My_3 \geq 100, \sum_{j=1}^{3}\sum_{k=1}^{4} x_{4jk} + \sum_{k=1}^{4} x_{4Ck} + My_3 \geq 120$$

**Refinery Capacity Restrictions**

$$\sum_{i=1}^{4}\sum_{k=1}^{4} x_{i1k} = 100 \quad , \sum_{i=1}^{4}\sum_{k=1}^{4} x_{i2k} = 60 \quad , \sum_{i=1}^{4}\sum_{k=1}^{4} x_{i3k} = 80, \quad .. \quad \textbf{(28)}$$

$$\sum_{i=1}^{4}\sum_{k=1}^{4} x_{iAk} + My_1 \geq 120 \quad , \sum_{i=1}^{4}\sum_{k=1}^{4} x_{iBk} + My_2 \geq 120, \quad \sum_{i=1}^{4}\sum_{k=1}^{4} x_{iCk} + My_3 \geq 120$$

**<u>Distribution Center Restrictions</u>**

$$\sum_{i=1}^{4}\sum_{j=1}^{3} x_{ij1} + \sum_{i=1}^{4} x_{iA1} - My_1 \leq 100, \quad \sum_{i=1}^{4}\sum_{j=1}^{3} x_{ij2} + \sum_{i=1}^{4} x_{iA2} - My_1 \leq 80$$

$$\sum_{i=1}^{4}\sum_{j=1}^{3} x_{ij1} + \sum_{i=1}^{4} x_{iB1} - My_2 \le 100\,, \quad \sum_{i=1}^{4}\sum_{j=1}^{3} x_{ij2} + \sum_{i=1}^{4} x_{iB2} - My_2 \le 80$$

$$\sum_{i=1}^{4}\sum_{j=1}^{3} x_{ij1} + \sum_{i=1}^{4} x_{iC1} - My_3 \le 100, \sum_{i=1}^{4}\sum_{j=1}^{3} x_{ij2} + \sum_{i=1}^{4} x_{iC2} - My_3 \le 80$$

$$\sum_{i=1}^{4}\sum_{j=1}^{3} x_{ij3} + \sum_{i=1}^{4} x_{iA3} - My_1 \le 80\,, \sum_{i=1}^{4}\sum_{j=1}^{3} x_{ij3} + \sum_{i=1}^{4} x_{iA3} - My_1 \le 100 \quad \textbf{(29)}$$

$$\sum_{i=1}^{4}\sum_{j=1}^{3} x_{ij3} + \sum_{i=1}^{4} x_{iB3} - My_2 \le 100\,, \sum_{i=1}^{4}\sum_{j=1}^{3} x_{ij4} + \sum_{i=1}^{4} x_{iB4} - My_2 \le 80$$

$$\sum_{i=1}^{4}\sum_{j=1}^{3} x_{ij3} + \sum_{i=1}^{4} x_{iC3} - My_3 \le 100, \sum_{i=1}^{4}\sum_{j=1}^{3} x_{ij4} + \sum_{i=1}^{4} x_{iC4} - My_3 \le 80$$

$$y_1 + y_2 + y_3 = 2, x_{ijk} \ge 0\,, y_i \in \{0,1\}\ , i = 1,2,3 \quad M \gg 0$$

Using the ready-made computer program WINQSB, the results of the optimal solution were obtained (in units of measure, million barrels / year) as follows:

$x_{111} = 35\,, x_{1c2} = 25, x_{234} = 60\,, x_{323} = 5\,, x_{1C3} = 75\ , x_{334} = 20\,, x_{3c4} = 20\,, x_{411} = 65, x_{422} = 55$

The total annual cost = 2707 million dollars / year, and site C (Missouri) was chosen for the construction of the new refinery, which is to be entered within the company's expansion plan.

## 10.Explanation of the Post-Optimal Solution

One refinery capacity constraint with a shadow price of 8.5 means that if the value of this resource, represented by the capacity of the first refinery, is decreased by one unit from 100 to 99, the objective function will improve (decrease) by 8.5 million dollars. On the other hand, if we take a constraint with a shadow price of (-3), it means that if the value of this resource, represented by the capacity of the second refinery, is decreased by one unit from 100 to 90.

## 11.Conclusions

In this article, we discussed a few topics connected to sensitivity analysis, which is also known as post optimality analysis. Administrations utilize this sort of analysis to respond to a number of hypothetical inquiries concerning the values used in the linear programming model, such as with this kind of analysis, it is determined how the profit and values on the right-hand side of the constraint vary in relation to the best possible solution. The idea of prime numbers was also helpful to us in resolving the fuzzy linear programming model

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