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*Research Article*

# Near Approximations in $G_m$-Closure Spaces

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Most real-life situations need some sort of approximation to fit mathematical models. The beauty of using topology in approximation is achieved via obtaining approximation for qualitative subgraphs without coding or using assumption. The aim of this paper is to apply near concepts in the $G_m$-closure approximation spaces. The basic notions of near approximations are introduced and sufficiently illustrated. Near approximations are considered as mathematical tools to modify the approximations of graphs. Moreover, proved results, examples, and counterexamples are provided.

## 1. Introduction

The theory of rough sets, proposed by Pawlak [1], is an extension of set theory for the study of intelligent systems characterized by insufficient and incomplete information. Using the concepts of lower and upper approximation in rough set theory, knowledge hidden in information systems may be unraveled and expressed in the form of decision rules. The notions of closure operator and closure system are very useful tools in several sections of mathematics, as an example, in algebra [2–4], topology [5–7], and computer science theory [8, 9]. Many works have appeared recently, for example, in structural analysis [10, 11], in chemistry [12], and in physics [13]. The purpose of the present work is to put a starting point for the application of abstract topological graph theory in the rough set analysis. Also, we will integrate some ideas in terms of concept in topological graph theory. Topological graph theory is a branch of mathematics, whose concepts exist not only in almost all branches of mathematics but also in many real-life applications. We believe that topological graph structure will be an important base for modification of knowledge extraction and processing.

## 2. Preliminaries

This section presents a review of some fundamental notions of Pawlak's rough sets [1, 14, 15] and $G_m$-closure spaces [10, 11].

### *2.1. Fundamental Notions of Uncertainty*

Motivation for rough set theory has come from the need to represent subsets of a universe in terms of equivalence classes of a partition of that universe. The partition characterizes a topological space, called approximation space $K = (X, R)$, where $X$ is a set called the universe and $R$ is an equivalence relation [15, 16]. The equivalence classes of $R$ are also known as the granules, elementary sets or blocks; we will use $R_x \subseteq X$ to denote the equivalence class containing $x \in X$. In the approximation space, we consider two operators, the upper and lower approximations of subsets: let $A \subseteq X$, then the lower approximation (resp., the upper approximation) of $A$ is given by

$$L(A) = \{x \in X : R_x \subseteq A\} \qquad \left(\text{resp., } U(A) = \left\{x \in X : R_x \cap A \neq \phi\right\}\right). \tag{2.1}$$

Boundary, positive, and negative regions are also defined:

$$\mathrm{Bd}_R(A) = U(A) - L(A), \qquad \mathrm{POS}_R(A) = L(A), \qquad \mathrm{NEG}_R(A) = X - U(A). \tag{2.2}$$

In an approximation space $K = (X, R)$, if $A$ and $B$ are two subsets of $X$, then directly from the definitions of lower and upper approximations, we can get the following properties of the lower and upper approximations [15]:

(1) $L(A) \subseteq A \subseteq U(A)$,

(2) $L(\phi) = U(\phi) = \phi$ and $L(X) = U(X) = X$,

(3) $U(A \cup B) = U(A) \cup U(B)$,

(4) $L(A \cap B) = L(A) \cap L(B)$,

(5) If $A \subseteq B$, then $L(A) \subseteq L(B)$,

(6) If $A \subseteq B$, then $U(A) \subseteq U(B)$,

(7) $L(A \cup B) \supseteq L(A) \cup L(B)$,

(8) $U(A \cap B) \subseteq U(A) \cap U(B)$,

(9) $L(A^c) = [U(A)]^c$,

(10) $U(A^c) = [L(A)]^c$,

(11) $L(L(A)) = U(L(A)) = L(A)$,

(12) $U(U(A)) = L(U(A)) = U(A)$.

The inexactness of a set is due to the existence of a boundary region. The greater of the boundary region of a set, means the Pawlak [1], introduced the accuracy measure which is considered as a numerical characterization of imprecision. The following definition gives the accuracy measure of a subset $A \subseteq X$ in approximation space $K = (X, R)$.

*Definition 2.1.* Let $K = (X, R)$ be an approximation space. The accuracy measure of a subset $A \subseteq X$ is defined by $\eta(A)$ and define by

$$\eta(A) = \frac{|L(A)|}{|U(A)|}, \quad \text{where } |U(A)| \neq 0. \tag{2.3}$$

The accuracy measure is also called the accuracy of approximation.

## 2.2. Fundamental Notions of $G_m$-Closure Spaces

In this section, we introduce the concepts of closure operators on digraphs; several known topological properties on the obtained $G_m$-closure spaces are studied.

*Definition 2.2* (see [10, 11]). Let $G = (V(G), E(G))$ be a digraph, $P(V(G))$ its power set of all subgraphs of $G$, and $\mathrm{Cl}_G : P(V(G)) \to P(V(G))$ a mapping associating with each subgraph $H = (V(H), E(H))$; a subgraph $\mathrm{Cl}_G(V(H)) \subseteq V(G)$ is called the closure subgraph of $H$ such that

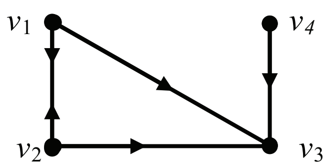
$$\mathrm{Cl}_G(V(H)) = V(H) \cup \left\{ v \in V(G) - V(H); \overrightarrow{\mathrm{hv}} \in E(G) \forall h \in V(H) \right\}. \tag{2.4}$$

The operation $\mathrm{Cl}_G$ is called graph closure operator, and the pair $(G, \mathcal{F}_G)$ is called $G$-closure space, where $\mathcal{F}_G$ is the family of elements of $\mathrm{Cl}_G$. Evidently $\mathrm{Cl}_G(V(H)) = \cap\{V(F);\ V(F) \in \mathcal{F}_G \text{ and } V(H) \subseteq V(F)\}$. The dual of the graph closure operator $\mathrm{Cl}_G$ is the graph interior operator $\mathrm{Int}_G :\ P(V(G)) \to P(V(G))$ defined by $\mathrm{Int}_G(V(H)) = V(G) - \mathrm{Cl}_G(V(G) - V(H))$ for all subgraph $H \subseteq G$. A family of elements of $\mathrm{Int}_G$ is called interior subgraph of $H$ and denoted by $\mathcal{T}_G$. It is clear that $(G, \mathcal{T}_G)$ is a topological space. Evidently $\mathrm{Int}_G(V(H)) = \cup\{V(\mathrm{O});\ V(\mathrm{O}) \in \mathcal{T}_G \text{ and } V(\mathrm{O}) \subseteq V(H)\}$. Then the domain of $\mathrm{Cl}_G$ is equal to the domain of $\mathrm{Int}_G$ and also $\mathrm{Cl}_G(V(H)) = V(G) - \mathrm{Int}_G(V(G) - V(H))$. A subgraph $H$ of $G$-closure space $(G, \mathcal{T}_G)$ is called closed subgraph if $\mathrm{Cl}_G(V(H)) = V(H)$. It is called open subgraph if its complement is closed subgraph, that is, $\mathrm{Cl}_G(V(G) - V(H)) = V(G) - V(H)$, or equivalently $\mathrm{Int}_G(V(H)) = V(H)$.

*Example 2.3.* Let $G = (V(G), E(G))$ be a digraph such that:

$V(G) = \{v_1, v_2, v_3, v_4\}$,

$E(G) = \{(v_1, v_2), (v_1, v_3), (v_2, v_1), (v_2, v_3), (v_4, v_3)\}$, for more details (Table 1)




$\mathcal{F}_G = \{V(G), \phi, \{v_3\}, \{v_3, v_4\}, \{v_1, v_2, v_3\}\}$,

$\mathcal{T}_G = \{V(G), \phi, \{v_4\}, \{v_1, v_2\}, \{v_1, v_2, v_4\}\}$.

We obtain a new definition to construct topological closure spaces from $G$-closure spaces by redefining graph closure operator on the resultant subgraphs as a domain of the graph closure operator and stop when the operator transfers each subgraph to itself.

**Table** 1

| $V(H)$ | $\mathrm{Cl}_G(V(H))$ | $V(H)$ | $\mathrm{Cl}_G(V(H))$ |
|---|---|---|---|
| $V(G)$ | $V(G)$ | $\{v_1, v_4\}$ | $V(G)$ |
| $\phi$ | $\phi$ | $\{v_2, v_3\}$ | $\{v_1, v_2, v_3\}$ |
| $\{v_1\}$ | $\{v_1, v_2, v_3\}$ | $\{v_2, v_4\}$ | $V(G)$ |
| $\{v_2\}$ | $\{v_1, v_2, v_3\}$ | $\{v_3, v_4\}$ | $\{v_3, v_4\}$ |
| $\{v_3\}$ | $\{v_3\}$ | $\{v_1, v_2, v_3\}$ | $V(G)$ |
| $\{v_4\}$ | $\{v_3, v_4\}$ | $\{v_1, v_2, v_4\}$ | $V(G)$ |
| $\{v_1, v_2\}$ | $\{v_1, v_2, v_3\}$ | $\{v_1, v_3, v_4\}$ | $V(G)$ |
| $\{v_1, v_3\}$ | $\{v_1, v_2, v_3\}$ | $\{v_2, v_3, v_4\}$ | $V(G)$ |

*Definition 2.4* (see [10, 11]). Let $G = (V(G), E(G))$ be a digraph and $\mathrm{Cl}_{Gm} : P(V(G)) \to P(V(G))$ an operator such that:

(a) It is called $G_m$-closure operator if $\mathrm{Cl}_{Gm}(V(H)) = \mathrm{Cl}_G(\mathrm{Cl}_G(\ldots \mathrm{Cl}_G(V(H))))$, $m$-times, for every subgraph $H \subseteq G$,

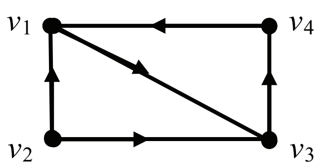
(b) it is called $G_m$-topological closure operator if $\mathrm{Cl}_{Gm+1}(V(H)) = \mathrm{Cl}_{Gm}(V(H))$ for all subgraph $H \subseteq G$.

The space $(G, \mathcal{F}_{Gm})$ is called $G_m$-closure space.

*Example 2.5.* Let $G = (V(G), E(G))$ be a digraph such that:

$$V(G) = \{v_1, v_2, v_3, v_4\},$$

$$E(G) = \{(v_1, v_3), (v_2, v_1), (v_2, v_3), (v_3, v_4), (v_4, v_1)\}, \text{ for more details (see Table 2)}$$




$$\mathcal{F}_{G2} = \{V(G), \phi, \{v_1, v_3, v_4\}\},$$

$$\mathcal{T}_{G2} = \{V(G), \phi, \{v_2\}\}.$$

**Proposition 2.6** (see [10]). *Let $(G, \mathcal{F}_{Gm})$ be a $G_m$-closure space. If $H$ and $K$ are two subgraphs of $G$ such that $H \subseteq K \subseteq G$, then*

$$\mathrm{Cl}_{Gm}(V(H)) \subseteq \mathrm{Cl}_{Gm}(V(K)), \qquad \mathrm{Int}_{Gm}(V(H)) \subseteq \mathrm{Int}_{Gm}(V(K)). \tag{2.5}$$

**Proposition 2.7** (see [10]). *Let $(G, \mathcal{F}_{Gm})$ be a $G_m$-closure space. If $H$ and $K$ are two subgraphs of $G$, then*

(a) $\mathrm{Cl}_{Gm}(V(H) \cup V(K)) = \mathrm{Cl}_{Gm}(V(H)) \cup \mathrm{Cl}_{Gm}(V(K))$,

(b) $\mathrm{Int}_{Gm}(V(H) \cap V(K)) = \mathrm{Int}_{Gm}(V(H)) \cap \mathrm{Int}_{Gm}(V(K))$.

**Table** 2

| $V(H)$ | $\mathrm{Cl}_G(V(H))$ | $\mathrm{Cl}_{G2}(V(H))$ | $V(H)$ | $\mathrm{Cl}_G(V(H))$ | $\mathrm{Cl}_{G2}(V(H))$ |
|---|---|---|---|---|---|
| $V(G)$ | $V(G)$ | $V(G)$ | $\{v_1, v_4\}$ | $\{v_1, v_3, v_4\}$ | $\{v_1, v_3, v_4\}$ |
| $\phi$ | $\phi$ | $\phi$ | $\{v_2, v_3\}$ | $V(G)$ | $V(G)$ |
| $\{v_1\}$ | $\{v_1, v_3\}$ | $\{v_1, v_3, v_4\}$ | $\{v_2, v_4\}$ | $V(G)$ | $V(G)$ |
| $\{v_2\}$ | $\{v_1, v_2, v_3\}$ | $V(G)$ | $\{v_3, v_4\}$ | $\{v_1, v_3, v_4\}$ | $\{v_1, v_3, v_4\}$ |
| $\{v_3\}$ | $\{v_3, v_4\}$ | $\{v_1, v_3, v_4\}$ | $\{v_1, v_2, v_3\}$ | $V(G)$ | $V(G)$ |
| $\{v_4\}$ | $\{v_1, v_4\}$ | $\{v_1, v_3, v_4\}$ | $\{v_1, v_2, v_4\}$ | $V(G)$ | $V(G)$ |
| $\{v_1, v_2\}$ | $\{v_1, v_2, v_3\}$ | $V(G)$ | $\{v_1, v_3, v_4\}$ | $\{v_1, v_3, v_4\}$ | $\{v_1, v_3, v_4\}$ |
| $\{v_1, v_3\}$ | $\{v_1, v_3, v_4\}$ | $\{v_1, v_3, v_4\}$ | $\{v_2, v_3, v_4\}$ | $V(G)$ | $V(G)$ |

**Proposition 2.8** (see [10]). *Let* $(G, \digamma_{Gm})$ *be a* $G_m$*-closure space. If* $H$ *and* $K$ *are two subgraphs of* $G$*, then*

(a) $\mathrm{Cl}_{Gm}(V(H) \cap V(K)) \subseteq \mathrm{Cl}_{Gm}(V(H)) \cap \mathrm{Cl}_{Gm}(V(K))$*, and*

(b) $\mathrm{Int}_{Gm}(V(H)) \cup \mathrm{Int}_{Gm}(V(K)) \subseteq \mathrm{Int}_{Gm}(V(H) \cup V(K))$.

*Remark 2.9.* The converse of Proposition 2.8 need not be true in general, as the following example (Example 2.3 in [10]).

*Definition 2.10* (see [10]). Let $(G, \digamma_{Gm})$ be a $G_m$-closure space and $H \subseteq G$; the boundary of $H$ is denoted by $\mathrm{Bd}_{Gm}(V(H))$ and is defined by

$$\mathrm{Bd}_{Gm}(V(H)) = \mathrm{Cl}_{Gm}(V(H)) - \mathrm{Int}_{Gm}(V(H)). \tag{2.6}$$

**Proposition 2.11** (see [10]). *Let* $(G, \digamma_{Gm})$ *be a* $G_m$*-closure space and* $H \subseteq G$*, then*

(a) $\mathrm{Bd}_{Gm}(V(H)) = \mathrm{Cl}_{Gm}(V(H)) \cap \mathrm{Cl}_{Gm}(V(G) - V(H))$,

(b) $\mathrm{Bd}_{Gm}(V(H)) = \mathrm{Bd}_{Gm}(V(G) - V(H))$,

(c) $\mathrm{Cl}_{Gm}(V(H)) = V(H) \cup \mathrm{Bd}_{Gm}(V(H))$,

(d) $\mathrm{Int}_{Gm}(V(H)) = V(H) - \mathrm{Bd}_{Gm}(V(H))$.

By a similar way of definitions of regular open set [17], semiopen set [18], preopen set [19], $\gamma$-open set [20] (b-open set [21]), $\alpha$-open set [22], and $\beta$-open set [23] (=semi-pre-open set [24]), we introduce the following definitions which are essential for our present study. In $G_m$-closure space $(G, \digamma_{Gm})$ the subgraph $H$ of $G$ is called

(a) regular open subgraph [10] (briefly $R$-osg) if $V(H) = \mathrm{Int}_{Gm}(\mathrm{Cl}_{Gm}(V(H)))$,

(b) semiopen subgraph [10] (briefly $S$-osg) if $V(H) \subseteq \mathrm{Cl}_{Gm}(\mathrm{Int}_{Gm}(V(H)))$,

(c) preopen subgraph [10] (briefly $P$-osg) if $V(H) \subseteq \mathrm{Int}_{Gm}(\mathrm{Cl}_{Gm}(V(H)))$,

(d) $\gamma$-open subgraph (briefly $\gamma$-osg) if $V(H) \subseteq \mathrm{Cl}_{Gm}(\mathrm{Int}_{Gm}(V(H))) \cup \mathrm{Int}_{Gm}(\mathrm{Cl}_{Gm}(V(H)))$,

(e) $\alpha$-open subgraph [10] (briefly $\alpha$-osg) if $V(H) \subseteq \mathrm{Int}_{Gm}(\mathrm{Cl}_{Gm}(\mathrm{Int}_{Gm}(V(H))))$,

(f) $\beta$-open subgraph [10] (briefly $\beta$-osg) if $V(H) \subseteq \mathrm{Cl}_{Gm}(\mathrm{Int}_{Gm}(\mathrm{Cl}_{Gm} V(H)))$.

The complement of an $R$-osg (resp., $S$-osg, $P$-osg, $\gamma$-osg, $\alpha$-osg, and $\beta$-osg) is called $R$-closed subgraph (briefly $R$-csg) (resp., $S$-csg, $P$-csg, $\gamma$-csg, $\alpha$-csg, and $\beta$-csg).

The family of all $R$-osgs (resp., $S$-osgs, $P$-osgs, $\gamma$-osgs, $\alpha$-osgs, and $\beta$-osgs) of $(G, \digamma_{Gm})$ is denoted by $\mathrm{RO}_{Gm}(G)$ (resp., $\mathrm{SO}_{Gm}(G)$, $\mathrm{PO}_{Gm}(G)$, $\gamma\mathrm{O}_{Gm}(G)$, $\alpha\mathrm{O}_{Gm}(G)$, and $\beta\mathrm{O}_{Gm}(G)$). All of $\mathrm{SO}_{Gm}(G)$, $\mathrm{PO}_{Gm}(G)$, $\gamma\mathrm{O}_{Gm}(G)$, $\alpha\mathrm{O}_{Gm}(G)$, and $\beta\mathrm{O}_{Gm}(G)$ are larger than $\tau_{Gm}$ and closed under forming arbitrary union.

The family of all $R$-csgs (resp., $S$-csgs, $P$-csgs, $\gamma$-csgs, $\alpha$-csgs, and $\beta$-csgs) of $(G, \digamma_{Gm})$ is denoted by $\mathrm{RC}_{Gm}(G)$ (resp., $\mathrm{SC}_{Gm}(G)$, $\mathrm{PC}_{Gm}(G)$, $\gamma\mathrm{C}_{Gm}(G)$, $\alpha\mathrm{C}_{Gm}(G)$, and $\beta\mathrm{C}_{Gm}(G)$).

The near closure (resp., near interior and near boundary) of a subgraph $H$ of $G$ in a $G_m$-closure space $(G, \digamma_{Gm})$ is denoted by $\mathrm{Cl}^j_{Gm}(V(H))$ (resp. $\mathrm{Int}^j_{Gm}(V(H))$ and $\mathrm{Bd}^j_{Gm}(V(H))$) and defined by

$$\begin{aligned}
&\mathrm{Cl}^j_{Gm}(V(\mathrm{H})) = \cap \{V(F);\ V(F) \text{ is } j\text{-csg and } V(H) \subseteq V(F)\},\\
&\Big(\text{resp., } \mathrm{Int}^j_{Gm}(V(H)) = V(G) - \mathrm{Cl}^j_{Gm}(V(G) - V(H)) \text{ and}\\
&\mathrm{Bd}^j_{Gm}(V(H)) = \mathrm{Cl}^j_{Gm}(V(H)) - \mathrm{Int}^j_{Gm}(V(H))\Big), \quad \text{where } j \in \{R, S, P, \gamma, \alpha, \beta\}.
\end{aligned} \tag{2.7}$$

**Proposition 2.12** (see [10]). *Let $(G, \digamma_{Gm})$ be $G_m$-closure space, the implication $\tau_{Gm}$ and the families of near-open and near-closed graphs are given by following statements:*

(a) $\mathrm{RO}_{Gm}(G) \subseteq \tau_{Gm} \subseteq \alpha\mathrm{O}_{Gm}(G) \subseteq \mathrm{SO}_{Gm}(G) \subseteq \gamma\mathrm{O}_{Gm}(G) \subseteq \beta\mathrm{O}_{Gm}(G)$,

(b) $\mathrm{RO}_{Gm}(G) \subseteq \tau_{Gm} \subseteq \alpha\mathrm{O}_{Gm}(G) \subseteq \mathrm{PO}_{Gm}(G) \subseteq \gamma\mathrm{O}_{Gm}(G) \subseteq \beta\mathrm{O}_{Gm}(G)$,

(c) $\mathrm{RC}_{Gm}(G) \subseteq \digamma_{Gm} \subseteq \alpha\mathrm{C}_{Gm}(G) \subseteq \mathrm{SC}_{Gm}(G) \subseteq \gamma\mathrm{C}_{Gm}(G) \subseteq \beta\mathrm{C}_{Gm}(G)$,

(d) $\mathrm{RC}_{Gm}(G) \subseteq \digamma_{Gm} \subseteq \alpha\mathrm{C}_{Gm}(G) \subseteq \mathrm{PC}_{Gm}(G) \subseteq \gamma\mathrm{C}_{Gm}(G) \subseteq \beta\mathrm{C}_{Gm}(G)$.

## 3. Generalization of Pawlak Approximation Spaces

In this section we will generalize Pawlak's concepts in the case of general relations. Hence, the approximation space $G_m = (G, \mathrm{Cl}_{Gm})$ with general relation $\mathrm{Cl}_{Gm}$ on $G$ (i.e., closure operator $\mathrm{Cl}_{Gm}$ on $G$) defines a uniquely $G_m$-closure space $(G, \digamma_{Gm})$, where $\digamma_{Gm}$ is the $G_m$-closure space associated with $G_m$. We will give this hypothesis in the following definition.

*Definition 3.1.* Let $G_m = (G, \mathrm{Cl}_{Gm})$ be an approximation space, where $G$ is a finite and nonempty universe graph, $\mathrm{Cl}_{Gm}$ is a general relation on $G$, and $\digamma_{Gm}$ is the $G_m$-closure space associated with $G_m$. Then the triple $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \digamma_{Gm})$ is called a $G_m$-closure approximation space.

The following definition introduces the lower and the upper approximations in a $G_m$-closure approximation space $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \digamma_{Gm})$.

*Definition 3.2.* Let $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \digamma_{Gm})$ be a $G_m$-closure approximation space and $H \subseteq G$. The lower approximation (resp., the upper approximation) of $H$ is denoted by $L(V(H))$ (resp., $U(V(H))$) and is defined by

$$L(V(H)) = \mathrm{Int}_{Gm}(V(H)) \qquad \big(\text{resp., } U(V(H)) = \mathrm{Cl}_{Gm}(V(H))\big). \tag{3.1}$$

The following definition introduces new concepts of definability for a subgraph $H \subseteq G$ in a $G_m$-closure approximation space $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \mathcal{F}_{Gm})$.

*Definition 3.3.* Let $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \mathcal{F}_{Gm})$ be a $G_m$-closure approximation space. If $H \subseteq G$, then $H$ is called

(a) totally $\mathcal{G}_m$-definable ($\mathcal{G}_m$-exact) graph if $L(V(H)) = V(H) = U(V(H))$,

(b) internally $\mathcal{G}_m$-definable graph if $L(V(H)) = V(H), U(V(H)) \neq V(H)$,

(c) externally $\mathcal{G}_m$-definable graph if $L(V(H)) \neq V(H), U(V(H)) = V(H)$,

(d) $\mathcal{G}_m$-indefinable ($\mathcal{G}_m$-rough) graph if $L(V(H)) \neq V(H), U(V(H)) \neq V(H)$.

**Proposition 3.4.** *Let $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \mathcal{F}_{Gm})$ be a $G_m$-closure approximation space. If $H$ and $K$ are subgraphs of $G$, then*

(1) $L(V(H)) \subseteq V(H) \subseteq U(V(H))$,

(2) $L(\phi) = U(\phi) = \phi$ *and* $L(V(G)) = U(V(G)) = V(G)$,

(3) $U(V(H) \cup V(K)) = U(V(H)) \cup U(V(K))$,

(4) $L(V(H) \cap V(K)) = L(V(H)) \cap L(V(K))$,

(5) *if* $H \subseteq K$, *then* $L(V(H)) \subseteq L(V(K))$,

(6) *if* $H \subseteq K$, *then* $U(V(H)) \subseteq U(V(K))$,

(7) $L(V(H) \cup V(K)) \supseteq L(V(H)) \cup L(V(K))$,

(8) $U(V(H) \cap V(K)) \subseteq U(V(H)) \cap U(V(K))$,

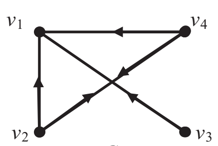
(9) $L(V(G) - V(H)) = V(G) - U(V(H))$,

(10) $U(V(G) - V(H)) = V(G) - L(V(H))$.

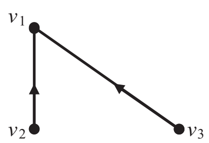
*Proof.* By using properties of $G_m$-interior and $G_m$-closure, the proof is obvious. □

The following example illustrates that properties 11 and 12 which are introduced in Section 2.1 cannot be applied for this new generalization.

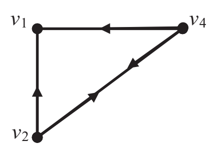
*Example 3.5.* Let $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \mathcal{F}_{Gm})$ be a $G_m$-closure approximation space such that $G = (V(G), E(G))$: $V(G) = \{v_1, v_2, v_3, v_4\}$, $E(G) = \{(v_2, v_1), (v_2, v_4), (v_3, v_1), (v_4, v_1), (v_4, v_1)\}$,



G



H



K

$$\mathcal{F}_G = \{V(G), \phi, \{v_1\}, \{v_1, v_3\}, \{v_1, v_2, v_4\}\},$$

$$\mathcal{T}_G = \{V(G), \phi, \{v_3\}, \{v_2, v_4\}, \{v_2, v_3, v_4\}\}.$$

Let $H = (V(H), E(H))$: $V(H) = \{v_1, v_2, v_3\}, E(H) = \{(v_2, v_1), (v_3, v_1)\}$, and $K = (V(K), E(K))$: $V(K) = \{v_1, v_2, v_4\}, E(K) = \{(v_2, v_1), (v_2, v_4), (v_4, v_1), (v_4, v_2)\}$. Then

$$L(L(V(H))) = L(V(H)) = \{v_3\}, \qquad U(L(V(H))) = \{v_1, v_3\}. \tag{3.2}$$

Thus,

$$L(L(V(H))) = L(V(H)) \neq U(L(V(H))). \tag{3.3}$$

Also,

$$U(U(V(H))) = U(V(H)) = \{v_1, v_2, v_4\}, \qquad L(U(V(H))) = \{v_2, v_4\}. \tag{3.4}$$

Thus,

$$U(U(V(H))) = U(V(H)) \neq L(U(V(H))). \tag{3.5}$$

**Lemma 3.6.** *Let $(G, \mathcal{F}_{Gm})$ be a $G_m$-closure space. Then*

$$\mathrm{Int}_{Gm}(V(G) - V(H)) = V(G) - \mathrm{Cl}_{Gm}(V(H)) \quad \forall \text{subgraph } \mathrm{H} \subseteq \mathrm{G}. \tag{3.6}$$

*Proof.* It follows from definition of $G_m$-closure space. □

**Lemma 3.7.** *Let $H$ be a subgraph of $G$ in the $G_m$-closure space $(G, \mathcal{F}_{Gm})$. Then $v \in \mathrm{Cl}_{Gm}(V(H))$ if and only if for each subgraph $K \subseteq G$ and $v \in \mathrm{Int}_{Gm}(V(K))$, then $\mathrm{Int}_{Gm}(V(K)) \cap V(H) \neq \phi$.*

*Proof.* (⇒) Let $v \in \mathrm{Cl}_{Gm}(V(H))$ and $v \in \mathrm{Int}_{Gm}(V(K))$ for some $K \subseteq G$. Assume $\mathrm{Int}_{Gm}(V(K)) \cap V(H) = \phi$. This implies that $V(H) \subseteq V(G) - \mathrm{Int}_{Gm}(V(K))$ which is closed graph. Hence, $v \in V(G) - \mathrm{Int}_{Gm}(V(K))$, since $v \in \mathrm{Cl}_{Gm}(V(H))$ and this leads to a contradiction. Therefore, $\mathrm{Int}_{Gm}(V(K)) \cap V(H) \neq \phi$.

(⇐) Suppose that for each $K \subseteq G$ and $v \in \mathrm{Int}_{Gm}(V(K))$, $\mathrm{Int}_{Gm}(V(K)) \cap V(H) \neq \phi$. Let $v \notin \mathrm{Cl}_{Gm}(V(H))$ which is closed. Then there exists a closed graph $F \subseteq G$ such that $F \supseteq H$ and $v \notin V(F)$. Hence, $V(G) - V(F)$ is open subgraph containing $v$. Thus, $v \in \mathrm{Int}_{Gm}(V(G) - V(F)) = V(G) - V(F)$ and $\mathrm{Int}_{Gm}(V(G) - V(F)) \cap V(H) = \phi$, that is, there exists a subgraph $K = G - F$ of $G$ such that $\mathrm{Int}_{Gm}(V(K)) \cap V(H) = \phi$, which leads to a contradiction. Therefore, $v \in \mathrm{Cl}_{Gm}(V(H)$. □

**Lemma 3.8.** *Let $H$ and $K$ be two subgraphs of $G$ in the $G_m$-closure space $(G, \mathcal{F}_{Gm})$. If $H$ is open subgraph, then $V(H) \cap \mathrm{Cl}_{Gm}(V(K)) \subseteq \mathrm{Cl}_{Gm}(V(H) \cap V(K))$.*

*Proof.* Let $v \in V(H) \cap \mathrm{Cl}_{Gm}(V(K))$. If $O$ is open subgraph such that $v \in V(O)$, then $V(O) \cap V(H)$ is an open subgraph and $v \in V(O) \cap V(H)$. Therefore, $V(O) \cap (V(H) \cap V(K)) \neq \phi$ and $v \in \mathrm{Cl}_{Gm}(V(H) \cap V(K))$. Hence, the result. □

**Proposition 3.9.** *Let* $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \mathcal{F}_{Gm})$ *be a* $G_m$*-closure approximation space. If* $H$ *and* $K$ *are subgraphs of* $G$*, then*

(a) $L(V(H) - V(K)) \subseteq L(V(H)) - L(V(K))$,

(b) $U(V(H) - V(K)) \supseteq U(V(H) - U(V(K))$.

*Proof.* (a) We need to show that $\mathrm{Int}_{Gm}(V(H) - V(K)) \subseteq \mathrm{Int}_{Gm}(V(H)) - \mathrm{Int}_{Gm}(V(K))$. Now,

$$V(H) - V(K) = V(H) \cap (V(G) - V(K)). \tag{3.7}$$

Then,

$$\begin{aligned} \mathrm{Int}_{Gm}(V(H) - V(K)) &= \mathrm{Int}_{Gm}(V(H) \cap (V(G) - V(K))) \\ &= \mathrm{Int}_{Gm}(V(H)) \cap \mathrm{Int}_{Gm}(V(G) - V(K)). \end{aligned} \tag{3.8}$$

Thus, by Lemma 3.6, we have

$$\begin{aligned} \mathrm{Int}_{Gm}(V(H) - V(K)) &= \mathrm{Int}_{Gm}(V(H)) \cap (V(G) - \mathrm{Cl}_{Gm}(V(K))) \\ &= \mathrm{Int}_{Gm}(V(H)) - \mathrm{Cl}_{Gm}(V(K)) \\ &\subseteq \mathrm{Int}_{Gm}(V(H)) - \mathrm{Int}_{Gm}(V(K)). \end{aligned} \tag{3.9}$$

Therefore,

$$\begin{aligned} L(V(H) - V(K)) = \mathrm{Int}_{Gm}(V(H) - V(K)) &\subseteq \mathrm{Int}_{Gm}(V(H)) - \mathrm{Int}_{Gm}(V(K)) \\ &= L(V(H)) - L(V(K)). \end{aligned} \tag{3.10}$$

(b) We need to show that

$$\mathrm{Cl}_{Gm}(V(H) - V(K)) \supseteq \mathrm{Cl}_{Gm}(V(H)) - \mathrm{Cl}_{Gm}(V(K)). \tag{3.11}$$

Now,

$$\mathrm{Cl}_{Gm}(V(H)) - \mathrm{Cl}_{Gm}(V(K)) = \mathrm{Cl}_{Gm}(V(H)) \cap (V(G) - \mathrm{Cl}_{Gm}(V(K))). \tag{3.12}$$

Thus, by Lemma 3.6, we have

$$\mathrm{Cl}_{Gm}(V(H)) - \mathrm{Cl}_{Gm}(V(K)) = \mathrm{Cl}_{Gm}(V(H)) \cap \mathrm{Int}_{Gm}(V(G) - V(K)). \tag{3.13}$$

Hence, by Lemma 3.8, we have

$$\begin{aligned}\mathrm{Cl}_{Gm}(V(H)) - \mathrm{Cl}_{Gm}(V(K)) &= \mathrm{Cl}_{Gm}(V(H)) \cap \mathrm{Int}_{Gm}(V(G) - V(K))\\ &\subseteq \mathrm{Cl}_{Gm}[V(H) \cap \mathrm{Int}_{Gm}(V(G) - V(K))]\\ &= \mathrm{Cl}_{Gm}[V(H) \cap V(G) - \mathrm{Cl}_{Gm}(V(K))]\\ &= \mathrm{Cl}_{Gm}[V(H) - \mathrm{Cl}_{Gm}(V(K))],\end{aligned} \tag{3.14}$$

Thus,

$$\mathrm{Cl}_{Gm}(V(H)) - \mathrm{Cl}_{Gm}(V(K)) \subseteq \mathrm{Cl}_{Gm}(V(H) - V(K)). \tag{3.15}$$

Therefore,

$$\begin{aligned}U(V(H) - V(K)) = \mathrm{Cl}_{Gm}(V(H) - V(K)) &\supseteq \mathrm{Cl}_{Gm}(V(H)) - \mathrm{Cl}_{Gm}(V(K))\\ &= U(V(H)) - U(V(K)).\end{aligned} \tag{3.16}$$

□

## 4. Near Lower and Near Upper in $G_m$-Closure Approximation Spaces

In this section, we study approximation spaces from $G_m$-closure view. We obtain some rules to find lower and upper approximations in several ways in approximation spaces with general relations. We will recall and introduce some definitions and propositions about some classes of near-open graphs which are essential for our present study.

*Definition 4.1.* Let $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \mathcal{F}_{Gm})$ be a $G_m$-closure approximation space and $H \subseteq G$. The near-lower approximation ($j$-lower approximation) (resp., near-upper approximation ($j$-upper approximation)) of $H$ is denoted by $L^j(V(H))$ (resp., $U^j(V(H))$) and is defined by

$$\begin{aligned}L^j(V(H)) = \mathrm{Int}^j_{Gm}(V(H)) \qquad &\left(\text{resp., } U^j(V(H)) = \mathrm{Cl}^j_{Gm}(V(H))\right),\\ &\text{where } j \in \{R, S, P, \gamma, \alpha, \beta\}.\end{aligned} \tag{4.1}$$

**Proposition 4.2.** *Let $\mathcal{G}_m = (G,\ Cl_{Gm},\ \mathcal{F}_{Gm})$ be a $G_m$-closure approximation space. If $H \subseteq G$, then $L(V(H)) \subseteq L^j(V(H)) \subseteq V(H) \subseteq U^j(V(H)) \subseteq U(V(H))$, for all $j \in \{S, P, \gamma, \alpha, \beta\}$.*

*Proof.* The proofs of the five cases are similar, so we will only prove the case when $j = S$. Now,

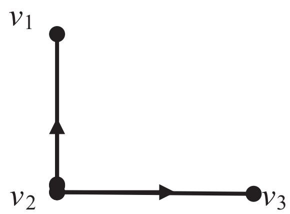
$$\begin{aligned}U(V(H)) = \mathrm{Cl}_{Gm}(V(H)) &= \cap\{V(F); V(F) \in \mathcal{F}_{Gm} \text{ and } V(H) \subseteq V(F)\}\\ &\supseteq \cap\{V(F); V(F) \in \mathrm{SC}_{Gm}(G) \text{ and } V(H) \subseteq V(F)\}\\ &\text{since } \mathcal{F}_{Gm} \subseteq \mathrm{SC}_{Gm}(G)\\ &= \mathrm{Cl}^S_{Gm}(V(H)) = U^S(V(H)) \supseteq V(H),\end{aligned} \tag{4.2}$$

$$\begin{aligned} L(V(H)) &= \mathrm{Int}_{Gm}(V(H)) = V(G) - \mathrm{Cl}_{Gm}(V(G) - V(H)) \\ &\subseteq V(G) - \mathrm{Cl}^{S}_{Gm}(V(G) - V(H)) \\ &\text{since } \mathcal{T}_{Gm} \subseteq \mathrm{SO}_{Gm}(G) \\ &= \mathrm{Int}^{S}_{Gm}(V(H)) = L^{S}(V(H)) \subseteq V(H). \end{aligned} \tag{4.3}$$

From (4.2) and (4.3) we get $L(V(H)) \subseteq L^{S}(V(H)) \subseteq V(H) \subseteq U^{S}(V(H)) \subseteq U(V(H))$. □

In general the above proposition is not true in the case of $j = R$ as the following example illustrates.

*Example 4.3.* Let $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \mathcal{F}_{Gm})$ be a $G_m$-closure approximation space such that $G = (V(G), E(G))$: $V(G) = \{v_1, v_2, v_3\}$, $E(G) = \{(v_2, v_1), (v_2, v_3)\}$,




$$\mathcal{F}_G = \{V(G), \phi, \{v_1\}, \{v_3\}, \{v_1, v_3\}\},$$

$$\mathcal{T}_G = \{V(G), \phi, \{v_2\}, \{v_1, v_2\}, \{v_2, v_3\}\}.$$

Hence, $\mathrm{RO}_{Gm}(G) = \{V(G), \phi\}$ and $\mathrm{RC}_{Gm}(G) = \{V(G), \phi\}$. If $H = (V(H), E(H))$: $V(H) = \{v_1, v_3\}, E(H) = \phi$, then

$$\begin{aligned} L(V(H)) = \mathrm{Int}_{Gm}(V(H)) = \phi, \qquad & U(V(H)) = \mathrm{Cl}_{Gm}(V(H)) = \{v_1, v_3\}, \\ L^{R}(V(H)) = \mathrm{Int}^{R}_{Gm}(V(H)) = \phi, \qquad & U^{R}(V(H)) = \mathrm{Cl}^{R}_{Gm}(V(H)) = V(G). \end{aligned} \tag{4.4}$$

Therefore,

$$L^{R}(V(H)) = L(V(H)), \qquad U(V(H)) \subseteq U^{R}(V(H)). \tag{4.5}$$

**Proposition 4.4.** *Let $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \mathcal{F}_{Gm})$ be a $G_m$- closure approximation space. If $H \subseteq G$, then the implication between lower approximation and $j$-lower approximation of $H$ are given by the following statement for all $j \in \{S, P, \gamma, \alpha, \beta\}$:*

(a) $L(V(H)) \subseteq L^{\alpha}(V(H)) \subseteq L^{S}(V(H)) \subseteq L^{\gamma}(V(H)) \subseteq L^{\beta}(V(H))$,

(b) $L(V(H)) \subseteq L^{\alpha}(V(H)) \subseteq L^{P}(V(H)) \subseteq L^{\gamma}(V(H)) \subseteq L^{\beta}(V(H))$.

*Proof.* By using Proposition 4.2, we get $L(V(H)) \subseteq L^{\alpha}(V(H))$. We will prove $L^{\alpha}(V(H)) \subseteq L^{S}(V(H))$. Now,

$$\begin{aligned} L^{\alpha}(V(H)) &= \mathrm{Int}^{\alpha}_{Gm}(V(H)) = V(G) - \mathrm{Cl}^{\alpha}_{Gm}(V(G) - V(H)) \\ &\subseteq V(G) - \mathrm{Cl}^{S}_{Gm}(V(G) - V(H)), \end{aligned} \tag{4.6}$$

since $\alpha\mathrm{O}_{Gm}(G) \subseteq \mathrm{SO}_{Gm}(G)$. Thus,

$$L^{\alpha}(V(H)) = \mathrm{Int}^{\alpha}_{Gm}(V(H)) \subseteq \mathrm{Int}^{S}_{Gm}(V(H)) = L^{S}(V(H)). \tag{4.7}$$

Similarly we can prove the other cases. □

**Proposition 4.5.** *Let $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \mathcal{F}_{Gm})$ be a $G_m$-closure approximation space. If $H \subseteq G$, then the implication between upper approximation and $j$-upper approximation of $H$ are given by the following statement for all $j \in \{S, P, \gamma, \alpha, \beta\}$,*

(a) $U^{\beta}(V(H)) \subseteq U^{\gamma}(V(H)) \subseteq U^{S}(V(H)) \subseteq U^{\alpha}(V(H)) \subseteq U(V(H))$,

(b) $U^{\beta}(V(H)) \subseteq U^{\gamma}(V(H)) \subseteq U^{P}(V(H)) \subseteq U^{\alpha}(V(H)) \subseteq U(V(H))$.

*Proof.* By using Proposition 4.2, we get $U^{\alpha}(V(H)) \subseteq U(V(H))$. We will prove $U^{P}(V(H)) \subseteq U^{\alpha}(V(H))$. Now,

$$\begin{aligned} U^{P}(V(H)) = \mathrm{Cl}^{P}_{Gm}(V(H)) &= \cap\{V(F); V(F) \in \mathrm{PC}_{Gm}(G) \text{ and } V(H) \subseteq V(F)\} \\ &\subseteq \cap\{V(F); V(F) \in \alpha\mathrm{C}_{Gm}(G) \text{ and } V(H) \subseteq V(F)\} \end{aligned} \tag{4.8}$$

since $\alpha\mathrm{C}_{Gm}(G) \subseteq \mathrm{PC}_{Gm}(G)$. Thus,

$$U^{P}(V(H)) = \mathrm{Cl}^{P}_{Gm}(V(H)) \subseteq \mathrm{Cl}^{\alpha}_{Gm}(V(H)) = U^{\alpha}(V(H)). \tag{4.9}$$

Similarly we can prove the other cases. □

**Proposition 4.6.** *Let $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \mathcal{F}_{Gm})$ be a $G_m$-closure approximation space. If $H$ and $K$ are two subgraphs of $G$, then, for all $j \in \{R, S, P, \gamma, \alpha, \beta\}$,*

(1) $L^{j}(\phi) = U^{j}(\phi) = \phi$ *and* $L^{j}(V(G)) = U^{j}(V(G)) = V(G)$,

(2) *if* $V(H) \subseteq V(K)$, *then* $L^{j}(V(H)) \subseteq L^{j}(V(K))$,

(3) *if* $V(H) \subseteq V(K)$, *then* $U^{j}(V(H)) \subseteq U^{j}(V(K))$,

(4) $L^{j}(V(H) \cup V(K)) \supseteq L^{j}(V(H)) \cup L^{j}(V(K))$,

(5) $U^{j}(V(H) \cup V(K)) \supseteq U^{j}(V(H)) \cup U^{j}(V(K))$,

(6) $L^{j}(V(H) \cap V(K)) \subseteq L^{j}(V(H)) \cap L^{j}(V(K))$,

(7) $U^{j}(V(H) \cap V(K)) \subseteq U^{j}(V(H)) \cap U^{j}(V(K))$,

(8) $L^{j}(V(H)^{c}) = [U^{j}(V(H))]^{c}$,

(9) $U^{j}(V(H)^{c}) = [L^{j}(V(H))]^{c}$.

*Proof.* By using properties of $j$-interior and $j$-closure for all $j \in \{R, S, P, \gamma, \alpha, \beta\}$, the proof is obvious. □

In general, properties 3 and 4 which are introduced in Section 2.1 cannot be applied for $j$-lower and $j$-upper approximations, where $j \in \{S, P, \gamma, \beta\}$. The following example illustrates this fact in the case of $j = \beta$.

*Example 4.7.* Let $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \mathcal{F}_{Gm})$ be a $G_m$-closure approximation space which is given in Example 2.3:

$$\mathcal{F}_G = \{V(G), \phi, \{v_3\}, \{v_3, v_4\}, \{v_1, v_2, v_3\}\},$$
$$\mathcal{T}_G = \{V(G), \phi, \{v_4\}, \{v_1, v_2\}, \{v_1, v_2, v_4\}\}.$$

$$\begin{aligned}\beta\mathrm{O}_{G1}(G) = \{&V(G), \phi, \{v_1\}, \{v_2\}, \{v_4\}, \{v_1, v_2\}, \{v_1, v_3\}\{v_1, v_4\}, \{v_2, v_3\}, \{v_2, v_4\},\\ &\{v_3, v_4\}, \{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}, \{v_2, v_3, v_4\}\},\\ \beta\mathrm{C}_{G1}(G) = \{&V(G), \phi, \{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_1, v_2\}\{v_1, v_3\}, \{v_1, v_4\}, \{v_2, v_3\},\\ &\{v_2, v_4\}, \{v_3, v_4\}, \{v_1, v_2, v_3\}, \{v_1, v_3, v_4\}, \{v_2, v_3, v_4\}\}.\end{aligned} \tag{4.10}$$

If

$$\begin{aligned}H &= (V(H), E(H)); \quad V(H) = \{v_1, v_3\}, \quad E(H) = \{(v_1, v_3)\},\\ K &= (V(K), E(K)); \quad V(K) = \{v_2, v_3\}, \quad E(K) = \{(v_2, v_3)\},\end{aligned} \tag{4.11}$$

then

$$L^{\beta}(V(H)) \cap L^{\beta}(V(K)) = \{v_1, v_3\} \cap \{v_2, v_3\} = \{v_3\}, \tag{4.12}$$

but

$$L^{\beta}(V(H) \cap V(K)) = \phi. \tag{4.13}$$

Thus,

$$L^{\beta}(V(H) \cap V(K)) \neq L^{\beta}(V(H)) \cap L^{\beta}(V(K)). \tag{4.14}$$

Also, if

$$\begin{aligned}H &= (V(H), E(H)); \quad V(H) = \{v_1, v_2\}, \quad E(H) = \{(v_1, v_2), \{v_2, v_1\}\},\\ K &= (V(K), E(K)); \quad V(K) = \{v_1, v_4\}, \quad E(K) = \phi,\end{aligned} \tag{4.15}$$

then

$$U^{\beta}(V(H)) \cup U^{\beta}(V(K)) = \{v_1, v_2\} \cup \{v_1, v_4\} = \{v_1, v_2, v_4\}, \tag{4.16}$$

but

$$U^{\beta}(V(H) \cup V(K)) = V(G). \tag{4.17}$$

Thus,

$$U^{\beta}(V(H) \cup V(K)) \neq U^{\beta}(V(H)) \cup L^{\beta}(V(K)). \tag{4.18}$$

In general, properties 11 and 12 which are introduced in Section 2.1 cannot be applied for $j$-lower and $j$-upper approximations, where $j \in \{S, P, \gamma, \beta\}$. The following example illustrates this fact in the case of $j = \beta$.

*Example 4.8.* Let $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \mathcal{F}_{Gm})$ be a $G_m$-closure approximation space which is given in Example 2.3. If

$$\begin{aligned} H &= (V(H), E(H)); \quad V(H) = \{v_1, v_2, v_4\}, \quad E(H) = \{(v_1, v_2), (v_2, v_1)\}, \\ K &= (V(K), E(K)); \quad V(K) = \{v_3\}, \quad E(K) = \phi, \end{aligned} \tag{4.19}$$

then

$$L^{\beta}\left(L^{\beta}(V(H))\right) = L^{\beta}(V(H)) = \{v_1, v_2, v_4\}, \qquad U^{\beta}\left(L^{\beta}(V(H))\right) = V(G). \tag{4.20}$$

Thus,

$$L^{\beta}\left(L^{\beta}(V(H))\right) = L^{\beta}(V(H)) \neq U^{\beta}\left(L^{\beta}(V(H))\right). \tag{4.21}$$

Also,

$$U^{\beta}\left(U^{\beta}(V(K))\right) = U^{\beta}(V(K)) = \{v_3\}, \qquad L^{\beta}\left(U^{\beta}(V(K))\right) = \phi. \tag{4.22}$$

Hence,

$$U^{\beta}\left(U^{\beta}(V(K))\right) = U^{\beta}(V(K)) \neq L^{\beta}\left(U^{\beta}(V(K))\right). \tag{4.23}$$

**Lemma 4.9.** *Let $(G, \mathcal{F}_{Gm})$ be a $G_m$-closure space. Then* $\mathrm{Int}^{j}_{Gm}(V(G) - V(H)) = V(G) - \mathrm{Cl}^{j}_{Gm}(V(H))$ *for all subgraph $H \subseteq G$ and $j \in \{R, S, P, \gamma, \alpha, \beta\}$.*

*Proof.* It follows from definition near-open subgraphs in $G_m$-closure space. □

**Proposition 4.10.** *Let $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \mathcal{F}_{Gm})$ be a $G_m$-closure approximation space. If $H$ and $K$ are subgraphs of $G$, then*

$$L^{j}(V(H) - V(K)) \subseteq L^{j}(V(H)) - L^{j}(V(K)), \quad \forall j \in \{R, S, P, \gamma, \alpha, \beta\}. \tag{4.24}$$

*Proof.* We need to show that

$$\mathrm{Int}^{j}_{Gm}(V(H)-V(K)) \subseteq \mathrm{Int}^{j}_{Gm}(V(H))-\mathrm{Int}^{j}_{Gm}(V(K)). \tag{4.25}$$

Now,

$$V(H)-V(K) = V(H) \cap (V(G)-V(K)). \tag{4.26}$$

Then

$$\begin{aligned}\mathrm{Int}^{j}_{Gm}(V(H)-V(K)) &= \mathrm{Int}^{j}_{Gm}(V(H)\cap (V(G)-V(K)))\\ &\subseteq \mathrm{Int}^{j}_{Gm}(V(H)) \cap \mathrm{Int}^{j}_{Gm}(V(G)-V(K)).\end{aligned} \tag{4.27}$$

Thus, by Lemma 4.9, we have

$$\begin{aligned}\mathrm{Int}^{j}_{Gm}(V(H)-V(K)) &\subseteq \mathrm{Int}^{j}_{Gm}(V(H)) \cap \left(V(G)-\mathrm{Cl}^{j}_{Gm}(V(K))\right)\\ &= \mathrm{Int}^{j}_{Gm}(V(H))-\mathrm{Cl}^{j}_{Gm}(V(K)) \subseteq \mathrm{Int}^{j}_{Gm}(V(H))-\mathrm{Int}^{j}_{Gm}(V(K)).\end{aligned} \tag{4.28}$$

Therefore

$$\begin{aligned}L^{j}(V(H)-V(K)) &= \mathrm{Int}^{j}_{Gm}(V(H)-V(K)) \subseteq \mathrm{Int}^{j}_{Gm}(V(H))-\mathrm{Int}^{j}_{Gm}(V(K))\\ &= L^{j}(V(H))-L^{j}(V(K)).\end{aligned} \tag{4.29}$$

□

In general, part (b) in Proposition 3.9 cannot be applied for $j$-upper approximations for all $j \in \{R,S,P,\gamma,\alpha,\beta\}$. Example 4.11 (resp., Example 4.12) illustrates that part (b) in Proposition 3.9 cannot be applied in the case of $j = \beta$ (resp., $j = R$).

*Example 4.11.* Let $\mathcal{G}_m = (G,\mathrm{Cl}_{Gm},\mathcal{F}_{Gm})$ be a $G_m$-closure approximation space which is given in Example 2.3. If

$$\begin{aligned}H &= (V(H),E(H)); \quad V(H)=\{v_1,v_2,v_4\}, \quad E(H)=\{(v_1,v_2),(v_2,v_1)\},\\ K &= (V(K),E(K)); \quad V(K)=\{v_1,v_2\}, \quad E(K)=\{(v_1,v_2),(v_2,v_1)\},\end{aligned} \tag{4.30}$$

then

$$U^{\beta}(V(H)-V(K)) = U^{\beta}(\{v_4\}) = \{v_4\}, \tag{4.31}$$

but

$$U^{\beta}(V(H))-U^{\beta}(V(K)) = V(G)-\{v_1,v_2\} = \{v_3,v_4\}. \tag{4.32}$$

Hence,

$$U^{\beta}(V(H)-V(K)) \subseteq U^{\beta}(V(H)) - U^{\beta}(V(K)). \tag{4.33}$$

*Example 4.12.* Let $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \mathcal{F}_{Gm})$ be a $G_m$-closure approximation space which is given in Example 2.3:

$$\begin{aligned} \mathrm{RO}_{G1}(G) &= \{V(G), \phi, \{v_4\}, \{v_1, v_2\}\}, \\ \mathrm{RC}_{G1}(G) &= \{V(G), \phi, \{v_3, v_4\}, \{v_1, v_2, v_3\}\}. \end{aligned} \tag{4.34}$$

If

$$\begin{aligned} H &= (V(H),\ E(H)); \quad V(H) = \{v_1\}, \quad E(H) = \phi, \\ K &= (V(K),\ E(K)); \quad V(K) = \{v_3\}, \quad E(K) = \phi, \end{aligned} \tag{4.35}$$

then

$$U^{R}(V(H)-V(K)) = U^{R}(\phi) = \phi, \tag{4.36}$$

but

$$U^{R}(V(H)) - U^{R}(V(K)) = \{v_1, v_2, v_3\} - \{v_3\} = \{v_1, v_2\}. \tag{4.37}$$

Hence,

$$U^{R}(V(H)-V(K)) \ \subseteq\ U^{R}(V(H)) - U^{R}(V(K)). \tag{4.38}$$

## 5. Near-Boundary Regions and Near Accuracy in $G_m$-Closure Approximation Spaces

In this section we divide the boundary region into several levels. These levels help to decrease the boundary region. In the following definition we introduce the near boundary region of a subgraph $H$ of $G$ in a $G_m$-closure approximation space $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \mathcal{F}_{Gm})$.

*Definition 5.1.* Let $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \mathcal{F}_{Gm})$ be a $G_m$-closure approximation space and $H \subseteq G$. The near-boundary ($j$-boundary) region of $H$ is denoted by $\mathrm{Bd}^{j}_{\mathcal{G}m}(V(H))$ and is defined by

$$\mathrm{Bd}^{j}_{\mathcal{G}m}(V(H)) = U^{j}(V(H)) - L^{j}(V(H)), \quad \text{where } j \in \{R,\ S,\ P,\ \gamma,\ \alpha,\ \beta\}. \tag{5.1}$$

*Definition 5.2.* Let $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \mathcal{F}_{Gm})$ be a $G_m$-closure approximation space and $H \subseteq G$. The near-positive ($j$-positive) region of $H$ is denoted by $\mathrm{POS}^{j}_{\mathcal{G}m}(V(H))$ and is defined by

$$\mathrm{POS}^{j}_{\mathcal{G}m}(V(H)) = L^{j}(V(H)), \quad \text{where } j \in \{R, S, P, \gamma, \alpha, \beta\}. \tag{5.2}$$

*Definition 5.3.* Let $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \digamma_{Gm})$ be a $G_m$-closure approximation space and $H \subseteq G$. The near negative (briefly $j$-negative) region of $H$ is denoted by $\mathrm{NEG}^j_{\mathcal{G}_m}(V(H))$ and is defined by

$$\mathrm{NEG}^j_{\mathcal{G}_m}(V(H)) = V(G) - U^j(V(H)), \quad \text{where } j \in \{R, S, P, \gamma, \alpha, \beta\}. \tag{5.3}$$

**Proposition 5.4.** *Let $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \digamma_{Gm})$ be a $G_m$-closure approximation space. If $H \subseteq G$, then*

$$\mathrm{Bd}^j_{\mathcal{G}_m}(V(H)) \subseteq \mathrm{Bd}_{\mathcal{G}_m}(V(H)) \quad \forall j \in \{S, P, \gamma, \alpha, \beta\}. \tag{5.4}$$

*Proof.* By using Proposition 4.2, the proof is obvious. □

In general, the above proposition is not true in the case of $j = R$ as illustrated in the following example.

*Example 5.5.* Let $\mathcal{G}_m = (G,\ \mathrm{Cl}_{Gm},\ \digamma_{Gm})$ be a $G_m$-closure approximation space which is given in Example 2.3. If

$$H = (V(H), E(H)): \quad V(H) = \{v_1, v_3\}, E(H) = \{(v_1, v_3)\}, \tag{5.5}$$

then

$$\begin{aligned} \mathrm{Bd}_{\mathcal{G}_m}(V(H)) &= U(V(H)) - L(V(H)) = \mathrm{Cl}_{Gm}(V(H)) - \mathrm{Int}_{Gm}(V(H)) \\ &= \mathrm{Cl}_{Gm}(\{v_1, v_3\}) - \mathrm{Int}_{Gm}(\{v_1, v_3\}) = \{v_1, v_3\} - \phi = \{v_1, v_3\}, \\ \mathrm{Bd}^R_{\mathcal{G}_m}(V(H)) &= U^R(V(H)) - L^R(V(H)) = \mathrm{Cl}^R_{\mathcal{G}_m}(V(H)) - \mathrm{Int}^R_{\mathcal{G}_m}(V(H)) \\ &= \mathrm{Cl}^R_{\mathcal{G}_m}(\{v_1, v_3\}) - \mathrm{Int}^R_{\mathcal{G}_m}(\{v_1, v_3\}) = V(G) - \phi = V(G). \end{aligned} \tag{5.6}$$

**Proposition 5.6.** *Let $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \digamma_{Gm})$ be a $G_m$-closure approximation space. If $H \subseteq G$, then the implication between boundary and $j$-boundary of $H$ given by the following statement for all $j \in \{S, P, \gamma, \alpha, \beta\}$:*

(a) $\mathrm{Bd}^\beta_{\mathcal{G}_m}(V(H)) \subseteq \mathrm{Bd}^\gamma_{\mathcal{G}_m}(V(H)) \subseteq \mathrm{Bd}^S_{\mathcal{G}_m}(V(H)) \subseteq \mathrm{Bd}^\alpha_{\mathcal{G}_m}(V(H)) \subseteq \mathrm{Bd}_{\mathcal{G}_m}(V(H))$,

(b) $\mathrm{Bd}^\beta_{\mathcal{G}_m}(V(H)) \subseteq \mathrm{Bd}^\gamma_{\mathcal{G}_m}(V(H)) \subseteq \mathrm{Bd}^P_{\mathcal{G}_m}(V(H)) \subseteq \mathrm{Bd}^\alpha_{\mathcal{G}_m}(V(H)) \subseteq \mathrm{Bd}_{\mathcal{G}_m}(V(H))$.

*Proof.* By using Propositions 4.4 and 4.5, the proof is obvious. □

*Definition 5.7.* Let $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \digamma_{Gm})$ be a $G_m$-closure approximation space and $H$ a finite nonempty subgraph of $G$. The near accuracy ($j$-accuracy) of $H$ is denoted by $\eta^j_{\mathcal{G}_m}(V(H))$ and is defined by

$$\eta^j_{\mathcal{G}_m}(V(H)) = \frac{|L^j(V(H))|}{|U^j(V(H))|}, \quad \text{where } \left|U^j(V(H))\right| \neq 0\ \forall j \in \{R, S, P, \gamma, \alpha, \beta\}. \tag{5.7}$$

**Proposition 5.8.** *Let $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \mathcal{F}_{Gm})$ be a $G_m$-closure approximation space. If $H$ is a finite nonempty subgraph of G, then $\eta_{\mathcal{G}m}(V(H)) \leq \eta^{j}_{\mathcal{G}m}(V(H))$ for all $j \in \{S, P, \gamma, \alpha, \beta\}$, where $\eta_{\mathcal{G}m}(V(H)) = |L(V(H))|/|U(V(H))|$ is the accuracy of $H$.*

*Proof.* By using Proposition 4.2, the proof is obvious. □

In general, the above proposition is not true in the case of $j = R$. This fact is illustrated in the following example.

*Example 5.9.* Let $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \mathcal{F}_{Gm})$ be a $G_m$-closure approximation space which is given in Example 4.3. If

$$H = (V(H), E(H)) : \quad V(H) = \{v_1, v_2\}, E(H) = \{(v_2, v_1)\}, \tag{5.8}$$

then

$$\eta_{\mathcal{G}m}(V(H)) = \frac{2}{3}, \qquad \eta^{R}_{\mathcal{G}m}(V(H)) = 0. \tag{5.9}$$

Thus,

$$\eta^{R}_{\mathcal{G}m}(V(H)) < \eta_{\mathcal{G}m}(V(H)). \tag{5.10}$$

**Proposition 5.10.** *Let $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \mathcal{F}_{Gm})$ be a $G_m$-closure approximation space. If $H \subseteq G$, then the implication between accuracy and $j$-accuracy of $H$ is given by the following statement for all $j \in \{S,\ P,\ \gamma,\ \alpha,\ \beta\}$:*

(a) $\eta_{\mathcal{G}m}(V(H)) \leq \eta^{\alpha}_{\mathcal{G}m}(V(H)) \leq \eta^{S}_{\mathcal{G}m}(V(H)) \leq \eta^{\gamma}_{\mathcal{G}m}(V(H)) \leq \eta^{\beta}_{\mathcal{G}m}(V(H))$,

(b) $\eta_{\mathcal{G}m}(V(H)) \leq \eta^{\alpha}_{\mathcal{G}m}(V(H)) \leq \eta^{P}_{\mathcal{G}m}(V(H)) \leq \eta^{\gamma}_{\mathcal{G}m}(V(H)) \leq \eta^{\beta}_{\mathcal{G}m}(V(H))$.

*Proof.* By using Propositions 4.4 and 4.5, the proof is obvious. □

## 6. Rough and Near-Rough Cluster Vertices in $G_m$-Closure Approximation Spaces

In this section, we introduce the definitions of definability of graphs, rough cluster vertices and near-rough cluster vertices in approximation spaces with general relations. The following definition introduces new concepts of definability for a subgraph $H \subseteq G$ in a $G_m$-closure approximation space $\mathcal{G}_m = (G,\ \mathrm{Cl}_{Gm},\ \mathcal{F}_{Gm})$.

*Definition 6.1.* Let $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \mathcal{F}_{Gm})$ be a $G_m$-closure approximation space. If $H \subseteq G$, then $H$ is called

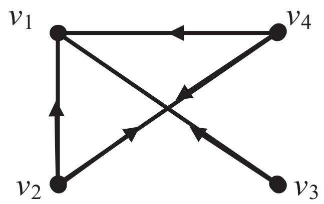
(a) totally ${}_j\mathcal{G}_m$-definable (${}_j\mathcal{G}_m$-exact) graph if $L^j(V(H)) = V(H) = U^j(V(H))$,

(b) internally ${}_j\mathcal{G}_m$-definable graph if $L^j(V(H)) = V(H)$, $U^j(V(H)) \neq V(H)$,

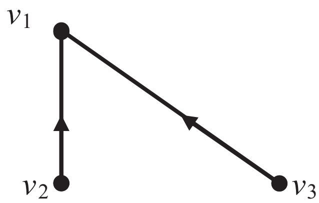
(c) externally ${}_j\mathcal{G}_m$-definable graph if $L^j(V(H)) \neq V(H)$, $U^j(V(H)) = V(H)$,

(d) ${}_j\mathcal{G}_m$-indefinable (${}_j\mathcal{G}_m$-rough) graph if $L^j(V(H)) \neq V(H)$, $U^j(V(H)) \neq V(H)$, where $j \in \{R, S, P, \gamma, \alpha, \beta\}$.

*Example 6.2.* Let $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \mathcal{F}_{Gm})$ be a $G_m$-closure approximation space such that $G = (V(G), E(G)) : V(G) = \{v_1, v_2, v_3, v_4\}, E(G) = \{(v_2, v_1), (v_2, v_4), (v_3, v_1), (v_4, v_1), (v_4, v_2)\}$,




$G$




$H$

$$\mathcal{F}_G = \{V(G), \phi, \{v_1\}, \{v_1, v_3\}, \{v_1, v_2, v_4\}\},$$
$$\mathcal{T}_G = \{V(G), \phi, \{v_3\}, \{v_2, v_4\}, \{v_2, v_3, v_4\}\}.$$

Let $H = (V(H), E(H))$: $V(H) = \{v_1, v_2, v_3\}$, $E(H) = \{(v_2, v_1), (v_3, v_1)\}$, then, for $j \in \{S, P\}$, we get

$$\begin{aligned}
\mathrm{POS}^S_{\mathcal{G}m}(V(H)) &= L^S(V(H)) = \mathrm{Int}^S_{Gm}(V(H)) = \{v_1, v_3\}, \\
U^S(V(H)) &= \mathrm{Cl}^S_{Gm}(V(H)) = V(G), \\
\mathrm{Bd}^S_{Gm}(V(H)) &= \mathrm{Bd}^S_{\mathcal{G}m}(V(H)) = \{v_2, v_4\}, \mathrm{NEG}^S_{\mathcal{G}m}(V(H)) = \phi, \\
\mathrm{POS}^P_{\mathcal{G}m}(V(H)) &= L^P(V(H)) = \mathrm{Int}^P_{Gm}(V(H)) = \{v_1, v_2, v_3\}, \\
U^P(V(H)) &= \mathrm{Cl}^P_{Gm}(V(H)) = \{v_1, v_2, v_3\}, \\
\mathrm{Bd}^S_{Gm}(V(H)) &= \mathrm{Bd}^P_{\mathcal{G}m}(V(H)) = \phi, \mathrm{NEG}^P_{\mathcal{G}m}(V(H)) = \{v_4\}.
\end{aligned} \tag{6.1}$$

Thus, $H$ is an ${}_S\mathcal{G}_m$-indefinable (${}_S\mathcal{G}_m$-rough) graph and ${}_P\mathcal{G}_m$-definable (${}_P\mathcal{G}_m$-exact) graph.

The following definition introduces the concept of rough cluster vertices of a subgraph $H$ of $G$ in a $G_m$-closure approximation $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \mathcal{F}_{Gm})$.

*Definition 6.3.* Let $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \mathcal{F}_{Gm})$ be a $G_m$-closure approximation space. The vertex $v \in G$ is said to be a rough cluster vertex of a subgraph $H$ of $G$ if, for all subgraph $K$ of $G$ such that $v \in L(V(K))$, $(L(V(K)) - \{v\}) \cap V(H) \neq \phi$.

The graph of all rough cluster vertices of $H$ is denoted by $R'(V(H))$ and is called the rough derived graph of $H$.

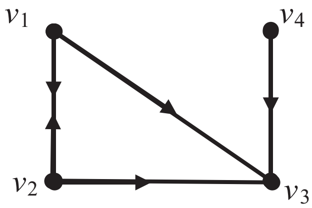
**Theorem 6.4.** *Let $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \mathcal{F}_{Gm})$ be a $G_m$-closure approximation space. Then a subgraph $H$ of $G$ is closed if and only if $R'(V(H)) \subseteq V(H)$.*

*Proof.* ($\Rightarrow$) Suppose that $H$ is a closed subgraph of $G$, and let $v \notin V(H)$ (i.e., $v \in V(G) - V(H)$). Then $V(G) - V(H)$ is open subgraph. Thus, $v \in L(V(G) - V(H)) = \mathrm{Int}_{Gm}(V(G) - V(H)) = V(G) - V(H)$ and $L(V(G) - V(H)) \cap V(H) = \phi$. Hence, $v \notin R'(V(H))$. Therefore, $R'(V(H)) \subseteq V(H)$.

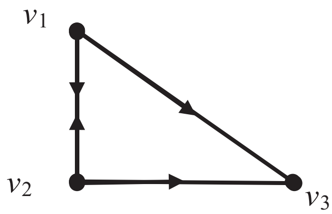
($\Leftarrow$) Let $R'(V(H)) \subseteq V(H)$. To show that $H$ is a closed subgraph of $G$, let $v \in V(G) - V(H)$. Then $v \notin R'(V(H))$, and hence there exists a subgraph $K_v \subseteq G$ such that $v \in L(V(K_v))$ and $L(V(K_v) - V(v)) \cap V(H) = \phi$. But $v \notin V(H)$, hence $L(V(K_v)) \cap V(H) = \phi$. So $v \in L(V(K_v) \subseteq V(G) - V(H)$ and $V(G) - V(H) = \bigcup_{v\in V(G)-V(H)}\{v\} \subseteq \bigcup_{v\in V(G)-V(H)} L(V(K_v) \subseteq \bigcup_{v\in V(G)-V(H)} \mathrm{Int}_{Gm}(V(K_v)) \subseteq V(G) - V(H)$.

Thus, $V(G) - V(H)$ is a union of open graphs, which is open. Hence, $H$ is closed subgraph of $G$. □

*Example 6.5.* Let $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \mathcal{F}_{Gm})$ be a $G_m$-closure approximation space such that $G = (V(G), E(G))$: $V(G) = \{v_1, v_2, v_3, v_4\}$, $E(G) = \{(v_1, v_2), (v_1, v_3), (v_2, v_1), (v_2, v_3), (v_4, v_3)\}$,



$G$



$H$

$$\mathcal{F}_G = \{V(G), \phi, \{v_3\}, \{v_3, v_4\}, \{v_1, v_2, v_3\}\},$$

$$\mathcal{T}_G = \{V(G), \phi, \{v_4\}, \{v_1, v_2\}, \{v_1, v_2, v_4\}\}.$$

If $H = (V(H), E(H))$: $V(H) = \{v_1, v_2, v_3\}$, $E(H) = \{(v_1, v_2), (v_1, v_3), (v_2, v_1), (v_2, v_3)\}$, then $R'(V(H)) = \{v_1, v_2, v_3\}$. Thus, $R'(V(H)) \subseteq V(H)$ and $H$ is closed subgraph of $G$.

The following definition introduces the concept of near-rough ($j$-rough) cluster vertices of a subgraph $H$ of $G$ in a $G_m$-closure approximation space $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \mathcal{F}_{Gm})$ for all $j \in \{R, S, P, \gamma, \alpha, \beta\}$.

*Definition 6.6.* Let $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \mathcal{F}_{Gm})$ be a $G_m$-closure approximation space. The vertex $v \in G$ is said to be near-rough ($j$-rough) cluster vertex of a subgraph $H$ of $G$ for all $j \in \{R, S, P, \gamma, \alpha, \beta\}$, if, for all subgraph $K$ of $G$ such that $v \in L^j(V(K))$, $(L^j(V(K)) - \{v\}) \cap V(H) \neq \phi$.

The graph of all $j$-rough cluster vertices of $H$ is denoted by $R'_j(V(H))$ and is called the $j$-rough derived graph of $H$ for all $j \in \{R, S, P, \gamma, \alpha, \beta\}$.

**Theorem 6.7.** *Let $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \mathcal{F}_{Gm})$ be a $G_m$-closure approximation space. Then a subgraph $H$ of $G$ is a $j$-closed for all $j \in \{S, P, \gamma, \alpha, \beta\}$ if and only if $R'_j(V(H)) \subseteq V(H)$.*

*Proof.* The proofs of the five cases are similar, so we will only prove the case when $j = \beta$.

($\Rightarrow$) Suppose that $H$ is a $\beta$-closed subgraph of $G$, and let $v \notin V(H)$ (i.e., $v \in (V(G) - V(H))$. Then $V(G) - V(H) \in \beta O_{Gm}(G)$. Thus, $v \in L^{\beta}(V(G) - V(H)) = \mathrm{Int}^{\beta}_{Gm}(V(G) - V(H)) = V(G) - V(H)$ and $L^{\beta}(V(G) - V(H)) \cap V(H) = \phi$. Hence, $v \notin R'_{\beta}(V(H))$. Therefore, $R'_{\beta}(V(H)) \subseteq V(H)$.

($\Leftarrow$) Let $R'_{\beta}(V(H)) \subseteq V(H)$. To show that $H$ is a $\beta$-closed subgraph of $G$, let $v \in (V(G) - V(H))$, then $v \notin R'(V(H))$, and hence there exists a subgraph $K_v \subseteq G$ such that $v \in L^{\beta}(V(K_v))$ and $L^{\beta}(V(K_v) - V(v)) \cap V(H) = \phi$. But $v \notin V(H)$, hence $L^{\beta}(V(K_v)) \cap V(H) = \phi$. So $v \in L^{\beta}(V(K_v) \subseteq V(G) - V(H)$ and $V(G) - V(H) = \bigcup_{v \in V(G)-V(H)} \{v\} \subseteq \bigcup_{v \in V(G)-V(H)} L^{\beta}(V(K_v) \subseteq \bigcup_{v \in V(G)-V(H)} \mathrm{Int}^{\beta}_{Gm}(V(K_v)) \subseteq V(G) - V(H)$.

Thus, $V(G) - V(H)$ is a union of $\beta$-open graphs, which is $\beta$-open. Hence, $H$ is $\beta$-closed subgraph of $G$. □

*Example 6.8.* Let $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \mathcal{F}_{Gm})$ be a $G_m$-closure approximation space which is given in Example 6.5.

If $H = (V(H), E(H)); V(H) = \{v_1, v_2\}, E(H) = \{(v_1, v_2), (v_2, v_1)\}$. Then $R'_S(V(H)) = \{v_1, v_2\}$, thus $R'_S(V(H)) \subseteq V(H)$ and $H$ is $S$-closed subgraph of $G$.


In general, Theorem 6.7 cannot be satisfied in the case of $j = R$, as the following example illustrates.

*Example 6.9.* Let $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \mathcal{F}_{Gm})$ be a $G_m$-closure approximation space which is given in Example 6.5.

If $H = (V(H), E(H)); V(H) = \{v_3\}, E(H) = \phi$. Then $R'_R(V(H)) = \{v_3\}$, thus $R'_R(V(H)) \subseteq V(H)$. But $H$ is not an $R$-closed subgraph of $G$, since $\mathrm{RC}_{Gm}(G) = \{V(G), \phi, \{v_3, v_4\}, \{v_1, v_2, v_3\}\}$.

**Theorem 6.10.** *Let $H$ be a subgraph of $G$ in the $G_m$-closure approximation space $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \mathcal{F}_{Gm})$. Then $v \in U(V(H))$ if and only if, for each $K \subseteq G$ and $v \in L(V(K)$, $L(V(K)) \cap V(H) \neq \phi$.*

*Proof.* ($\Rightarrow$) Let $v \in U(V(H))$ and $v \in L(V(K))$ for some $K \subseteq G$. Assume $L(V(K)) \cap V(H) = \phi$. This implies that $V(G) \subseteq V(G) - L(V(K))$. But $V(G) - L(V(K)) = V(G) - \mathrm{Int}_{Gm}(V(K))$ which is closed graph. Hence, $v \in V(G) - L(V(K))$, since $v \in U(V(H))$ and this leads to a contradiction. Therefore, $L(V(K)) \cap V(H) \neq \phi$.

($\Leftarrow$) Suppose that, for each $K \subseteq G$ and $v \in L(V(K))$, $L(V(K)) \cap V(H) \neq \phi$. Let $v \notin U(V(H))$. But $U(V(H)) = \mathrm{Cl}_{Gm}(V(H))$ which is closed. Then there exists a closed graph $F \subseteq G$ such that $F \supseteq H$ and $v \notin V(F)$. Hence, $V(G)-V(F)$ is open graph containing $v$. Thus,

$$\begin{aligned} v \in L(V(G) - V(F)) &= \mathrm{Int}_{Gm}(V(G) - V(F)) \\ &= V(G) - V(F), \end{aligned}$$

$$L(V(G) - V(F)) \cap V(H) = \phi. \tag{6.2}$$

that is, there exists a subgraph $K = G - F$ such that $L(V(K) \cap V(H) = \phi$, which leads to a contradiction. Therefore, $v \in U(V(H))$. □

**Theorem 6.11.** *Let $H$ be a subgraph of $G$ in the $G_m$-closure approximation space $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \digamma_{Gm})$. Then $v \in U^j(V(H))$ for all $j \in \{S, P, \gamma, \alpha, \beta\}$ if and only if, for each $K \subseteq G$ and $v \in L^j(V(K))$, $L(V(K)) \cap V(H) \neq \phi$.*

*Proof.* The proof is similar to the proof of Theorem 6.10. □

**Theorem 6.12.** *Let $H$ be a subgraph of $G$ in the $G_m$-closure approximation space $\mathcal{G}_m = (G, Cl_{Gm}, \digamma_{Gm})$. Then $U(V(H) = V(H) \cup R'(V(H))$.*

*Proof.* By Theorem 6.4, we get $R'(V(H)) \subseteq U(V(H))$. Then

$$V(H) \cup R'(V(H)) \subseteq V(H) \cup U(V(H)) = U(V(H)). \tag{6.3}$$

For the converse inclusion, let $v \in U(V(H))$, then either $v \in V(H)$ and hence $v \in V(H) \cup R'(V(H))$ or $v \notin V(H)$. Hence, by Theorem 6.10 for each $K \subseteq G$, $v \in L(V(K))$, we get $L(V(K)) \cap V(H) \neq \phi$. Then $v \in R'(V(H))$ and hence $v \in V(H) \cup R'_j(V(H))$. Thus, $U(V(H)) \subseteq V(H) \cup R'_j(V(H))$. Therefore, $U(V(H) = V(H) \cup R'_j(V(H))$. □

**Theorem 6.13.** *Let $H$ be a subgraph of $G$ in the $G_m$-closure approximation space $\mathcal{G}_m = (G, \mathrm{Cl}_{Gm}, \digamma_{Gm})$. Then $U^j(V(H)) = V(H) \bigcup R_j{}'(V(H))$ for all $j \in \{S, P, \gamma, \alpha, \beta\}$.*

*Proof.* The proof is similar to the proof of Theorem 6.12. □

## 7. Conclusions

In this paper, we used $G_m$-topological concepts to introduce a generalization of Pawlak approximation space. Concepts of definability for subgraphs in $G_m$-approximation spaces are introduced. Several types of approximations which are called near approximations are mathematical tools to modify the approximations. The suggested methods of near approximations open way for constructing new types of lower and upper approximations.

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