DOI: 10.24996/ijs.2024.65.3.27

ISSN: 0067-2904

The Simulation Technique to Estimate the Parameters of Generalized Exponential Rayleigh Model

Rehab Noori Shalan *[1], Iden Hassan Alkanani[2]

[1]*Department of mathematics, college of science. University of, Basrah, Iraq.*
[2]*Department of mathematics, college of science for women. University of Baghdad, Baghdad, Iraq*

Received: 17/12/2022 Accepted: 21/3/2023 Published: 30/3/2024

Abstract:
The paper shows how to estimate the three parameters of the generalized exponential Rayleigh distribution by utilizing the three estimation methods, namely, the moment employing estimation method (MEM), ordinary least squares estimation method (OLSEM), and maximum entropy estimation method (MEEM). The simulation technique is used for all these estimation methods to find the parameters for the generalized exponential Rayleigh distribution. In order to find the best method, we use the mean squares error criterion. Finally, in order to extract the experimental results, one of object oriented programming languages visual basic. net was used

Keywords: Moment estimator method, Ordinary least squares estimation method, Maximum entropy estimation method

رحاب نوري[1*], ايدن حسن[2]

كليه العلوم , قسم الرياضيات, جامعه البصره, البصره, العراق
كليه العلوم للبنات, قسم الرياضيات, جامعه بغداد, بغداد, العراق

الخلاصه :
توضح الورقة كيفية تقدير المعلمات الثلاثة لتوزيع رايلي الأسي المعمم باستخدام طرق التقدير الثلاثة وهي. الأولى هي طريقه العزوم ((MEM)، والثانية هي طريقة تقدير المربعات الصغرى العادية ((OLSEM)، والثالثة هي طريقة تقدير الانتروبيا القصوى .(MEEM),تُستخدم تقنية المحاكاة لجميع طرق التقدير هذه لإيجاد معلمات توزيع رايلي الأسي المعمم. باستخدام معيار خطأ المربعات المتوسطة للمقارنة بين طرق التقدير هذه للعثور على أي من هذه الطرق هو الأفضل للطرق الأخرى. أخيرًا ، من أجل استخراج النتائج التجريبية ، تم استخدام إحدى لغات البرمجة المرئيه

1. Introduction

The application of statistics plays an important role in our divergent phenomenon life, especially in engineering and medicine. The researchers depend on many procedures to find the new or mixture and compose distributions [1]. In 2021, Iden H.H. and Lamyaa K.H. introduced the mixed distribution where mixing between the reliability function for the exponential model

*Email: rehab.shallan@sc.uoBaghdad.edu.iq

and the reliability function for the Rayleigh model by using X=min (Y, Z), the model is called the Exponential – Rayleigh model.

In 2021 Lamyaa K.H. and I den H.H. applied the maximum likelihood method to estimate the values of the survival function and hazard function based on real data for lung cancer and stomach cancer obtained from Iraq [2,3,4]. In the last research, we introduced the (GER) distribution that strutting is to add the shape parameter to the exponential Rayleigh distribution to get the (GER). The aim of this paper is to compare between (ME) method, (OLSE) method and (MES) method by utilizing the mean squares error criterion for the Monte Carlo simulation technique [5] to find the best one from the others. The organization of this paper is as follows: In section two, the generalized exponential Rayleigh distribution is given. In section three, the estimation method is discussed. In section four, a simulation technique is used. In section five, Numerical results are done. In section six, the conclusion is given.

2. The Generalized exponential Rayleigh distribution

2.1 The probability density function of the generalized Exponential- Rayleigh distribution [6] is given as follows:

$$f(x;\alpha,\beta,\lambda)=\left(\frac{\alpha}{\lambda}x^{\frac{1}{\lambda}-1}+\frac{\beta}{\lambda}x^{\frac{2}{\lambda}-1}\right)e^{-\left(\alpha x^{\frac{1}{\lambda}}+\frac{\beta}{2}x^{\frac{2}{\lambda}}\right)}\qquad x\geq 0\ .$$

2.2 The cumulative distribution function for generalized Exponential- Rayleigh distribution is:

$$F(x;\alpha,\beta,\lambda)=1-e^{-\left(\alpha x^{\frac{1}{\lambda}}+\frac{\beta}{2}x^{\frac{2}{\lambda}}\right)}\ .$$

2.3 The reliability function of this distribution is $R(x;\alpha,\beta,\lambda)=e^{-\left(\alpha x^{\frac{1}{\lambda}}+\frac{\beta}{2}x^{\frac{2}{\lambda}}\right)}\quad x\geq 0.$

Then, the hazard rate function for this distribution is: $h(x;\alpha,\beta,\lambda)=\frac{\alpha}{\lambda}x^{\frac{1}{\lambda}-1}+\frac{\beta}{\lambda}x^{\frac{2}{\lambda}-1}\quad x\geq 0$.

3-Estimation Methods: In this section, we describe the estimation of the parameters of the generalized exponential Rayleigh distribution by employing three methods, namely the moment estimation method, ordinary least squares estimation method and the maximum entropy estimation method.

3.1 The Moment Estimator Method(MEM):

The method of the moment is one of the simplest techniques which is commonly used in the field of parameter estimation. The idea of this method is to equate the sample moments and population moments.

$M_1=E(X)=\int x\, f(x;\alpha,\beta,\lambda)dx$.

$M_1=E(X)=e^{-\frac{\alpha}{2}}\left[\frac{\alpha}{2}\left(\frac{2}{\beta}\right)^{\frac{n+\lambda+1}{2}}\Gamma\frac{n+\lambda+1}{2}+\left(\frac{2}{\beta}\right)^{\frac{n+\lambda}{2}}\Gamma\frac{n+\lambda+2}{2}\right]$ is the first population moment

$M_1'=\frac{\sum_{i=1}^{n}x_i}{n}=\bar{x}$ is the sample moment ,so

$M_1=M_1'$. Thus, we have

$$e^{-\frac{\alpha}{2}}\left[\frac{\alpha}{2}\left(\frac{2}{\beta}\right)^{\frac{n+\lambda+1}{2}}\Gamma\frac{n+\lambda+1}{2}+\left(\frac{2}{\beta}\right)^{\frac{n+\lambda}{2}}\Gamma\frac{n+\lambda+2}{2}\right]=\bar{x}\ .$$

$$f(\alpha)=e^{-\frac{\alpha}{2}}\left(\frac{\alpha}{2}\right)\left(\frac{2}{\beta}\right)^{\frac{n+\lambda+1}{2}}\Gamma\frac{n+\lambda+1}{2}+e^{-\frac{\alpha}{2}}\left(\frac{2}{\beta}\right)^{\frac{n+\lambda}{2}}\Gamma\frac{n+\lambda+2}{2}-\bar{x}\ . \qquad (3.1.1)$$

$M_2=E(X^2)=\int x^2\, f(x,\alpha,\beta,\lambda)dx$.

Consider that $M_2 = E(X^2) = e^{-\frac{\alpha}{2}}\left[\frac{\alpha}{2}\left(\frac{2}{\beta}\right)^{\frac{n+2\lambda+1}{2}}\Gamma\frac{n+2\lambda+1}{2} + \left(\frac{2}{\beta}\right)^{\frac{n+2\lambda}{2}}\Gamma\frac{n+2\lambda+1}{2}\right]$ is the second population moment and $M_2' = \frac{\sum_{i=1}^{n} x_i^2}{n}$ is the second sample moment, so that if we let $M_2 = M_2'$, then we have

$$e^{-\frac{\alpha}{2}}\left[\frac{\alpha}{2}\left(\frac{2}{\beta}\right)^{\frac{n+2\lambda+1}{2}}\Gamma\frac{n+2\lambda+1}{2} + \left(\frac{2}{\beta}\right)^{\frac{n+2\lambda}{2}}\Gamma\frac{n+2\lambda+1}{2}\right] = \frac{\sum_{i=1}^{n} x_i^2}{n}.$$

$$f(B) = e^{-\frac{\alpha}{2}}\left(\frac{\alpha}{2}\right)\left(\frac{2}{\beta}\right)^{\frac{n+2\lambda+1}{2}}\Gamma\frac{n+2\lambda+1}{2} + e^{-\frac{\alpha}{2}}\left(\frac{2}{\beta}\right)^{\frac{n+2\lambda}{2}}\Gamma\frac{n+2\lambda+1}{2} - \frac{\sum_{i=1}^{n} x_i^2}{n} \tag{3.1.2}$$

$$M_3 = E(x^3) = \int_{allx} x^3 f(x;\alpha,\beta,\lambda)\, dx .$$

$$M_3 = E(x^3) = e^{-\frac{\alpha}{2}}\left[\frac{\alpha}{2}\left(\frac{2}{\beta}\right)^{\frac{n+3\lambda+1}{2}}\Gamma\frac{n+3\lambda+1}{2} + \left(\frac{2}{\beta}\right)^{\frac{n+3\lambda}{2}}\Gamma\frac{n+3\lambda+1}{2}\right].$$

$$M_3' = \frac{\sum_{i=1}^{n} x_i^3}{n} .$$

$$M_3 = M_3' .$$

$$e^{-\frac{\alpha}{2}}\left[\frac{\alpha}{2}\left(\frac{2}{\beta}\right)^{\frac{n+3\lambda+1}{2}}\Gamma\frac{n+3\lambda+1}{2} + \left(\frac{2}{\beta}\right)^{\frac{n+3\lambda}{2}}\Gamma\frac{n+3\lambda+1}{2}\right] = \frac{\sum_{i=1}^{n} x_i^3}{n} .$$

$$f(\lambda) = e^{-\frac{\alpha}{2}}\left(\frac{\alpha}{2}\right)\left(\frac{2}{\beta}\right)^{\frac{n+2\lambda+1}{2}}\Gamma\frac{n+3\lambda+1}{2} + \left(\frac{2}{\beta}\right)^{\frac{n+3\lambda}{2}}\Gamma\frac{n+3\lambda}{2} e^{-\frac{\alpha}{2}} - \frac{\sum_{i=1}^{n} x_i^3}{n} \tag{3.1.3}$$

$$\begin{bmatrix}\alpha_{k+1}\\ \beta_{k+1}\\ \lambda_{k+1}\end{bmatrix} - \begin{bmatrix}\alpha_k\\ \beta_k\\ \lambda_k\end{bmatrix} - J^{-1}\begin{bmatrix}f(\alpha)\\ f(\beta)\\ f(\lambda)\end{bmatrix}$$

$$J = \begin{bmatrix}\frac{\partial f(\alpha)}{\partial\alpha} & \frac{\partial f(\alpha)}{\partial\beta} & \frac{\partial f(\alpha)}{\partial\lambda}\\ \frac{\partial f(\beta)}{\partial\alpha} & \frac{\partial f(\beta)}{\partial\beta} & \frac{\partial f(\beta)}{\partial\lambda}\\ \frac{\partial f(\lambda)}{\partial\alpha} & \frac{\partial f(\lambda)}{\partial\beta} & \frac{\partial f(\lambda)}{\partial\lambda}\end{bmatrix}$$

$$\frac{\partial f(\alpha)}{\partial\alpha} = \frac{1}{2} e^{-\frac{\alpha}{2}}\left(\frac{2}{\beta}\right)^{\frac{n+\lambda+1}{2}}\Gamma\frac{n+\lambda+1}{2} - \frac{1}{2} e^{-\frac{\alpha}{2}}\left(\frac{\alpha}{2}\right)\left(\frac{2}{\beta}\right)^{\frac{n+\lambda+1}{2}}\Gamma\frac{n+\lambda+1}{2}$$
$$- \frac{1}{2} e^{-\frac{\alpha}{2}}\left(\frac{2}{\beta}\right)^{\frac{n+\lambda}{2}}\Gamma\frac{n+\lambda+2}{2}$$

$$\frac{\partial f(\alpha)}{\partial\beta} = -e^{-\frac{\alpha}{2}}\left[\frac{\alpha}{2}\Gamma\frac{n+\lambda+1}{2}\cdot\frac{2^{\left(\frac{n+\lambda+1}{2}\right)}}{\beta^{\left(\frac{n+\lambda+3}{2}\right)}}\cdot\left(\frac{n+\lambda+1}{2}\right)\right.$$
$$\left.+\Gamma\frac{n+\lambda+2}{2}\left(\frac{n+\lambda}{2}\right)\cdot\frac{2^{\left(\frac{n+\lambda}{2}\right)}}{\beta^{\left(\frac{n+\lambda+2}{2}\right)}}\right]$$

$$\frac{\partial f(\alpha)}{\partial\lambda} = e^{-\frac{\alpha}{2}}\left(\frac{2}{\beta}\right)^{\frac{n+1}{2}}\left[\left(\frac{2}{\beta}\right)^{\frac{\lambda}{2}}\ln\left(\frac{2}{\beta}\right)\cdot\frac{1}{2}\cdot\Gamma\frac{n+\lambda+1}{2} + \left(\frac{2}{\beta}\right)^{\frac{\lambda}{2}}\Gamma'\left(\frac{n+1}{2}+\frac{\lambda}{2}\right)\cdot\frac{1}{2}\right] + e^{-\frac{\alpha}{2}}\left(\frac{2}{\beta}\right)^{\frac{n}{2}}$$
$$\left[\left(\frac{2}{\beta}\right)^{\frac{\lambda}{2}}\ln\left(\frac{2}{\beta}\right)\cdot\frac{1}{2}\cdot\Gamma\frac{n+\lambda+2}{2} + \left(\frac{2}{\beta}\right)^{\frac{\lambda}{2}}\Gamma'\left(\frac{n+2}{2}+\frac{\lambda}{2}\right)\cdot\frac{1}{2}\right]$$

$$\frac{\partial f(\beta)}{\partial\alpha} = -\frac{1}{2} e^{-\frac{\alpha}{2}}\left(\frac{2}{\beta}\right)^{\frac{n+2\lambda}{2}}\Gamma\frac{n+2\lambda+1}{2}\left[\left(\frac{2}{\beta}\right)^{\frac{1}{2}} - \frac{\alpha}{2}\left(\frac{2}{\beta}\right)^{\frac{1}{2}} - 1\right]$$

$$\frac{\partial f(\beta)}{\partial\beta} = -e^{-\frac{\alpha}{2}}\left[\frac{\alpha}{2}\Gamma\frac{n+2\lambda+1}{2}\cdot\left(\frac{n+2\lambda+1}{2}\right)\cdot\frac{2^{\left(\frac{n+2\lambda+1}{2}\right)}}{\beta^{\left(\frac{n+2\lambda+3}{2}\right)}} + \Gamma\frac{n+2\lambda+1}{2}\left(\frac{n+2\lambda}{2}\right)\cdot\frac{2^{\left(\frac{n+2\lambda}{2}\right)}}{\beta^{\left(\frac{n+2\lambda+2}{2}\right)}}\right]$$

$$\frac{\partial f(\beta)}{\partial\lambda}=e^{-\frac{\alpha}{2}}\left(\frac{\alpha}{2}\right)\cdot\left(\frac{2}{\beta}\right)^{\frac{n+2\lambda+1}{2}}\ln\left(\frac{2}{\beta}\right)\Gamma\frac{n+2\lambda+1}{2}+e^{-\frac{\alpha}{2}}\left(\frac{\alpha}{2}\right)\cdot\left(\frac{2}{\beta}\right)^{\frac{n+2\lambda+1}{2}}\Gamma'(\frac{n+1}{2}+\lambda)$$
$$+e^{-\frac{\alpha}{2}}\left(\frac{2}{\beta}\right)^{\frac{n+2\lambda}{2}}\ln\left(\frac{2}{\beta}\right)\Gamma\frac{n+2\lambda+1}{2}+e^{-\frac{\alpha}{2}}\left(\frac{2}{\beta}\right)^{\frac{n+2\lambda}{2}}\Gamma'(\frac{n+2\lambda+1}{2}+\lambda)$$
$$\frac{f(\lambda)}{\partial\alpha}=\frac{1}{2}\,e^{-\frac{\alpha}{2}}\left(\frac{2}{\beta}\right)^{\frac{n+3\lambda+1}{2}}\left(1-\frac{\alpha}{2}\right)-\frac{1}{2}\,e^{-\frac{\alpha}{2}}\left(\frac{2}{\beta}\right)^{\frac{n+3\lambda}{2}}\Gamma\frac{n+3\lambda+1}{2}$$
$$\frac{f(\lambda)}{\partial\beta}=e^{-\frac{\alpha}{2}}\left(\frac{\alpha}{2}\right)\Gamma\frac{n+3\lambda+1}{2}\cdot-\left(\frac{n+3\lambda+1}{2}\right)\cdot\frac{2^{\left(\frac{n+3\lambda+1}{2}\right)}}{\beta^{\left(\frac{n+3\lambda+3}{2}\right)}}+e^{-\frac{\alpha}{2}}\Gamma\frac{n+3\lambda+1}{2}\cdot-\left(\frac{n+3\lambda}{2}\right)\cdot\frac{2^{\left(\frac{n+3\lambda}{2}\right)}}{\beta^{\left(\frac{n+3\lambda+2}{2}\right)}}$$
$$=-\,e^{-\frac{\alpha}{2}}\Gamma\frac{n+3\lambda+1}{2}\left[\left(\frac{\alpha}{2}\right)\cdot\left(\frac{n+3\lambda+1}{2}\right)\frac{2^{\left(\frac{n+3\lambda+1}{2}\right)}}{\beta^{\left(\frac{n+3\lambda+3}{2}\right)}}+\left(\frac{n+3\lambda}{2}\right)\cdot\frac{2^{\left(\frac{n+3\lambda}{2}\right)}}{\beta^{\left(\frac{n+3\lambda+2}{2}\right)}}\right]$$
$$\frac{f(\lambda)}{\partial\lambda}=e^{-\frac{\alpha}{2}}\left(\frac{\alpha}{2}\right)\left(\frac{2}{\beta}\right)^{\frac{n+1}{2}}\left[\left(\frac{2}{\beta}\right)^{\frac{3\lambda}{2}}\ln\left(\frac{2}{\beta}\right)\cdot\frac{3}{2}\cdot\Gamma\frac{n+3\lambda+1}{2}+\left(\frac{2}{\beta}\right)^{\frac{3\lambda}{2}}\Gamma'(\frac{n+1}{2}+\frac{3\lambda}{2})\cdot\frac{3}{2}\right]$$
$$+e^{-\frac{\alpha}{2}}\left(\frac{2}{\beta}\right)^{\frac{n}{2}}\left[\left(\frac{2}{\beta}\right)^{\frac{3\lambda}{2}}\ln\left(\frac{2}{\beta}\right)\cdot\frac{3}{2}\cdot\Gamma\frac{n+3\lambda+1}{2}+\left(\frac{2}{\beta}\right)^{\frac{3\lambda}{2}}\cdot\Gamma'(\frac{n+1}{2}+\frac{3\lambda}{2})*\frac{3}{2}\right.$$

3.2 The Ordinary Least Squares Estimation Method(OLSEM):

The ordinary least squares method is one of the most popular procedures in estimating the parameters when the model is linear or nonlinear in variables.

The idea of this method is to minimize the sum of squared differences between the observed sample value and the estimated expected value by linear approximation:

$$\sum_{i=1}^{n}\epsilon_i^2=\sum_{i=1}^{n}\left(y_i-E\,(y_i)\right)^2=\sum_{i=1}^{n}(y_i-\hat{y})^2$$
$$\sum_{i=1}^{n}\epsilon_i^2=\sum_{i=1}^{n}\left[\hat{F}(X_i)-F(X_i)\right]^2$$

The empirical cdF is $F(X_i)=\frac{i-0.5}{n}$.

$$\sum_{i=1}^{n}\epsilon_i^2=\sum_{i=1}^{n}\left[1-e^{-\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)}-\frac{i-0.5}{n}\right]^2.$$
$$\sum_{i=1}^{n}\epsilon_i^2=\sum_{i=1}^{n}\left[\frac{n-i+0.5}{n}-e^{-\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)}\right]^2$$
$$\frac{\partial\sum\epsilon_i^2}{\partial\alpha}=\sum_{i=1}^{n}\frac{n-i+0.5}{n}\,x_i^{\frac{1}{\lambda}}\,e^{-\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)}-\sum_{i=1}^{n}x_i^{\frac{1}{\lambda}}\,e^{-2\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)}=f(\alpha)\qquad(3.2.1)$$
$$\frac{\partial\sum\epsilon_i^2}{\partial\beta}=\frac{1}{4}\sum_{i=1}^{n}\frac{n-i+0.5}{n}\,x_i^{\frac{2}{\lambda}}\,e^{-\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)}-\frac{1}{2}\sum_{i=1}^{n}x_i^{\frac{2}{\lambda}}\,e^{-2\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)}=g(\beta)\qquad(3.2.2)$$
$$\frac{\partial\sum\epsilon_i^2}{\partial\lambda}=\sum_{i=1}^{n}\frac{\alpha}{\lambda^2}x_i^{\frac{1}{\lambda}}\ln x_i\;e^{-2\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)}+\sum_{i=1}^{n}\frac{\beta}{\lambda^2}\,x_i^{\frac{2}{\lambda}}\,\ln x_i\;e^{-2\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)}$$
$$-\sum_{i=1}^{n}\frac{n-i+0.5}{n}\left(\frac{\alpha}{\lambda^2}x_i^{\frac{1}{\lambda}}\ln x_i\;e^{-\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)}\right)$$
$$-\sum_{i=1}^{n}\frac{n-i+0.5}{n}\left(\frac{\beta}{\lambda^2}\,x_i^{\frac{2}{\lambda}}\,\ln x_i\,e^{-\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)}\right)=L(\lambda)\qquad(3.2.3)$$

$$\frac{\partial f(\alpha)}{\partial\alpha} = -\sum_{i=1}^{n}\frac{n-i+0.5}{n}\, x_i^{\frac{2}{\lambda}}\, e^{-\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)} + 2\sum_{i=1}^{n} x_i^{\frac{4}{\lambda}}\, e^{-2\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)}$$

$$\frac{\partial g(\beta)}{\partial\beta} = -\frac{1}{4}\sum_{i=1}^{n}\frac{n-i+0.5}{n}\, x_i^{\frac{4}{\lambda}}\, e^{-\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)} + \frac{1}{2}\sum_{i=1}^{n} x_i^{\frac{4}{\lambda}}\, e^{-2\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)}$$

$$\begin{aligned}
\frac{\partial \ln(\lambda)}{\partial\lambda} = {} & \sum_{i=1}^{n}\frac{2\alpha^2}{\lambda^4} - \sum_{i=1}^{n}\frac{\alpha}{\lambda^4}\, x_i^{\frac{1}{\lambda}}(\ln x_i)^2\, e^{-2\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)} - \sum_{i=1}^{n}\frac{2\alpha}{\lambda^3} x_i^{\frac{1}{\lambda}}\ln x_i\, e^{-2\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)} \\
& + \sum_{i=1}^{n}\frac{2\alpha\beta}{\lambda^4}\, x_i^{\frac{3}{\lambda}}(\ln x_i)^2 e^{-2\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)} + \sum_{i=1}^{n}\frac{2\beta^2}{\lambda^4}\, x_i^{\frac{4}{\lambda}}(\ln x_i)^2 e^{-2\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)} \\
& - \sum_{i=1}^{n}\frac{2\beta}{\lambda^4}\, x_i^{\frac{2}{\lambda}}(\ln x_i)^2 e^{-2\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)} - \sum_{i=1}^{n}\frac{2\beta}{\lambda^3}\, x_i^{\frac{2}{\lambda}}\ln x_i\, e^{-2\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)} \\
& - \sum_{i=1}^{n}\frac{\alpha^2}{\lambda^4}\, x_i^{\frac{2}{\lambda}}\, e^{-\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)}\frac{n-i+0.5}{n} \\
& - \sum_{i=1}^{n}\frac{\alpha\beta}{\lambda^4}\, x_i^{\frac{3}{\lambda}}(\ln x_i)^2 e^{-\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)}\frac{n-i+0.5}{n} \\
& + \sum_{i=1}^{n}\frac{\alpha}{\lambda^4}\, x_i^{\frac{1}{\lambda}}(\ln x_i)^2 e^{-\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)}\frac{n-i+0.5}{n} + \sum_{i=1}^{n}\frac{2\alpha}{\lambda^3}\, x_i^{\frac{1}{\lambda}}\ln x_i\, e^{-\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)}\frac{n-i+0.5}{n} \\
& - \sum_{i=1}^{n}\frac{\alpha\beta}{\lambda^4}\, x_i^{\frac{3}{\lambda}}(\ln x_i)^2\, e^{-\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)}\frac{n-i+0.5}{n} - \sum_{i=1}^{n}\frac{\beta^2}{\lambda^4}\, x_i^{\frac{4}{\lambda}}(\ln x_i)^2\, e^{-\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)}\frac{n-i+0.5}{n} + \\
& + \sum_{i=1}^{n}\frac{2\beta}{\lambda^4}\, x_i^{\frac{2}{\lambda}}(\ln x_i)^2\, e^{-\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)}\frac{n-i+0.5}{n} + \sum_{i=1}^{n}\frac{2\beta}{\lambda^3}\, x_i^{\frac{2}{\lambda}}\ln x_i\, e^{-\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)}\frac{n-i+0.5}{n}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial g(\beta)}{\partial\lambda} = {} & \frac{1}{2}\sum_{i=1}^{n}\frac{\alpha}{\lambda^2}\, x_i^{\frac{3}{\lambda}}\ln x_i\, e^{-\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)}\frac{n-i+0.5}{n} + \frac{1}{2}\sum_{i=1}^{n}\frac{1}{\lambda^2} x_i^{\frac{2}{\lambda}}\ln x_i\, e^{-\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)}\frac{n-i+0.5}{n} \\
& - \sum_{i=1}^{n}\frac{\alpha}{\lambda^2} x_i^{\frac{3}{\lambda}}\ln x_i\, e^{-2\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)} - \sum_{i=1}^{n}\frac{\beta}{\lambda^2}\, x_i^{\frac{4}{\lambda}}\ln x_i\, e^{-2\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)} \\
& + \sum_{i=1}^{n}\frac{1}{\lambda^2}\, x_i^{\frac{2}{\lambda}}\ln x_i\, e^{-2\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial L(\lambda)}{\partial\beta} = {} & -\sum_{i=1}^{n}\frac{\alpha}{\lambda^2} x_i^{\frac{1}{\lambda}}\ln x_i\, e^{-2\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)}\cdot\left(-x_i^{\frac{2}{\lambda}}\right) \\
& + \sum_{i=1}^{n} x_i^{\frac{2}{\lambda}}\left[\frac{\beta}{\lambda^2}\, e^{-2\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)}\left(-x_i^{\frac{2}{\lambda}}\right) + e^{-2\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)}\left(\frac{1}{\lambda^2}\right)\right] \\
& - \sum\frac{n-i+0.5}{n}\left[\frac{\alpha}{\lambda^2} x_i^{\frac{1}{\lambda}}\ln x_i\, e^{-\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)}\left(-\frac{1}{2}x_i^{\frac{2}{\lambda}}\right)\right] \\
& - \sum_{i=1}^{n}\frac{n-i+0.5}{n}\, x_i^{\frac{2}{\lambda}}\ln x_i\left[\frac{\beta}{\lambda^2}\, e^{-\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)}\cdot\left(-\frac{1}{2}x_i^{\frac{2}{\lambda}}\right) + e^{-\left(\alpha x_i^{\frac{1}{\lambda}}+\frac{\beta}{2}x_i^{\frac{2}{\lambda}}\right)}\left(\frac{1}{\lambda^2}\right)\right]
\end{aligned}$$

$$\frac{\partial L(\lambda)}{\partial \beta} = -\sum_{i=1}^{n} \frac{\alpha}{\lambda^2} x_i^{\frac{3}{\lambda}} \ln x_i \, e^{-2\left(\alpha x_i^{\frac{1}{\lambda}} + \frac{\beta}{2} x_i^{\frac{2}{\lambda}}\right)} - \sum_{i=1}^{n} \frac{\beta}{\lambda^2} x_i^{\frac{4}{\lambda}} \ln x_i \, e^{-2\left(\alpha x_i^{\frac{1}{\lambda}} + \frac{\beta}{2} x_i^{\frac{2}{\lambda}}\right)}$$
$$+ \sum_{i=1}^{n} \frac{1}{\lambda^2} x_i^{\frac{2}{\lambda}} \ln x_i \, e^{-2\left(\alpha x_i^{\frac{1}{\lambda}} + \frac{\beta}{2} x_i^{\frac{2}{\lambda}}\right)} + \frac{1}{2} \sum_{i=1}^{n} \frac{\alpha}{\lambda^2} x_i^{\frac{3}{\lambda}} \ln x_i \, e^{-\left(\alpha x_i^{\frac{1}{\lambda}} + \frac{\beta}{2} x_i^{\frac{2}{\lambda}}\right)} \frac{n-i+0.5}{n}$$
$$+ \frac{1}{2} \sum_{i=1}^{n} \frac{\beta}{\lambda^2} x_i^{\frac{4}{\lambda}} \ln x_i \, e^{-\left(\alpha x_i^{\frac{1}{\lambda}} + \frac{\beta}{2} x_i^{\frac{2}{\lambda}}\right)} \frac{n-i+0.5}{n} - \sum_{i=1}^{n} \frac{1}{\lambda^2} x_i^{\frac{2}{\lambda}} \ln x_i \, e^{-\left(\alpha x_i^{\frac{1}{\lambda}} + \frac{\beta}{2} x_i^{\frac{2}{\lambda}}\right)} \frac{n-i+0.5}{n}$$

$$\frac{\partial L(\lambda)}{\partial \alpha} = -\sum_{i=1}^{n} \frac{2\alpha}{\lambda^2} x_i^{\frac{2}{\lambda}} \ln x_i \, e^{-2\left(\alpha x_i^{\frac{1}{\lambda}} + \frac{\beta}{2} x_i^{\frac{2}{\lambda}}\right)} + \sum_{i=1}^{n} \frac{x_i^{\frac{1}{\lambda}} \ln x_i}{\lambda^2} e^{-2\left(\alpha x_i^{\frac{1}{\lambda}} + \frac{\beta}{2} x_i^{\frac{2}{\lambda}}\right)}$$
$$- \sum_{i=1}^{n} \frac{2\beta}{\lambda^2} x_i^{\frac{3}{\lambda}} \ln x_i \, e^{-2\left(\alpha x_i^{\frac{1}{\lambda}} + \frac{\beta}{2} x_i^{\frac{2}{\lambda}}\right)} + \sum_{i=1}^{n} \frac{n-i+0.5}{n} \frac{\alpha}{\lambda^2} x_i^{\frac{2}{\lambda}} \ln x_i \, e^{-\left(\alpha x_i^{\frac{1}{\lambda}} + \frac{\beta}{2} x_i^{\frac{2}{\lambda}}\right)}$$
$$- \sum_{i=1}^{n} \frac{n-i+0.5}{n} \frac{x_i^{\frac{1}{\lambda}} \ln x_i}{\lambda^2} e^{-\left(\alpha x_i^{\frac{1}{\lambda}} + \frac{\beta}{2} x_i^{\frac{2}{\lambda}}\right)} - \sum_{i=1}^{n} \frac{n-i+0.5}{n} \frac{\beta}{\lambda^2} x_i^{\frac{3}{\lambda}} \ln x_i \, e^{-\left(\alpha x_i^{\frac{1}{\lambda}} + \frac{\beta}{2} x_i^{\frac{2}{\lambda}}\right)}$$

$$\frac{\partial f(\alpha)}{\partial \lambda} = \sum_{i=1}^{n} \frac{n-i+0.5}{n} \frac{\alpha}{\lambda^2} x_i^{\frac{2}{\lambda}} \ln x_i \, e^{-\left(\alpha x_i^{\frac{1}{\lambda}} + \frac{\beta}{2} x_i^{\frac{2}{\lambda}}\right)}$$
$$+ \sum_{i=1}^{n} \frac{n-i+0.5}{n} \frac{\beta}{\lambda^2} x_i^{\frac{3}{\lambda}} \ln x_i \, e^{-\left(\alpha x_i^{\frac{1}{\lambda}} + \frac{\beta}{2} x_i^{\frac{2}{\lambda}}\right)} - \sum_{i=1}^{n} \frac{n-i+0.5}{n} \frac{x_i^{\frac{1}{\lambda}} \ln x_i}{\lambda^2} e^{-\left(\alpha x_i^{\frac{1}{\lambda}} + \frac{\beta}{2} x_i^{\frac{2}{\lambda}}\right)}$$
$$- \sum_{i=1}^{n} \frac{2\alpha}{\lambda^2} x_i^{\frac{2}{\lambda}} \ln x_i \, e^{-2\left(\alpha x_i^{\frac{1}{\lambda}} + \frac{\beta}{2} x_i^{\frac{2}{\lambda}}\right)} - \sum_{i=1}^{n} \frac{\beta}{\lambda^2} x_i^{\frac{3}{\lambda}} \ln x_i \, e^{-2\left(\alpha x_i^{\frac{1}{\lambda}} + \frac{\beta}{2} x_i^{\frac{2}{\lambda}}\right)} +$$
$$\sum_{i=1}^{n} \frac{x_i^{\frac{2}{\lambda}} \ln x_i}{\lambda^2} e^{-4\left(\alpha x_i^{\frac{1}{\lambda}} + \frac{\beta}{2} x_i^{\frac{2}{\lambda}}\right)}$$

$$\frac{\partial f(\alpha)}{\partial \beta} = \frac{-1}{2} \sum_{i=1}^{n} \frac{n-i+0.5}{n} x_i^{\frac{3}{\lambda}} e^{-\left(\alpha x_i^{\frac{1}{\lambda}} + \frac{\beta}{2} x_i^{\frac{2}{\lambda}}\right)} + \sum_{i=1}^{n} x_i^{\frac{3}{\lambda}} e^{-2\left(\alpha x_i^{\frac{1}{\lambda}} + \frac{\beta}{2} x_i^{\frac{2}{\lambda}}\right)}$$
$$\frac{\partial g(\beta)}{\partial \alpha} = \frac{-1}{2} \sum_{i=1}^{n} \frac{n-i+0.5}{n} x_i^{\frac{3}{\lambda}} e^{-\left(\alpha x_i^{\frac{1}{\lambda}} + \frac{\beta}{2} x_i^{\frac{2}{\lambda}}\right)} + \sum_{i=1}^{n} x_i^{\frac{3}{\lambda}} e^{-2\left(\alpha x_i^{\frac{1}{\lambda}} + \frac{\beta}{2} x_i^{\frac{2}{\lambda}}\right)}$$

3.3 The Maximum Entropy Estimation Method (MEE)

The entropy of statistics deals with a gauge of uncertainty or imbalance companion with probability function.

The idea of the maximum entropy estimation is a tool for inference under uncertainty. This technique produces the most convenient probability density function that gives the present information as seeking of the probability density function which maximizes the entropy information subject to the constraints of information that is denoted by the Lagrange multipliers method.

Finally, the outcome is probability density function which is consistent with the known constraints formulated as averages or expected values of one or more quantities. Otherwise, we obtain the least biased estimate possible on the given information [7,8]

There are four steps in the maximum entropy estimation method to estimate the three parameters of generalized exponential- Rayleigh distribution which is as follows.

1. Specification of suitable constraints.
2. Construction of the Lagrange multipliers.

3. Derivation of the entropy function of the distribution.
4. Derivation of the relation between the Lagrange multiplier and the constraints.

Step1

$$f(t;\ \alpha,\beta,\lambda) = \left(\frac{\alpha}{\lambda}\ t^{\frac{1}{\lambda}-1} + \frac{\beta}{\lambda}\ t^{\frac{2}{\lambda}-1}\right)\ e^{-\left(\alpha t^{\frac{1}{\lambda}} + \frac{\beta}{2} t^{\frac{2}{\lambda}}\right)} \qquad (3.3.1)$$

$\Omega = \{\alpha,\beta,\lambda; \alpha > 0, \beta > 0, \lambda > 0\}$

The Boltzmann- Gibbs- Shannon entropy is:

$$\delta = -\int_0^\infty f(t)\ \ln f(t) dt \qquad (3.3.2)$$

Where t is the random variable of a non – negative continuous, $f(t)$ is the probability density function of t.

Now, taking the natural algorithm, then we get

$$\ln f(t) = \ln\left[\left(\frac{\alpha}{\lambda}\ t^{\frac{1}{\lambda}-1} + \frac{\beta}{\lambda}\ t^{\frac{2}{\lambda}-1}\right)\ e^{-\left(\alpha t^{\frac{1}{\lambda}} + \frac{\beta}{2} t^{\frac{2}{\lambda}}\right)}\right].$$

$$\ln f(t) = \ln\left(\frac{\alpha}{\lambda}\ t^{\frac{1}{\lambda}-1} + \frac{\beta}{\lambda}\ t^{\frac{2}{\lambda}-1}\right) - \left(\alpha t^{\frac{1}{\lambda}} + \frac{\beta}{2}\ t^{\frac{2}{\lambda}}\right)$$

By substituting in equation (3.2) we get:

$$\delta = -\int_0^\infty \left[\ln\left(\frac{\alpha}{\lambda}\ t^{\frac{1}{\lambda}-1} + \frac{\beta}{\lambda}\ t^{\frac{2}{\lambda}-1}\right) - \left(\alpha t^{\frac{1}{\lambda}} + \frac{\beta}{2}\ t^{\frac{2}{\lambda}}\right)\right]\ f(t) \qquad (3.3.3)$$

The given information that is used in the fundamentals of maximum entropy is formulated as many constraints representing the expectations of function gi (t).

$$E[gi(t)] = \int_0^\infty gi(t)\ f(t)\ dt = C_i \qquad (3.3.4)$$

$$\int_0^\infty \ln\left(\frac{\alpha}{\lambda}\ t^{\frac{1}{\lambda}-1} + \frac{\beta}{\lambda}\ t^{\frac{2}{\lambda}-1}\right)\ f(t)\ dt = E\left[\ln\left(\frac{\alpha}{\lambda}\ t^{\frac{1}{\lambda}-1} + \frac{\beta}{\lambda}\ t^{\frac{2}{\lambda}-1}\right)\ \right] = C_2$$

$$\int_0^\infty \left(\alpha\ t^{\frac{1}{\lambda}} + \frac{\beta}{2}\ t^{\frac{2}{\lambda}}\right)\ f(t)\ dt = E\left[\left(\alpha\ t^{\frac{1}{\lambda}} + \frac{\beta}{2}\ t^{\frac{2}{\lambda}}\right)\right] = C_3$$

$$\int_0^\infty f(t),\alpha,\beta,\lambda) dt = 1 = C_1 \qquad (3.3.5)$$

Where $C_i, i = 1,2,3$ are constraints

Step2

The maximization is supplementary via to the Lagrange multipliers method. Then the solution form of the maximum entropy from maximizing the Boltzmann.

Gibbs – Shannon entropy (BGS) equation (3.3.2) is given by:

$$f(t) = \exp\left[-\lambda_0 - \sum_{j=1}^{n} \lambda_i g_i(t)\right] \qquad (3.3.6)$$

Where $\lambda_i, i = 1,2,\dots,n$ are the Lagrange multipliers linked to the constraints in equation (3.3.4) and λ_0 is the multiplier linked to the additional constraint equation (3.3.5). Thus the method yields:

$$f(t) = exp\left[-\lambda_0 - \left\{\lambda_1 \ln\left(\frac{\alpha}{\lambda}\ t^{\frac{1}{\lambda}-1} + \frac{\beta}{\lambda}\ t^{\frac{2}{\lambda}-1}\right) - \lambda_2\ \left(\alpha\ t^{\frac{1}{\lambda}} + \frac{\beta}{2}\ t^{\frac{2}{\lambda}}\right)\right\}\right] \qquad (3.3.7)$$

Where $\lambda_0, \lambda_1,\ \lambda_2$ are the lagrange multipliers substituting equation (3.3.7) into equation (3.3.5) then we get:

$$\int_0^\infty f(t)\ dt\ =\ \int_0^\infty exp\left[-\lambda_0 - \left\{\lambda_1 \ln\left(\frac{\alpha}{\lambda}\ t^{\frac{1}{\lambda}-1} + \frac{\beta}{\lambda}\ t^{\frac{2}{\lambda}-1}\right) - \lambda_2\ \left(\alpha\ t^{\frac{1}{\lambda}} + \frac{\beta}{2}\ t^{\frac{2}{\lambda}}\right)\right\}\right]\ dt = 1$$

$$\int_0^\infty exp\ (-\lambda_0) \exp\left[-\lambda_1 \left(\frac{\alpha}{\lambda}\ t^{\frac{1}{\lambda}-1} + \frac{\beta}{\lambda}\ t^{\frac{2}{\lambda}-1}\right) + \lambda_2\ \left(\alpha\ t^{\frac{1}{\lambda}} + \frac{\beta}{2}\ t^{\frac{2}{\lambda}}\right)\right] dt = 1$$

$$e^{-\lambda_0}\ \int_0^\infty exp\ \left[-\lambda_1 \ln\left(\frac{\alpha}{\lambda}\ t^{\frac{1}{\lambda}-1} + \frac{\beta}{\lambda}\ t^{\frac{2}{\lambda}-1}\right) \exp\left(\lambda_2\ \left(\alpha\ t^{\frac{1}{\lambda}} + \frac{\beta}{2}\ t^{\frac{2}{\lambda}}\right)\right)\right]\ dt = 1$$

$$\int_0^\infty e^{\ln\left(\frac{\alpha}{\lambda}t^{\frac{1}{\lambda}-1}+\frac{\beta}{\lambda}t^{\frac{2}{\lambda}-1}\right)^{-\lambda_1}} . e^{\lambda_2\left(\alpha\, t^{\frac{1}{\lambda}}+\frac{\beta}{2}t^{\frac{2}{\lambda}}\right)}dt = e^{\lambda_0}$$

$$\int_0^\infty \frac{e^{\lambda_2\left(\alpha\, t^{\frac{1}{\lambda}}+\frac{\beta}{2}t^{\frac{2}{\lambda}}\right)}}{\left(\frac{\alpha}{\lambda}t^{\frac{1}{\lambda}-1}+\frac{\beta}{\lambda}t^{\frac{2}{\lambda}-1}\right)^{\lambda_1}}\, dt = e^{\lambda_0} \tag{3.3.8}$$

Recall that

$$(a+b)^n = \sum_{j=0}^{n} C_j^n\ a^j\ b^{n-j}$$

$$\begin{aligned}\left(\frac{\alpha}{\lambda}\ t^{\frac{1}{\lambda}-1}+\frac{\beta}{\lambda}\ t^{\frac{2}{\lambda}-1}\right)^{\lambda_1} &= \sum_{j=0}^{\lambda_1} C_j^{\lambda_1}\ \left(\frac{\alpha}{\lambda}\ t^{\frac{1}{\lambda}-1}\right)^j\ \left(\frac{\beta}{\lambda}\ t^{\frac{2}{\lambda}-1}\right)^{\lambda_1-j}\\ &= \sum_{j=0}^{\lambda_1} C_j^{\lambda_1}\ \left(\frac{\alpha}{\lambda}\right)^j\left(\frac{\beta}{\lambda}\right)^{\lambda_1-j}\left(t^{\frac{2}{\lambda}-1}\right)^{\lambda_1-j}\\ &= \sum_{j=0}^{\lambda_1} C_j^{\lambda_1}\ \left(\frac{\alpha}{\lambda}\right)^j\left(\frac{\beta}{\lambda}\right)^{\lambda_1-j}\left(t^{\frac{1}{\lambda}-1}\right)^j\left(t^{\frac{2}{\lambda}-1}\right)^{\lambda_1-j}\end{aligned}$$

$$\begin{aligned}\left(\frac{\alpha}{\lambda}\ t^{\frac{1}{\lambda}-1}+\frac{\beta}{\lambda}\ t^{\frac{2}{\lambda}-1}\right)^{\lambda_1} &= \sum_{j=0}^{\lambda_1} C_j^{\lambda_1}\ \left(\frac{\alpha}{\lambda}\right)^j\left(\frac{\beta}{\lambda}\right)^{\lambda_1-j} t^{\frac{\lambda_1(2-\lambda)-j}{\lambda}}\\ &= \left(\frac{\alpha}{\lambda}+\frac{\beta}{\lambda}\right)^{\lambda_1} t^{\frac{\lambda_1(2-\lambda)-j}{\lambda}}\\ &= \left(\frac{\alpha+\beta}{\lambda}\right)^{\lambda_1} t^{\frac{\lambda_1(2-\lambda)-j}{\lambda}}\end{aligned}$$

By substituting this formula in equation (3.3.8), we have

$$\int_0^\infty \frac{e^{\lambda_2\left(\alpha t^{\frac{1}{\lambda}}+\frac{\beta}{2}\ t^{\frac{2}{\lambda}}\right)}}{\left(\frac{\alpha+\beta}{\lambda}\right)^{\lambda_1} t^{\frac{\lambda_1(2-\lambda)-j}{\lambda}}} = dt =\ e^{\lambda_0}\ , \qquad \frac{1}{\alpha+\beta_1+\lambda_1}\int_0^\infty t^{\frac{j}{\lambda}-\frac{\lambda_1(2-\lambda)}{\lambda}}\ e^{\lambda_2\left(\alpha t^{\frac{1}{\lambda}}+\frac{\beta}{2}t^{\frac{2}{\lambda}}\right)}\ dt = e^{\lambda_0}$$

$$\frac{1}{\left(\frac{\alpha+\beta}{\lambda}\right)^{\lambda_1}}\ \int_0^\infty t^{\frac{j}{\lambda}-\frac{\lambda_1(2-\lambda)}{\lambda}} e^{\alpha\lambda_2 t^{\frac{1}{\lambda}}} e^{\frac{\beta\lambda_2}{2}\,t^{\frac{2}{\lambda}}}\, dt = e^{\lambda_0} \tag{3.3.9}$$

Recall that

$e^x = \sum_{r=0}^{\infty}\frac{x^r}{r!}$ Then, $\quad e^{\alpha\lambda_2 t^{\frac{1}{\lambda}}} = \sum_{r=0}^{\infty}\frac{\left(\alpha\lambda_2 t^{\frac{1}{\lambda}}\right)^r}{r!} = \sum_{r=0}^{\infty}\frac{(\alpha\lambda_2)^r t^{\frac{r}{\lambda}}}{r!}$

By substituting this formula in equation (3.3.9), we get:

$$\frac{1}{\left(\frac{\alpha+\beta}{\lambda}\right)^{\lambda_1}}\ \int_0^\infty \sum_{r=0}^{\infty}\frac{(\alpha\lambda_2)^r t^{\frac{r}{\lambda}}}{r!} t^{\frac{j}{\lambda}-\frac{\lambda_1(2-\lambda)}{\lambda}}\ e^{\frac{\beta\lambda_2}{2}\,t^{\frac{2}{\lambda}}}\, dt = e^{\lambda_0}$$

$$\frac{\sum_{r=0}^{\infty}\frac{(\alpha\lambda_2)^r}{r!}}{\left(\frac{\alpha+\beta}{\lambda}\right)^{\lambda_1}}\ \int_0^\infty t^{\frac{r}{\lambda}+\frac{j}{\lambda}-\frac{\lambda_1(2-\lambda)}{\lambda}}\ e^{\frac{\beta\lambda_2}{2}\,t^{\frac{2}{\lambda}}}\, dt = e^{\lambda_0}$$

$$\frac{e^{\alpha\lambda_2}}{\left(\frac{\alpha+\beta}{\lambda}\right)^{\lambda_1}}\ \int_0^\infty t^{\frac{r+j}{\lambda}-\frac{\lambda_1(2-\lambda)}{\lambda}}\ e^{\frac{\beta\lambda_2}{2}\,t^{\frac{2}{\lambda}}}\, dt = e^{\lambda_0} \tag{3.3.10}$$

Let $z = t^{\frac{2}{\lambda}} \Rightarrow z^{\frac{\lambda}{2}} = t\ \Rightarrow dt = \frac{\lambda}{2} z^{\frac{\lambda}{2}-1}dz$

By substituting in equation (3.3.10), we have

$$e^{\alpha\lambda_2}\left(\frac{\alpha+\beta}{\lambda}\right)^{-\lambda_1}\ \int_0^\infty (\ z^{\frac{\lambda}{2}})^{\frac{r+j}{\lambda}-\frac{\lambda_1(2-\lambda)}{\lambda}}\ e^{\frac{\beta\lambda_2}{2}z}\ \frac{\lambda}{2} z^{\frac{\lambda}{2}-1}dz = e^{\lambda_0}$$

$e^{\alpha\lambda_2}\left(\frac{\alpha+\beta}{\lambda}\right)^{-\lambda_1}\left(\frac{\lambda}{2}\right)\ \int_0^\infty z^{\frac{r+j}{2}-\frac{\lambda_1(2-\lambda)}{2}}\ z^{\frac{\lambda}{2}-1} e^{\frac{\beta\lambda_2}{2}z}\, dz = e^{\lambda_0}\,\frac{\beta^\alpha}{\Gamma\alpha}$ we have

$\alpha - 1 = \frac{r+j}{2} - \frac{\lambda_1(2-\lambda)}{2} + \frac{\lambda}{2} - 1, \qquad \alpha = \frac{r+j}{2} - \frac{\lambda_1(2-\lambda)}{2} + \frac{\lambda}{2}, \qquad \beta = \frac{-\beta\lambda_2}{2}$

$$e^{\alpha\lambda_2}\left(\frac{\alpha+\beta}{\lambda}\right)^{-\lambda_1}\left(\frac{\lambda}{2}\right)\frac{\sqrt{\frac{r+j}{2}-\frac{\lambda_1(2-\lambda)}{2}+\frac{\lambda}{2}}}{\left(\frac{-\beta\lambda_2}{2}\right)^{\frac{r+j}{2}-\frac{\lambda_1(2-\lambda)}{2}+\frac{\lambda}{2}}}=e^{\lambda_0}$$

$$e^{\alpha\lambda_2}\left(\frac{\alpha+\beta}{\lambda}\right)^{-\lambda_1}\left(\frac{\lambda}{2}\right)\int_0^\infty z^{\frac{r+j}{2}-\frac{\lambda_1(2-\lambda)}{2}+\frac{\lambda}{2}-1}e^{-\left(\frac{-\beta\lambda_2}{2}z\right)}dz=e^{\lambda_0}$$

The border between integral is negative gamma distribution, then we must multiply and divided by

$$\frac{\lambda e^{\alpha\lambda_2}\sqrt{\frac{r+j}{2}-\frac{\lambda_1(2-\lambda)}{2}+\frac{\lambda}{2}}}{2\left(\frac{\alpha+\beta}{\lambda}\right)^{\lambda_1}\left(\frac{-\beta\lambda_2}{2}\right)^{\frac{r+j}{2}-\frac{\lambda_1(2-\lambda)}{2}+\frac{\lambda}{2}}}=e^{\lambda_0}\quad. \tag{3.3.11}$$

Equation (3.3.11) expresses the zeroth Lagrange λ_0 multiplier as a function of Lagrange multipliers λ_1 and λ_2.

Step 3: Derivation of the entropy function of the distribution. Substituting equation (3.3.11) into equation (3.2.7) we have

$$f(t)=exp\left[-\lambda_0-\left\{\lambda_1\ln\left(\frac{\alpha}{\lambda}\,t^{\frac{1}{\lambda}-1}+\frac{\beta}{\lambda}\,t^{\frac{2}{\lambda}-1}\right)-\lambda_2\left(\alpha\,t^{\frac{1}{\lambda}}+\frac{\beta}{2}\,t^{\frac{2}{\lambda}}\right)\right\}\right]$$

$$f(t)=\frac{e^{-\lambda_0}\,e^{\lambda_2\left(\alpha\,t^{\frac{1}{\lambda}}+\frac{\beta}{2}t^{\frac{2}{\lambda}}\right)}}{\left(\frac{\alpha}{\lambda}t^{\frac{1}{\lambda}-1}+\frac{\beta}{\lambda}t^{\frac{2}{\lambda}-1}\right)^{-\lambda}},\qquad f(t)=\frac{2\left(\frac{\alpha+\beta}{\lambda}\right)^{\lambda_1}\left(-\frac{\beta\lambda_2}{2}\right)^{\frac{r+j}{2}-\frac{\lambda_1(2-\lambda)}{2}+\frac{\lambda}{2}}e^{\lambda_2\left(\alpha\,t^{\frac{1}{\lambda}}+\frac{\beta}{2}t^{\frac{2}{\lambda}}\right)}}{\lambda\,e^{\alpha\lambda_2}\sqrt{\frac{r+j}{2}-\frac{\lambda_1(2-\lambda)}{2}+\frac{\lambda}{2}}\left(\frac{\alpha}{\lambda}t^{\frac{1}{\lambda}-1}+\frac{\beta}{\lambda}t^{\frac{2}{\lambda}-1}\right)^{-\lambda}}$$

By taking the natural logarithms, then we get

$$\ln f(t)=\ln\left[\frac{2\left(\frac{\alpha+\beta}{\lambda}\right)^{\lambda_1}\left(-\frac{\beta\lambda_2}{2}\right)^{\frac{r+j}{2}-\frac{\lambda_1(2-\lambda)}{2}+\frac{\lambda}{2}}e^{\lambda_2\left(\alpha\,t^{\frac{1}{\lambda}}+\frac{\beta}{2}t^{\frac{2}{\lambda}}\right)}}{\lambda\,e^{\alpha\lambda_2}\sqrt{\frac{r+j}{2}-\frac{\lambda_1(2-\lambda)}{2}+\frac{\lambda}{2}}\left(\frac{\alpha}{\lambda}t^{\frac{1}{\lambda}-1}+\frac{\beta}{\lambda}t^{\frac{2}{\lambda}-1}\right)^{-\lambda}}\right]$$

$$\ln f(t)=\ln 2+\lambda_1\ln\left(\frac{\alpha+\beta}{\lambda}\right)+\left(\frac{r+j}{2}-\frac{\lambda_1(2-\lambda)}{2}+\frac{\lambda}{2}\right)\ln\left(-\frac{\beta\lambda_2}{2}\right)$$
$$+\lambda_2\left(\alpha\,t^{\frac{1}{\lambda}}+\frac{\beta}{2}\,t^{\frac{2}{\lambda}}\right)-\ln\lambda-\alpha\,\lambda_2-\ln\Gamma\left(\frac{r+j}{2}-\frac{\lambda_1(2-\lambda)}{2}+\frac{\lambda}{2}\right)-\lambda_1\ln\left(\frac{\alpha}{\lambda}\,t^{\frac{1}{\lambda}-1}+\frac{\beta}{\lambda}\,t^{\frac{2}{\lambda}-1}\right)$$

Hence, from the definition of entropy, equation (3.3.2)

$$s=-\ln 2-\lambda_1\ln\left(\frac{\alpha+\beta}{\lambda}\right)-\left(\frac{r+j}{2}-\frac{\lambda_1(2-\lambda)}{2}+\frac{\lambda}{2}\right)\ln\left(-\frac{\beta\lambda_2}{2}\right)-\lambda_2\left(\alpha\,t^{\frac{1}{\lambda}}+\frac{\beta}{2}\,t^{\frac{2}{\lambda}}\right)+\ln\lambda+\alpha\lambda_2$$
$$+\ln\Gamma\left(\frac{r+j}{2}-\frac{\lambda_1(2-\lambda)}{2}+\frac{\lambda}{2}\right)+\lambda_1\ln\left(\frac{\alpha}{\lambda}\,t^{\frac{1}{\lambda}-1}+\frac{\beta}{\lambda}\,t^{\frac{2}{\lambda}-1}\right)$$

$$s=-\ln 2-\lambda_1\ln\left(\frac{\alpha+\beta}{\lambda}\right)-\left(\frac{r+j}{2}-\frac{\lambda_1(2-\lambda)}{2}+\frac{\lambda}{2}\right)\ln\left(-\frac{\beta\lambda_2}{2}\right)$$
$$-\lambda_2E\left(\alpha\,t^{\frac{1}{\lambda}}+\frac{\beta}{2}\,t^{\frac{2}{\lambda}}\right)+\ln\lambda+\alpha\lambda_2+\ln\Gamma\left(\frac{r+j}{2}-\frac{\lambda_1(2-\lambda)}{2}+\frac{\lambda}{2}\right)$$
$$+\lambda_1E\left[\ln\left(\frac{\alpha}{\lambda}\,t^{\frac{1}{\lambda}-1}+\frac{\beta}{\lambda}\,t^{\frac{2}{\lambda}-1}\right)\right] \tag{3.3.12}$$

Step 4: Derivation of the relation between the Lagrange multipliers and constraints

$let\ a=\frac{r+j}{2}-\frac{\lambda_1(2-\lambda)}{2}+\frac{\lambda}{2}=\frac{\lambda}{2}+\frac{j}{2}-\lambda_1+\frac{\lambda_1\lambda}{2}+\frac{\lambda}{2},\qquad da=\left(\frac{\lambda_1}{2}+\frac{1}{2}\right)d\lambda$

Since $\frac{\partial}{\partial t}\ln\Gamma(t)=\psi(t)$ is the digamma function, it follow that

$\frac{da}{d\lambda}=\ln\Gamma\left(\frac{r+j}{2}-\frac{\lambda_1(2-\lambda)}{2}+\frac{\lambda}{2}\right)=\left(\frac{\lambda_1}{2}+\frac{1}{2}\right)\psi(a)$

There are five parameters in equation (3.3.12) which are $\alpha,\beta,\lambda,\lambda_1$, and$\lambda_2$. To maximize equation (3.3.12), we need to set the following partial derivative as follows:

$$\frac{\partial s}{\partial \alpha}=\frac{-\lambda_1}{\alpha+\beta}+\lambda_2 E\left(t^{\frac{1}{\lambda}}\right)+\lambda_2+\lambda_1 E\left[\frac{t^{\frac{1}{\lambda}-1}}{\alpha t^{\frac{1}{\lambda}-1}+\beta t^{\frac{2}{\lambda}-1}}\right]$$

$$\frac{\partial s}{\partial \lambda}=\frac{\lambda_1}{\lambda}-\left(\frac{\lambda_1}{2}+\frac{1}{2}\right)\ln\left(\frac{-\beta\lambda_2}{2}\right)+\frac{1}{\lambda}+\left(\frac{\lambda_1}{2}+\frac{1}{2}\right)\psi(a)+\lambda_2 E\left[\frac{1}{\lambda^2}\left(\alpha t^{\frac{1}{\lambda}}\ln t+\beta\, t^{\frac{2}{\lambda}}\ln t\right]\right.$$

$$-\lambda_1 E\left[\frac{\frac{1}{\lambda_3}\left(\alpha t^{\frac{1}{\lambda}-1}\ln t+2\beta t^{\frac{2}{\lambda}-1}\ln t\right)}{\alpha t^{\frac{1}{\lambda}-1}+\beta t^{\frac{2}{\lambda}-1}}+\frac{1}{\lambda^2}\right]$$

$$\frac{\partial s}{\partial \beta}=\frac{-\lambda_1}{\alpha+\beta}-\frac{1}{\beta}\left(\frac{r+j}{2}+\frac{\lambda_1(2-\lambda)}{2}+\frac{\lambda}{2}\right)-\frac{\lambda_2}{2}E\left(t^{\frac{2}{\lambda}}\right)+\lambda_1 E\left[\frac{t^{\frac{2}{\lambda}-1}}{\alpha t^{\frac{1}{\lambda}-1}+\beta t^{\frac{2}{\lambda}-1}}\right]$$

$b=\frac{r+j}{2}-\frac{\lambda_1(2-\lambda)}{2}+\frac{\lambda}{2}=\frac{r}{2}+\frac{j}{2}-\lambda_1+\frac{\lambda_1\lambda}{2}+\frac{\lambda}{2}, \qquad db=\left(-1+\frac{\lambda}{2}\right)d\lambda_1$

Since $\frac{\partial}{\partial t}\ln\Gamma(t)=\psi(t)$ is the digamma function, it is follow that

$\frac{db}{\partial\lambda_1}=\left(-1+\frac{\lambda}{2}\right)\ \psi(b)=\left(\frac{\lambda}{2}-1\right)\ \psi(b)$

$\frac{\partial s}{\partial\lambda_1}=-\ln\left(\frac{\alpha+\beta}{\lambda}\right)-\left(\frac{r+j}{2}-\frac{2-\lambda}{2}\right)\ln\left(\frac{-\beta\lambda_2}{2}\right)+\left(\frac{\lambda}{2}-1\right)\psi(b)+E\left[\ln\left(\frac{\alpha}{\lambda}\,t^{\frac{1}{\lambda}-1}+\frac{\beta}{\lambda}t^{\frac{2}{\lambda}-1}\right)\right]$

$\frac{\partial s}{\partial\lambda_2}=-\frac{1}{\lambda_2}\left(\frac{r+j}{2}-\frac{\lambda_1(2-\lambda)}{2}+\frac{\lambda}{2}\right)-E\ \left(\alpha t^{\frac{1}{\lambda}}+\frac{\beta}{2}t^{\frac{2}{\lambda}}\right)+\alpha$

The values of parameters are $\alpha>0,\beta>0,\lambda>0,\lambda_2<0,\infty<\lambda_1<\infty$ By the definition of expectation of the random variable
$E\,(Y)=\sum_{ally} y\,p(Y=y)$. Assume that $p(Y=y)=1$,Then the equations become as follows:

$E\left(t^{\frac{1}{\lambda}}\right)\approx\sum_{i=1}^{n}t_i^{\frac{1}{\lambda}}$

$E\left[\frac{t^{\frac{1}{\lambda}-1}}{\alpha t^{\frac{1}{\lambda}-1}+\beta t^{\frac{2}{\lambda}-1}}\right]\approx\sum_{i=1}^{n}\frac{t_i^{\frac{1}{\lambda}-1}}{\alpha t_i^{\frac{1}{\lambda}-1}+\beta t_i^{\frac{2}{\lambda}-1}}$

$E\ \left[\frac{1}{\lambda^2}\left(\alpha t^{\frac{1}{\lambda}}\ln t+\beta t^{\frac{2}{\lambda}}\ln t\right)\right]\approx\sum_{i=1}^{n}\left[\frac{1}{\lambda^2}\left(\alpha t_i^{\frac{1}{\lambda}}\ln t+\ \beta t_i^{\frac{2}{\lambda}}\ln t_i\right)\right]$

$E\left[\frac{t^{\frac{2}{\lambda}-1}}{\alpha t^{\frac{1}{\lambda}-1}+\beta\, t^{\frac{2}{\lambda}-1}}\right]\approx\sum_{i=1}^{n}\frac{t^{\frac{2}{\lambda}-1}}{\alpha t_i^{\frac{1}{\lambda}-1}+\beta\, t_i^{\frac{2}{\lambda}-1}}$

$E\left[t^{\frac{2}{\lambda}}\right]\approx\sum_{i=1}^{n}t_i^{\frac{2}{\lambda}}$

$$f(\alpha)=\frac{\partial s}{\partial\alpha}=\frac{-\lambda_1}{\alpha+\beta}+\lambda_2\sum_{i=1}^{n}t_i^{\frac{1}{\lambda}}+\lambda_2+\lambda_1\sum_{i=1}^{n}\left[\frac{t_i^{\frac{1}{\lambda}-1}}{\alpha t_i^{\frac{1}{\lambda}-1}+\beta t_i^{\frac{2}{\lambda}-1}}\right]$$

$f(\lambda)=\frac{\partial s}{\partial\alpha}=\frac{\lambda_1}{\lambda}-\left(\frac{\lambda_1}{2}+\frac{1}{2}\right)\ln\left(\frac{-\beta\lambda_2}{2}\right)+\frac{1}{\lambda}+\left(\frac{\lambda_1}{2}+\frac{1}{2}\right)\psi\,(a)$

$+\lambda_2\sum_{i=1}^{n}\frac{1}{\lambda^2}\ (\alpha t_i^{\frac{1}{\lambda}}\ln t_i+\beta\, t_i^{\frac{2}{\lambda}}\ln t_i)-\lambda_1\sum_{i=1}^{n}\left[\frac{\frac{1}{\lambda^3}(\alpha t_i^{\frac{1}{\lambda}-1}\ln t_i+2\beta t_i^{\frac{2}{\lambda}-1}\ln t_i}{\alpha t_i^{\frac{1}{\lambda}-1}+\beta t_i^{\frac{2}{\lambda}-1}}+\frac{1}{\lambda^2}\right]$

$$f(\beta)=-\frac{\lambda_1}{\alpha+\beta}-\frac{1}{\beta}\left(\frac{r+j}{2}-\frac{\lambda_1(2-\lambda)}{2}+\frac{\lambda}{2}\right)-\frac{\lambda_2}{2}\sum_{i=1}^{n}t_i^{\frac{2}{\lambda}}+\lambda_1\sum_{i=1}^{n}\frac{t_i^{\frac{2}{\lambda}-1}}{\alpha t_i^{\frac{1}{\lambda}-1}+\beta t_i^{\frac{2}{\lambda}-1}}$$

$$\frac{\partial f(\alpha)}{\partial \alpha} = \frac{\lambda_1(1)}{(\alpha+\beta)^2} - \lambda_1 \sum_{i=1}^{n} \frac{t_i^{\frac{1}{\lambda}-1} * t_i^{\frac{1}{\lambda}-1}}{\left(\alpha t^{\frac{1}{\lambda}-1} + \beta\, t^{\frac{2}{\lambda}-1}\right)^2} = \frac{\lambda_1}{(\alpha+\beta)^2} - \lambda_1 \sum_{i=1}^{n} \frac{t_i^{\frac{2}{\lambda}-2}}{\left(\alpha t_i^{\frac{1}{\lambda}-1} + \beta\, t_i^{\frac{2}{\lambda}-1}\right)^2}$$

$$\frac{\partial f(\alpha)}{\partial \beta} = \frac{\lambda_1(1)}{(\alpha+\beta)^2} - \lambda_1 \sum_{i=1}^{n} \frac{t_i^{\frac{1}{\lambda}-1} * t_i^{\frac{2}{\lambda}-1}}{\left(\alpha t^{\frac{1}{\lambda}-1} + \beta\, t^{\frac{2}{\lambda}-1}\right)^2} = \frac{\lambda_1}{(\alpha+\beta)^2} - \lambda_1 \sum_{i=1}^{n} \frac{t_i^{\frac{3}{\lambda}-2}}{\left(\alpha t_i^{\frac{1}{\lambda}-1} + \beta\, t_i^{\frac{2}{\lambda}-1}\right)^2}$$

$$\frac{\partial f(\alpha)}{\partial \lambda} = \frac{-\lambda_2}{\lambda^2} \sum_{i=1}^{n} t_i^{\frac{1}{\lambda}} \ln t_i - \frac{\lambda_1}{\lambda^2} \sum_{i=1}^{n} \frac{\left(\alpha t_i^{\frac{2}{\lambda}-2} \ln t_i + \beta\, t_i^{\frac{3}{\lambda}-2} \ln t_i\right) + \frac{\lambda_1}{\lambda^2} \sum \left(\alpha t_i^{\frac{2}{\lambda}-2} \ln t_i + 2\beta\, t_i^{\frac{3}{\lambda}-2} \ln t_i\right)}{\left(\alpha t_i^{\frac{1}{\lambda}-1} + \beta\, t_i^{\frac{2}{\lambda}-1}\right)^2}$$

$$\frac{f(\lambda)}{\partial \alpha} = \frac{\lambda_2}{\lambda^2} \sum_{i=1}^{n} \left(t_i^{\frac{1}{\lambda}} \ln t_i\right) - \frac{\lambda_1}{\lambda^3}\, \frac{\sum_{i=1}^{n}\left(\alpha t_i^{\frac{2}{\lambda}-2} \ln t_i + \beta\, t_i^{\frac{3}{\lambda}-2} \ln t_i\right) + \frac{\lambda_1}{\lambda^3} \sum_{i=1}^{n} \left(\alpha t_i^{\frac{2}{\lambda}-2} \ln t_i + 2\beta\, t_i^{\frac{3}{\lambda}-2} \ln t_i\right)}{\left(\alpha t_i^{\frac{1}{\lambda}-1} + \beta\, t_i^{\frac{2}{\lambda}-1}\right)^2}$$

$$\frac{f(\lambda)}{\partial \beta} = -\left(\frac{\lambda_1}{2} + \frac{1}{2}\right) . \frac{1}{\beta} + \frac{\lambda_2}{\lambda^2} \sum_{i=1}^{n} (t_i^{\frac{2}{\lambda}} \ln t_i) - \frac{\lambda_1}{\lambda^3} \sum_{i=1}^{n} \frac{\left(2\alpha t_i^{\frac{3}{\lambda}-2} \ln t_i + 2\beta\, t_i^{\frac{4}{\lambda}-2} \ln t_i\right)}{\left(\alpha t_i^{\frac{1}{\lambda}-1} + \beta\, t_i^{\frac{2}{\lambda}-1}\right)^2}$$
$$- \frac{\lambda_1}{\lambda^3} \sum_{i=1}^{n} \frac{\left(\alpha t_i^{\frac{3}{\lambda}-2} \ln t_i + 2\beta\, t_i^{\frac{4}{\lambda}-2} \ln t_i\right)}{\left(\alpha t_i^{\frac{1}{\lambda}-1} + \beta\, t_i^{\frac{2}{\lambda}-1}\right)^2}$$

$$\frac{f(\lambda)}{\partial \lambda} = \frac{-(\lambda_1+1)}{\lambda^2} + \left(\frac{\lambda_1}{2} + \frac{1}{2}\right) \acute{\psi}(a) \frac{\partial a}{\partial \lambda} - \frac{\lambda_2}{\lambda^4} \sum_{i=1}^{n} \left(\alpha t_i^{\frac{1}{\lambda}} (\ln t_i)^2 + 2\beta\, t_i^{\frac{2}{\lambda}} (\ln t_i)^2\right)$$
$$- \frac{2\lambda_2}{\lambda^3} \sum_{i=1}^{n} \left(\alpha t_i^{\frac{1}{\lambda}} \ln t_i + \beta\, t_i^{\frac{2}{\lambda}} \ln t_i\right) + \frac{\lambda_1}{\lambda^5} \sum_{i=1}^{n} \frac{\left(\alpha t_i^{\frac{1}{\lambda}-1} + 2\beta\, t_i^{\frac{2}{\lambda}-1}\right) \sum_{i=1}^{n} \left(\alpha t_i^{\frac{1}{\lambda}-1} (\ln t_i)^2 + 4\beta\, t_i^{\frac{2}{\lambda}-1} (\ln t_i)^2\right)}{\left(\alpha t_i^{\frac{1}{\lambda}-1} + \beta\, t_i^{\frac{2}{\lambda}-1}\right)^2}$$
$$+ \frac{\frac{3\lambda_1}{\lambda^4} \sum_{i=1}^{n} \left(\alpha t_i^{\frac{1}{\lambda}-1} + \beta\, t_i^{\frac{2}{\lambda}-1}\right)\left(\alpha t_i^{\frac{1}{\lambda}-1} \ln t_i + 2\beta\, t_i^{\frac{2}{\lambda}-1} \ln t_i\right)}{\left(\alpha t_i^{\frac{1}{\lambda}-1} + \beta\, t_i^{\frac{2}{\lambda}-1}\right)^2} -$$
$$\frac{\frac{-\lambda_1}{\lambda^5} \sum_{i=1}^{n} \left(\alpha t_i^{\frac{1}{\lambda}-1} \ln t_i + 2\beta\, t_i^{\frac{2}{\lambda}-1} \ln t_i\right)\left(\alpha t_i^{\frac{1}{\lambda}-1} \ln t_i + 2\beta\, t_i^{\frac{2}{\lambda}-1} \ln t_i\right)}{\left(\alpha t_i^{\frac{1}{\lambda}-1} + \beta\, t_i^{\frac{2}{\lambda}-1}\right)^2}$$

$$\frac{f(\lambda)}{\partial \lambda} = \frac{-(\lambda_1+1)}{\lambda^2} + \left(\frac{\lambda_1}{2} + \frac{1}{2}\right)^2 \left(\frac{-j-r-2}{(\lambda-2)(j+\lambda\lambda_1+\lambda-2\lambda_1+r)^2} + \frac{-j-r-1}{(\lambda-2)(j+\lambda\lambda_1+\lambda-2\lambda_1+r)^2} + \frac{1}{\lambda-2}\right)$$
$$- \frac{\lambda_2}{\lambda^4} \sum_{i=1}^{n} \left(\alpha t_i^{\frac{1}{\lambda}} (\ln t_i)^2 + 2\beta\, t_i^{\frac{2}{\lambda}} (\ln t_i)^2 - \frac{2\lambda_2}{\lambda^3} \sum_{i=1}^{n} \left(\alpha t_i^{\frac{1}{\lambda}} \ln t_i + 2\beta\, t_i^{\frac{2}{\lambda}} \ln t_i\right)\right.$$

$$+\frac{\lambda_1}{\lambda^5}\sum_{i=1}^{n}\frac{\left(\alpha t_i^{\frac{1}{\lambda}-1}+\beta\, t_i^{\frac{2}{\lambda}-1}\right)\left[\alpha t_i^{\frac{1}{\lambda}-1}(\ln t_i)^2+4\beta\, t_i^{\frac{2}{\lambda}-1}(\ln t_i)^2\right]}{\left(\alpha t_i^{\frac{1}{\lambda}-1}+\beta\, t_i^{\frac{2}{\lambda}-1}\right)^2}+$$

$$\frac{3\lambda_1}{\lambda^4}\sum_{i=1}^{n}\frac{\left(\alpha t_i^{\frac{1}{\lambda}-1}+\beta\, t_i^{\frac{2}{\lambda}-1}\right)\left(\alpha t_i^{\frac{1}{\lambda}-1}\ln t_i+2\beta\, t_i^{\frac{2}{\lambda}-1}\ln t_i\right)}{\left(\alpha t_i^{\frac{1}{\lambda}-1}+\beta\, t_i^{\frac{2}{\lambda}-1}\right)^2}$$

$$-\frac{\lambda_1}{\lambda^5}\frac{\sum_{i=1}^{n}\left(\alpha t_i^{\frac{1}{\lambda}-1}\ln t_i+2\beta\, t_i^{\frac{2}{\lambda}-1}\ln t_i\right)^2}{\left(\alpha t_i^{\frac{1}{\lambda}-1}+\beta\, t_i^{\frac{2}{\lambda}-1}\right)^2}\quad,\qquad j=1, r=1,\ \lambda_1=1, \lambda_2=1$$

$$\frac{\partial f(\beta)}{\partial\alpha}=\frac{\lambda_1}{(\alpha+\beta)^2}-\frac{\lambda_1\sum t_i^{\frac{2}{\lambda}-1}*t_i^{\frac{1}{\lambda}-1}}{\left(\alpha t_i^{\frac{1}{\lambda}-1}+\beta\, t_i^{\frac{2}{\lambda}-1}\right)^2}=\frac{\lambda_1}{(\alpha+\beta)^2}-\lambda_1\sum_{i=1}^{n}\frac{t_i^{\frac{3}{\lambda}-2}}{\left(\alpha t_i^{\frac{1}{\lambda}-1}+\beta\, t_i^{\frac{2}{\lambda}-1}\right)^2}=\frac{\partial f(\alpha)}{\partial\beta}$$

$$\frac{\partial f(\beta)}{\partial\beta}=\frac{\lambda_1}{(\alpha+\beta)^2}+\frac{1}{\beta^2}\left(\frac{r+j}{2}-\frac{\lambda_1(2-\lambda)}{2}+\frac{\lambda}{2}\right)-\lambda_1\sum_{i=1}^{n}\frac{t_i^{\frac{2}{\lambda}-1}*t_i^{\frac{2}{\lambda}-1}}{\left(\alpha t_i^{\frac{1}{\lambda}-1}+\beta\, t_i^{\frac{2}{\lambda}-1}\right)^2}$$

$$=\frac{\lambda_1}{(\alpha+\beta)^2}+\frac{1}{\beta^2}\left(\frac{r+j}{2}-\frac{\lambda_1(2-\lambda)}{2}+\frac{\lambda}{2}\right)-\lambda_1\sum_{i=1}^{n}\frac{t_i^{\frac{4}{\lambda}-2}}{\left(\alpha t_i^{\frac{1}{\lambda}-1}+\beta\, t_i^{\frac{2}{\lambda}-1}\right)^2}$$

$$\frac{\partial(\beta)}{\partial\lambda}=-\frac{1}{\beta}\left(\frac{\lambda_1}{2}+\frac{1}{2}\right)+\frac{\lambda_2}{\lambda^2}\sum_{i=1}^{n}t_i^{\frac{2}{\lambda}}\ln t_i-\frac{2\lambda_1}{\lambda^2}\frac{\sum_{i=1}^{n}\left(\alpha t_i^{\frac{3}{\lambda}-2}\ln t_i+\beta\, t_i^{\frac{4}{\lambda}-2}\ln t_i\right)}{\left(\alpha t_i^{\frac{1}{\lambda}-1}+\beta\, t_i^{\frac{2}{\lambda}-1}\right)^2}+$$

$$\left.\frac{\lambda_1}{\lambda^2}\frac{\sum_{i=1}^{n}\left(\alpha t_i^{\frac{3}{\lambda}-2}\ln t_i+2\beta\, t_i^{\frac{4}{\lambda}-2}\ln t_i\right)}{\left(\alpha t_i^{\frac{1}{\lambda}-1}+\beta\, t_i^{\frac{2}{\lambda}-1}\right)^2}\right]$$

4. Simulation results technique:

It has been highlighted that the Monte Carlo approach is the most general and widely used method in simulation techniques for generating data (samples) for any distribution. This approach (the simulation process) is adaptable, allowing for several tests and experiments.

To apply the Monte Carlo approach to the generalized exponential Rayleigh distribution [5,9], we employ the cumulative distribution function as shown below:

$$F(x;\alpha,\beta,\lambda)=1-e^{-\left(\alpha x^{\frac{1}{\lambda}}+\frac{\beta}{2}x^{\frac{2}{\lambda}}\right)}\qquad Then\quad u=1-e^{-\left(\alpha x^{\frac{1}{\lambda}}+\frac{\beta}{2}x^{\frac{2}{\lambda}}\right)}.$$

$$x=\left|\frac{-\alpha-\sqrt{\alpha^2-2\ln\beta+2\ln u-\beta}}{\beta}\right|^{\lambda}.$$

Because x is positive, the negative values resulting from this generation are ignored. According to above equation, different sizes of samples n=15,25, 50,75, 100 are generated and the unknown parameters are estimated by using ME, OLSE, and MEE methods. Therefore, we calculate the value of the mean square error for each method. $MSE = \sum_{i=1}^{L} \frac{(\hat{\theta}-\theta)^2}{L}$, where n is the size of sample and L is the number of repeating the experiments, L=1000, n=15,25, 50, 75,100.

5.Numerical results:

In this section, a simulation technique is used, namely the Monte-Carlo approach to finding the mean squares error for all methods that are used in this research.

Table 1: Represent the value of the estimator and mean squares error for all estimation methods with (α=0.25)

β	λ	n	MEM		OLSEM		MEEM		$best$
			$\hat{\alpha}$	Mse	$\hat{\alpha}$	Mse	$\hat{\alpha}$	Mae	
0.5	1	15	0.258324	2.90E-04	0.238545	1.67E-03	0.289428	1.57E-03	1
		25	0.251484	9.37E-06	0.404535	4.76E-02	0.39999	4.52E-02	1
		50	0.24988	5.29E-08	0.250115	2.35E-07	0.250449	2.06E-07	1
		75	0.250013	1.70E-10	0.249924	5.87E-09	0.250001	5.01E-10	1
		100	0.250002	4.08E-11	0.249651	1.93E-07	0.249651	1.95E-07	1
	2	15	0.232904	1.90E-03	0.296521	1.24E-02	0.360122	2.31E-02	1
		25	0.248796	2.53E-06	0.250857	1.94E-06	0.249418	3.53E-05	2
		50	0.249968	1.75E-07	0.250037	3.84E-07	0.249844	3.77E-08	1
		75	0.249944	3.36E-09	0.249618	1.69E-07	0.249598	1.72E-07	1
		100	0.249998	3.97E-12	0.27615	1.31E-03	0.276138	1.37E-03	1
1	1	15	0.276422	7.03E-04	0.538638	0.147788	0.456966	0.095908	1
		25	0.248133	3.98E-05	0.409786	4.87E-02	0.404728	4.40E-02	1
		50	0.249227	6.04E-07	0.351697	2.09E-02	0.351949	2.12E-02	1
		75	0.249995	1.30E-10	0.251378	3.21E-05	0.251378	3.21E-05	1
		100	0.250001	4.09E-11	0.257994	8.36E-05	0.257993	8.37E-05	1
	2	15	0.24759	7.13E-04	0.26327	2.47E-04	0.275899	6.99E-04	2
		25	0.248801	6.47E-06	0.222289	1.06E-03	0.226468	1.15E-03	1
		50	0.249781	7.54E-08	0.248545	2.30E-06	0.247878	5.30E-06	1
		75	0.250022	9.44E-10	0.263944	4.78E-04	0.263946	4.77E-04	1
		100	0.249999	2.16E-11	0.249731	1.01E-07	0.249736	9.97E-08	1
1.5	1	15	0.261887	2.86E-03	0.286884	1.31E-02	0.301585	1.18E-02	1
		25	0.249314	1.83E-06	0.268849	3.62E-04	0.271853	4.85E-04	1
		50	0.249688	1.21E-07	0.25425	1.06E-04	0.254321	1.10E-04	1
		75	0.25001	1.14E-09	0.270253	6.68E-04	0.27027	6.71E-04	1
		100	0.249998	5.18E-12	0.266484	2.70E-04	0.266477	2.75E-04	1
	2	15	0.300486	3.85E-03	0.292259	4.50E-03	0.362519	1.91E-02	1
		25	0.251763	4.91E-06	0.289972	2.20E-03	0.291296	2.13E-03	1
		50	0.250015	1.12E-08	0.260471	2.36E-04	0.260345	2.23E-04	1
		75	0.249996	1.44E-09	0.261229	2.87E-04	0.261224	2.86E-04	2
		100	0.250001	2.58E-11	0.254817	7.35E-05	0.254816	7.35E-05	1

Note that the mean square error for the moment estimation method is the best estimation from the other methods indicating that $\frac{27}{30} * 100$ =90.

Table 2: Represent the value of the estimator and mean squares error for all estimation methods with (α=0.5).

β	λ	n	MEM		OLSEM		MEEM		$best$
			$\hat{\alpha}$	Mae	$\hat{\alpha}$	Mae	$\hat{\alpha}$	Mae	
0.5	1	15	0.471463	3.51E-03	1.709969	2.501662	1.709123	2.632765	1
		25	0.49674	1.22E-05	0.502448	1.18E-05	0.504834	2.54E-05	2
		50	0.500053	6.16E-07	0.517912	6.51E-04	0.518218	6.56E-04	1
		75	0.499971	8.62E-10	0.532651	2.11E-03	0.532678	2.14E-03	1
		100	0.500003	1.75E-11	0.4999	3.73E-12	0.500001	6.63E-12	2
	2	15	0.541988	2.15E-03	0.620726	3.83E-02	0.623052	3.18E-02	1
		25	0.494823	2.68E-05	0.755702	0.129802	0.760201	0.135574	1
		50	0.499973	8.44E-08	0.524439	1.18E-03	0.52492	1.24E-03	1
		75	0.499993	3.79E-09	0.499968	1.12E-09	0.499969	2.70E-09	2
		100	0.499997	1.88E-11	0.513907	3.80E-04	0.513915	3.86E-04	1
1	1	15	0.522962	9.82E-04	0.753422	6.60E-02	0.794002	8.79E-02	1
		25	0.502799	2.19E-05	0.520574	1.03E-03	0.518754	9.20E-04	1
		50	0.500024	2.61E-07	0.57714	1.10E-02	0.576853	1.19E-02	1
		75	0.499943	3.39E-09	0.56108	6.82E-03	0.56111	6.85E-03	1
		100	0.500001	3.34E-11	0.506756	9.19E-05	0.506754	9.18E-05	1
	2	15	0.521279	5.61E-04	0.618132	4.44E-02	0.648695	0.045076	1
		25	0.500872	2.20E-06	0.531316	9.71E-03	0.532465	9.30E-03	1
		50	0.499857	7.08E-07	0.533912	1.24E-03	0.533869	1.28E-03	1
		75	0.499975	1.03E-09	0.504058	3.60E-05	0.503992	3.49E-05	1
		100	0.499996	1.60E-11	0.583439	1.30E-02	0.583431	0.013169	1
1.5	1	15	0.502806	1.23E-04	0.882473	0.152273	0.949483	0.207236	1
		25	0.500774	4.14E-06	0.516089	1.35E-03	0.513475	9.77E-04	1
		50	0.499691	1.26E-07	0.545404	4.81E-03	0.545025	4.80E-03	1
		75	0.499985	4.49E-09	0.516768	6.15E-04	0.516846	6.20E-04	1
		100	0.50001	1.03E-11	0.498253	9.60E-04	0.498256	9.60E-04	1
	2	15	0.475719	8.10E-04	0.490032	1.69E-04	0.490915	1.26E-04	3
		25	0.50049	2.68E-07	0.550634	4.18E-03	0.545988	3.26E-03	1
		50	0.500079	1.91E-08	0.582901	6.97E-03	0.583011	6.98E-03	1
		75	0.499952	4.50E-09	0.505468	7.30E-04	0.505443	7.31E-04	1
		100	0.50001	7.95E-13	0.512014	5.42E-04	0.512012	5.52E-04	1

Note that the mean square error for moment estimation method is the best estimation from the other methods indicating that $\frac{26}{30} * 100$ =86.6.

Table 3: Represent the value of the estimator and mean squares error for all estimation methods with (α=0.25).

β	λ	n	MEM		OLSEM		MEEM		$best$
			$\hat{\hat{\beta}}$	Mse_{α}	$\hat{\hat{\beta}}$	Mse_{α}	$\hat{\hat{\beta}}$	Mse_{α}	
0.5	1	15	0.464726	2.13E-03	0.298903	4.65E-02	0.311585	4.57E-02	1
		25	0.49602	1.71E-05	0.472943	1.00E-03	0.471556	1.12E-03	1
		50	0.499896	1.32E-07	0.406926	8.68E-03	0.406937	8.68E-03	1
		75	0.500021	5.23E-10	0.528865	1.69E-03	0.528839	1.69E-03	1
		100	0.499999	1.60E-12	0.519604	9.39E-04	0.519611	9.39E-04	1
	2	15	0.503915	0.002633	0.493052	0.010819	0.429573	1.92E-02	3
		25	0.497068	1.61E-05	0.456984	6.28E-03	0.459483	6.33E-03	1
		50	0.499636	4.78E-07	0.537895	7.86E-03	0.537458	7.70E-03	1
		75	0.499982	1.10E-09	0.521549	4.67E-04	0.52154	4.67E-04	1
		100	0.499996	2.07E-11	0.495301	0.005176	0.495305	5.18E-03	1
1	1	15	1.000258	5.45E-05	1.000314	2.45E-04	0.935629	4.37E-03	1
		25	1.00157	1.63E-05	0.849312	5.41E-02	0.849248	5.41E-02	1
		50	0.999919	7.23E-09	0.831448	3.10E-02	0.831763	0.030859	1
		75	1.000016	2.28E-09	1.10135	1.62E-02	1.101412	0.016195	1
		100	1.000003	9.15E-12	0.964632	2.15E-03	0.964637	2.15E-03	1
	2	15	1.02124	7.32E-04	0.831074	6.41E-02	0.830722	6.04E-02	1
		25	0.997192	1.76E-05	1.297131	0.164546	1.298636	0.166754	1
		50	1.000228	6.03E-08	0.941784	3.69E-03	0.941795	3.70E-03	1
		75	1.000026	1.23E-09	1.106374	1.15E-02	1.106303	1.15E-02	1
		100	1.000002	4.93E-12	1.031293	1.43E-03	1.031295	1.43E-03	1
1.5	1	15	1.535848	2.07E-03	1.659731	0.357156	1.696778	0.358158	1
		25	1.501646	3.21E-06	1.414983	2.38E-02	1.417792	2.57E-02	1
		50	1.49972	1.82E-07	1.64324	2.05E-02	1.643706	2.07E-02	1
		75	1.499987	1.04E-09	1.527833	8.33E-03	1.527801	8.34E-03	1
		100	1.500001	3.60E-12	1.392978	1.57E-02	1.392976	1.57E-02	1
	2	15	1.513715	3.52E-04	1.317882	0.127016	1.351833	9.86E-02	1
		25	1.502052	9.46E-06	1.601467	1.07E-02	1.603839	1.15E-02	1
		50	1.50017	1.68E-07	1.729041	5.75E-02	1.729074	0.057522	1
		75	1.500005	1.68E-09	1.504686	8.73E-03	1.504709	8.71E-03	1
		100	1.500002	5.42E-12	1.4467	1.17E-02	1.446704	0.011732	1

Note that the mean square error for the moment estimation method is the best estimation from the other methods indicating that $\frac{29}{30} * 100$ =96.66.

Table 4: Represent the value of the estimator and mean squares error for all estimation methods with (α=0.5).

$\hat{\beta}$	λ	n	MEM		OLSEM		MEEM		$best$
			$\hat{\hat{\beta}}$	Mse_{α}	$\hat{\hat{\beta}}$	Mse_{α}	$\hat{\hat{\beta}}$	Mse_{α}	
0.5	1	15	0.494696	4.84E-04	0.611324	1.46E-02	0.630322	0.017983	1
		25	0.502311	2.13E-05	0.560568	1.27E-02	0.55937	0.014139	1
		50	0.500028	5.21E-08	0.547069	3.87E-03	0.547995	3.98E-03	1
		75	0.500003	2.73E-11	0.560517	5.60E-03	0.56056	5.71E-03	1
		100	0.499995	2.63E-11	0.49098	1.10E-03	0.490982	1.00E-03	1
	2	15	0.466132	2.05E-03	0.483197	8.60E-03	0.39012	1.72E-02	1
		25	0.495919	1.96E-05	0.542362	1.80E-03	0.544874	2.01E-03	1
		50	0.500254	2.29E-07	0.500213	4.51E-03	0.499865	4.47E-03	1
		75	0.49995	2.55E-09	0.470616	1.62E-03	0.470629	1.62E-03	1
		100	0.499998	1.18E-11	0.467584	1.13E-03	0.46758	1.13E-03	1
1	1	15	0.912091	7.75E-03	1.048313	3.10E-03	1.01827	9.50E-03	2
		25	0.994416	4.72E-05	1.120559	7.27E-02	1.11995	0.068775	1
		50	0.999855	2.11E-08	0.860475	3.81E-02	0.8604	3.82E-02	1
		75	0.999983	1.05E-09	0.914948	1.18E-02	0.914971	1.18E-02	1
		100	0.999994	4.35E-11	1.090971	1.45E-02	1.090974	1.45E-02	1
	2	15	1.027409	7.82E-04	1.130087	5.51E-02	1.056763	3.59E-02	1
		25	0.999809	8.26E-06	1.113395	0.032839	1.114184	3.20E-02	1
		50	0.999732	7.19E-08	0.832589	5.92E-02	0.832622	5.94E-02	1
		75	0.999991	2.99E-09	0.858467	2.00E-02	0.858469	2.00E-02	1
		100	1.000111	6.79E-12	0.931363	0.011595	0.931369	1.16E-02	1
1.5	1	15	1.534589	1.72E-03	1.394253	1.12E-02	1.417703	7.04E-03	1
		25	1.49877	1.31E-05	1.747498	6.28E-02	1.7513	6.45E-02	1
		50	1.500254	1.76E-07	1.428695	2.66E-02	1.429198	2.64E-02	1
		75	1.499987	1.93E-09	1.476898	3.17E-02	1.476922	3.17E-02	1
		100	1.499996	1.72E-11	1.478857	1.45E-03	1.478858	1.45E-03	1
	2	15	1.546281	2.55E-03	1.548145	1.66E-02	1.510827	8.71E-03	1
		25	1.49895	1.40E-05	1.451667	4.63E-02	1.446869	4.77E-02	1
		50	1.500443	3.85E-07	1.640215	2.26E-02	1.640212	2.26E-02	1
		75	1.499934	4.86E-09	1.503022	3.70E-03	1.502982	3.71E-03	1
		100	1.50001	7.55E-14	1.657452	2.58E-02	1.657447	2.58E-02	1

Note that the mean square error for the moment estimation method is the best estimation from the other methods indicating that $\frac{29}{30} * 100$ =96.66.

Table 5: Represent the value of the estimator and mean squares error for all estimation methods with (α=0.25).

β	λ	n	MEM		OLSEM		MEEM		$best$
			$\hat{\lambda}$	Mse_α	$\hat{\lambda}$	Mse_α	$\hat{\lambda}$	Mse_α	
0.5	1	15	0.471463	1.63E-03	0.971701	1.49E-03	1.02588	2.94E-03	2
		25	0.49674	4.96E-05	1.005886	5.54E-05	1.001948	5.47E-06	3
		50	0.500053	1.14E-05	1.006218	4.19E-05	1.00546	3.16E-05	1
		75	0.499971	8.28E-06	1.006138	4.17E-05	1.0081	6.60E-05	1
		100	0.500003	4.57E-05	1.006033	3.66E-05	1.002569	6.72E-06	3
	2	15	0.541988	1.37E-03	2.026064	4.20E-03	1.983939	2.90E-04	3
		25	0.494823	1.98E-06	2.005134	3.66E-05	2.003713	2.57E-05	1
		50	0.499973	3.54E-05	2.008925	7.97E-05	2.006714	5.01E-05	1
		75	0.499993	7.56E-07	2.004103	1.83E-05	2.004741	2.66E-05	1
		100	0.499997	1.49E-05	2.002715	1.12E-05	2.005242	3.58E-05	2
1	1	15	0.522962	2.20E-03	0.998096	1.76E-03	0.95081	2.42E-03	2
		25	0.502799	3.85E-05	0.998288	1.45E-05	1.007649	6.48E-05	2
		50	0.500024	5.72E-05	1.005456	2.98E-05	1.00459	3.39E-05	2
		75	0.499943	1.76E-05	1.0072	5.30E-05	1.006024	3.63E-05	1
		100	0.500001	3.64E-05	1.006704	4.63E-05	1.002972	9.48E-06	3
	2	15	0.521279	1.64E-03	1.993994	6.00E-04	1.963324	1.51E-03	2
		25	0.500872	4.05E-05	2.002167	4.75E-06	1.999184	1.40E-06	3
		50	0.499857	2.93E-05	2.00441	2.23E-05	2.003151	1.29E-05	3
		75	0.499975	2.92E-05	2.007994	6.40E-05	2.005395	2.96E-05	1
		100	0.499996	1.81E-06	2.005752	3.70E-05	2.008047	6.50E-05	1
1.5	1	15	0.502806	6.58E-04	0.961021	1.52E-03	1.03334	2.07E-03	1
		25	0.500774	1.80E-05	1.003783	4.02E-05	0.997744	2.36E-05	1
		50	0.499691	1.64E-05	1.004512	2.17E-05	1.003736	2.04E-05	1
		75	0.499985	5.61E-06	1.00879	7.73E-05	1.002577	9.77E-06	1
		100	0.50001	4.29E-05	1.004784	2.29E-05	1.004767	2.35E-05	3
	2	15	0.475719	1.65E-04	2.028829	1.55E-03	2.050377	2.54E-03	1
		25	0.50049	5.60E-05	2.002742	1.10E-05	2.002885	8.44E-06	3
		50	0.500079	3.34E-05	2.00473	2.58E-05	2.004168	3.41E-05	2
		75	0.499952	4.47E-05	2.003139	1.47E-05	2.00455	2.91E-05	2
		100	0.50001	6.78E-06	2.004902	2.99E-05	2.001175	1.40E-06	3

Note that the mean square error for the moment estimation method is the best estimation from the other methods indicating that $\frac{13}{30} * 100$ =43.33.

Table 6: Represent the value of the estimator and mean squares error for all estimation methods with(α=0.5) .

β	λ	n	MEM		OLSEM		MEEM		$best$
			$\hat{\lambda}$	Mse_{α}	$\hat{\lambda}$	Mse_{α}	$\hat{\lambda}$	Mse_{α}	
0.5	1	15	0.99244	4.25E-03	1.030723	1.75E-03	0.990125	3.65E-03	2
		25	1.004876	3.20E-05	1.002023	7.32E-06	1.003	9.73E-06	2
		50	1.005147	2.69E-05	1.003068	1.00E-05	1.001395	2.80E-06	3
		75	1.001836	3.41E-06	1.005125	4.10E-05	1.004478	2.62E-05	1
		100	1.004728	2.90E-05	1.006067	3.68E-05	1.006693	4.75E-05	1
	2	15	2.017247	2.27E-03	1.974458	2.90E-03	2.060386	3.75E-03	1
		25	2.000963	3.15E-05	2.005122	3.25E-05	2.001633	3.21E-06	3
		50	2.003722	1.50E-05	2.007404	5.54E-05	2.000793	2.51E-06	3
		75	2.005732	3.49E-05	2.004408	3.36E-05	2.00183	5.90E-06	3
		100	2.003336	1.65E-05	2.006074	3.95E-05	2.001752	6.01E-06	3
1	1	15	0.990828	8.83E-05	0.995052	7.90E-04	1.015829	6.73E-04	1
		25	1.007518	6.11E-05	1.003496	1.28E-05	1.001228	1.32E-05	2
		50	1.006165	3.84E-05	1.00579	3.55E-05	1.001349	1.82E-06	3
		75	1.002112	4.65E-06	1.007888	6.29E-05	1.005411	3.12E-05	1
		100	1.005808	3.44E-05	1.002082	5.28E-06	1.006789	4.75E-05	2
	2	15	2.044973	2.06E-03	2.056745	3.87E-03	2.011694	7.28E-04	3
		25	2.005273	2.94E-05	2.006248	4.22E-05	2.007167	8.89E-05	1
		50	2.005824	3.44E-05	2.005684	4.04E-05	2.007352	5.54E-05	1
		75	2.000635	4.57E-07	2.007355	5.42E-05	2.003029	1.02E-05	1
		100	2.005714	3.37E-05	2.005462	3.01E-05	2.003519	1.56E-05	3
1.5	1	15	1.023702	1.00E-03	0.979387	6.64E-04	0.971061	9.91E-04	2
		25	1.006682	4.47E-05	1.00733	5.37E-05	1.008308	6.96E-05	1
		50	1.004768	2.32E-05	1.002164	8.31E-06	1.006421	4.16E-05	2
		75	1.004042	2.99E-05	1.006904	5.24E-05	1.003018	1.50E-05	3
		100	1.002447	1.12E-05	1.004981	2.71E-05	1.003606	1.43E-05	1
	2	15	2.0764	6.10E-03	1.980636	8.46E-04	1.989361	1.19E-04	3
		25	1.998982	3.05E-06	1.999427	1.48E-06	2.003591	1.30E-05	2
		50	2.00293	1.56E-05	2.003362	1.13E-05	2.004742	2.25E-05	2
		75	2.004249	2.93E-05	2.001503	2.36E-06	2.006001	4.35E-05	2
		100	2.007059	5.37E-05	2.00649	4.61E-05	2.001955	4.05E-06	3

Note that the mean square error for the maximum entropy estimation method is the best estimation from the other methods indicating that $\frac{11}{30} * 100$ =36.66.

5.Conclusion: In this paper, we suggest a novel distribution. It studies characteristics. Classical estimating methods were investigated in order to estimate the three unknown parameters of the new distribution. A simulation study was also performed to produce alternative sample sizes. We compared the various estimation techniques using the mean squared error. It should be noted that the moment estimation approach is the best since it has the lowest error for all sample sizes.

References

[1] M.J. Mohammed and I.H. Hussein, "Study of new mixture distribution", *journal of engineering and applied sciences*, vol. 14, no. 20, PP. 7566 – 7573, 2019.

[2] I.H. Hussein and S.A. Zain, "An Estimation of Survival and Hazard Function of Weighted Rayleigh Distribution", *Iraqi Journal of Science*, vol. 61, no.11, pp.3059-3071, 2020.

[3] L.K. Hissein, I.H. Hussein and H.A. Rasheed, "An estimation of solving and hazard rate function of exponential Rayleigh *Distribution", ibn al_haitham journal for pure and applied science ,* vol. 34, no. 3, PP. 93-107,2021.

[4] P.E. Oguntunde, O.S. Balogun, H.I. Okagbue, and S.A. Bishop, "the Weibull- exponential distribution, its properties and applications", *journal of applied sciences*, vol. 15, no. 11, PP. 1305 – 1311, 2015.

[5] R.N.Shalan and I.H. Hussein " Some Methods to Estimate the Parameters of Generalized Exponential Rayleigh Model by Simulation ", Iraqi Journal for Computer Science and Mathematics, , Vol. 4 No. 2 (2023) p. 118-129 .

[6] R.N.Shalan and I.H. Hussein,**"** STATISTICAL PROPERTIES OF GENERALIZED EXPONENTIAL RAYLEIGH DISTRIBUTION" Palestine Journal of Mathematics, Vol. 12(Special Issue I)(2023) , PP.46–58

[7] A.A. Noaman, S.A. Ahmed, and M.K. Hawash, "construction a new mixed probability distribution with fuzzy reliability estimation", *periodicals of engineering and natural sciences*, vol. 8, no. 2, PP. 590 – 601,2020.

[8] A. Saghir, M. Saleem, A. Khadim, and S. Tazeem, " the modified double weighted exponential distribution with properties", *mathematical theory and modelling*, vol. 5, no. 8, PP. 78 – 91,2015.

[9]] R. H. Shamkhi, W. K.. Ghafil and A.A. Jaaze,"Estimation of parameter for the Pareto distribution based on right censoring", *Int. J. Nonlinear Anal. Appl*, vol 13, no.1, PP. 1873-1877, 2022.