Weak Essential Submodules

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Abstract:

A non-zero submodule N of M is called essential if $N \cap L \neq 0$ for each non-zero submodule L of M. And a non-zero submodule K of M is called semi-essential if $K \cap P \neq 0$ for each non-zero prime submodule P of M. In this paper we investigate a class of submodules that lies between essential submodules and semi-essential submodules, we call these class of submodules weak essential submodules.

Keywords: Semi-prime submodules, Essential submodules, Uniform modules.

ξ0. Introduction

Let R be a commutative ring with identity 1, and let M be a unitary (left) R-module.In this work we assume that every submodule of M contained in a semi-prime submodule of M. A non- non-zero submodule N of M is called essential if N∩L≠(0) for every non-zero submodule L of M [1], and a proper submodule P of M is called prime if for each m∈M and r∈R whenever rm∈M, then either m∈M or r∈[P:M] [2]. A non-zero submodule K of M is called semi-essential if K∩P≠(0) for each non-zero prime submodule P of M [3].In this paper we investigate a class of submodules that lies between essential submodules and semi-essential submodules, we call this class of submodules, weak essential submodules.

ξ1. Notations And Basic Results:

Recall that a submodule S of an R-module M is called semi-prime if for each r∈R and m∈M with $r^k x \in N, k \in Z_+$ then rx∈N [4].Equivalently, if $r^2 x \in N$ then rx∈N [5]. In this section we study some properties of weak essential submodules.

(1.1) Definition: Let M be an R-module. A non-zero W of M is called weak essential if W∩S ≠(0) for each non–zero semi-prime submodule S of M.

It is clear that every essential submodule is weak essential and the converse is not true in general for example: In the Z-module Z_{36}, the submodule $\overline{(9)}$ of Z_{36} is weak essential but not essential,in fact $\overline{(9)} \cap \overline{(2)} \neq (0)$, $\overline{(9)} \cap \overline{(3)} \neq (0)$ and $\overline{(9)} \cap \overline{(6)} \neq (0)$ where $\overline{(2)}, \overline{(3)}$ and $\overline{(6)}$ are the only non-zero semi-prime submodules of Z_{36}. But $\overline{(9)} \cap \overline{(12)} = (0)$, therefore $\overline{(9)}$ is not essential submodule of Z_{36}. On the other hand every weak essential submodule is semi-essential , but the converse is not true as in the following example: In the Z-module M=Z⊕Z, the only prime submodule are of the form Z⊕ PZ and PZ⊕ Z where P is the prime number. The submodule N= (0) ⊕Z of M is semi-essential but not weak essential, since N∩2Z⊕ (0) = (0) where 2Z⊕ (0) is semi-prime submodule of M not prime submodule.

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The following proposition is another characterization of weak essential submodules. Compare with[1].

(1.2) Proposition: Let M be an R-module. A non-zero submodule W of M is weak essential if and only if for each non-zero semi-prime submodule S of M there exists $x\in S$ and $r\in R$, such that $(0) \neq rx\in W$.

Proof: Suppose that for each non-zero semi-prime submodule S of M, there exists $x\in S$ and $r\in R$ such that $(0) \neq rx \in W$. Not that $rx\in S$, therefore $(0) \neq rx\in W\cap S$. Thus $W\cap S \neq (0)$, that is W is a weak essential. Conversely, suppose that W is a weak essential submodule of M. Then $W\cap S \neq (0)$ for each semi-prime submodule S of M, thus there exists $(0) \neq x \in W\cap S$. This implies that $x\in W$ and hence $(0)\neq 1.x\in W$.

A submodule N is called irreducible if for each two submodules L_1 and L_2 of M such that $L_1 \cap L_2 = N$, then either $L_1=N$ or $L_2=N$ [4].We can show that if every semi-prime submodule of M is irreducible then a semi-essential submodule is weak essential as in the following proposition. Before that we need the following lemma which the proof can be seen in [5].

(1.3) Lemma: Let S be an irreducible submodule of M. Then S is semi-prime if and only if S is prime submodule.

(1.4) Proposition: Let M be an R-module such that every semi-prime submodule of M is irreducible. If a submodule W of M is semi-essential then W is a weak essential submodule of M.

Proof: Let S be a non-zero semi-prime submodule of M with $W\cap S = (0)$. Since S is irreducible submodule then by (1.3), S is prime submodule. But W is semi-essential submodule of M, therefore $S = (0)$.

(1.5) Remarks:

1. If W is a weak essential submodule and N is a submodule of W then N need not be weak essential. For example: consider the Z-module Z_{36}, the submodule $\overline{(2)}$ of Z_{36} is weak essential but the submodule $\overline{(18)}$ of $\overline{(2)}$ is not weak essential since $\overline{(18)}\cap\overline{(12)} = \overline{(0)}$ where $\overline{(12)}$ is a semi-prime submodule of $\overline{(2)}$.

2. Let M be an R-module and let W_1 and W_2 be submodules of M such that $W_1\subseteq W_2$. If W_1 is a weak essential submodule of M then W_2 is weak essential submodule of M.

3. Let M be an R-module, and let W_1 and W_2 be submodules of M, if $W_1\cap W_2$ is a weak essential submodule of M, then both of W_1 and W_2 are weak essential submodules of M.

Proof:

(2). Assume that $W_2 \cap S= (0)$, for some semi-prime submodule S of M, then $W_1\cap S= (0)$. But W_1 is a weak essential submodule of M, therefore $S = (0)$ and hence we are done.

(3). Follows immediately from (2).

The converse of (3) is not true in general for example, in the Z-module Z_{36} the only non-zero semi-prime submodules are only $\overline{(2)}$, $\overline{(3)}$ and $\overline{(6)}$. Both of $\overline{(12)}$ and $\overline{(18)}$ are weak essential submodules, but the intersection $\overline{(12)}\cap\overline{(18)} = \overline{(0)}$ is not weak essential submodule of Z_{36}.

Under some conditions the converse of (3) will be true as in the following two propositions.

(1.6) Proposition: Let M be an R-module and let W_1 and W_2 be submodules of M such that W_1 is an essential submodule of M, and W_2 is weak essential submodule of M. Then $W_1 \cap W_2$ is weak essential submodule of M.

Proof: Since W_2 is a weak essential submodule of M, then $W_2 \cap S \neq (0)$ for each non-zero semi-prime submodule S of M. But W_1 is an essential submodule of M, so $W_1 \cap (W_2 \cap S) \neq (0)$, this implies that $(W_1 \cap W_2.) \cap S \neq (0)$, thus we get the result.

(1.7) Proposition: Let M be an R-module and let W_1 and W_2 be submodules of M such that one of them does not contained in any semi-prime submodule of M. If W_1 and W_2 are weak essential submodules of M, then $W_1 \cap W_2$ is weak essential submodule of M.

Proof: Suppose that there exists a semi-prime submodule S of M such that $(W_1 \cap W_2.) \cap S = (0)$ Then $W_1 \cap (W_2 \cap S) = (0)$. By assumption either W_1 or W_2 is not contained in S. If $W_2 \not\subset S$, then $W_2 \cap S$ is semi-prime submodule of W_2 [5]. But W_1 is weak essential submodule of M, so $W_2 \cap S = (0)$. Also W_2 is weak essential submodule of M, therefore S = (0).

ξ 2. Weak essential homomorphisms:

This section is devoted to study weak essential homomorphisms, we start by the following definition.

(2.1) Definition: Let M_1 and M_2 be two R-modules. An R-homomorphism f: $M_1 \to M_2$ is called essential homomorphism if f (M_1) is a weak essential submodule of M_2.

(2.2) Remark: Let M be an R-module and let W be a submodule of M. W is weak essential submodule if and only if the inclusion homomorphism i: $W \to M$ is weak essential homomorphism.

Compare the following proposition with [6].

(2.3) Proposition: Let $M_1$and M_2 be R-modules and let f: $M_1 \to M_2$ be an R-epimorphism, then:

1. If W_1 is a weak essential submodule of M_1,then $f(W_1)$ is weak essential submodule of M_2

2. If W_2 is a weak essential submodule of M_2 such that ker (f) $\subseteq S_1$ for each semi-prime submodule S_1 of M_1, then $f^{-1}(W_2)$ is weak essential submodule of M_1.

Proof:

1. Let S_2 be a non-zero semi-prime submodule of M_2, then $f^{-1}(S_2)$ is semi-prime submodule of M_1 [5]. But W_1 is weak essential submodule of M_1, thus $W_1 \cap f^{-1}(S_2) \neq (0)$ and hence f $(W_1) \cap S_2 \neq (0)$.

2. Suppose there exists a non-zero semi-prime submodule S_1 of M_1 such that $f^{-1}(W_2) \cap S_1 = (0)$, this implies that $W_2 \cap f(S_1) = (0)$.But S_1 is semi-prime submodule with ker(f)$\subseteq S_1$, so $f(S_1)$ is semi-prime submodule of M_2 [5]. But W_2 is weak essential submodule of M_2, therefore f $(S_1) = (0)$ which implies that $S_1 \subseteq \ker(f) \subseteq f^{-1}(W_2)$, and hence $S_1 = f^{-1}(W_2) \cap S_1 = (0)$ that is $S_1 = (0)$.

Analogue of proposition (2.3.6) in [7] we can prove the following lemma which we need it in the next theorem.

(2.4) Lemma: Let $M_1$and M_2 be R-modules and let W_2 be a semi-prime submodule of M_2 such that $Hom_R(M_1, W_2) \subset Hom_R(M_1, M_2)$, then $Hom_R(M_1,W_2)$ is semi-prime submodule of $Hom_R(M_1,M_2)$.

Proof: Let r∈R and f∈ $Hom_R(M_1,M_2)$ such that $r^2f \in Hom_R(M_1,W_2)$ then for each x∈ M_1, $r^2f(x) \in W_2$. But W_2 is semi-prime submodule of M_2, so $rf(x) \in W_2$, hence $rf \in Hom_R(M_1,W_2)$.

(2.5) Theorem: Let $M_1$and M_2 be R-modules, and let $Hom_R(M_1,W_2)$ be a proper submodule of $Hom_R(M_1,M_2)$ for any submodule W_2 of M_2. If $Hom_R(M_1,W_2)$ is weak essential submodule of $Hom_R(M_1,M_2)$, then W_2 is weak essential submodule of M_2.

Proof: Let S_2 be a non-zero semi-prime submodule of M_2.By (2.4), $Hom_R(M_1,S_2)$ is semi-prime submodule of $Hom_R(M_1,M_2)$.But $Hom_R(M_1,W_2)$ is weak essential submodule of $Hom_R(M_1,M_2)$ then by (1.2), there exists $0 \neq f \in Hom_R(M_1,S_2)$ and $0 \neq r \in R$ such that $0 \neq rf \in Hom_R(M_1,W_2)$, that is $rf(m) \in W_2$ for each $m \in M_1$.So for each non-zero semi-prime submodule S_2 of M_2 we find $f(m) \in S_2$ for each $m \in M_1$ and we find r∈R with $0 \neq rf(m) \in W_2$ i.e. W_2 is essential submodule of M_2.

(2.6) Corollary: Let M be an R-module and let W be a submodule of M .If $Hom_R(M,W)$ is weak essential submodule of $Hom_R(M,M)$, then W is weak essential submodule of M.

ξ 3. Weak essential submodules in multiplication modules

Recall that an R-module M is called multiplication if for each submodule N of M there exists an ideal I of R such that N=IM [8].].A non-zero ideal I of R is called weak essential if $I \cap S \neq (0)$ for each non-zero semi-prime ideal S of R.

(3.1) Proposition: Let M be a finitely generated faithful multiplication module. And let W be a submodule of M such that W=IM for some ideal I of R. If W is a weak essential submodule of M then I is weak essential ideal of R.

Proof: Suppose that $I \cap S = (0)$ for some non-zero semi-prime ideal S of R. Since M is a faithful multiplication module, then $(0) = (I \cap S) M = IM \cap SM$. Also since S is semi-prime submodule, and M is finitely generated multiplication module so by [5], SM is semi-prime submodule of M. On the other hand W=IM is weak essential submodule of M, therefore SM = (0). But M is faithful module then S = (0).

Under some conditions the converse of (3.2) is true as in the following two propositions.

(3.2) Proposition: Let M be a faithful multiplication module and let W be submodule of M such that W=IM. Suppose that every non-zero proper semi-prime submodule of M is irreducible. If I is weak essential ideal of R then W is a weak essential submodule of M.

Proof: Suppose that $W \cap S = (0)$ for some non-zero proper semi-prime submodule S of M. By assumption S is an irreducible submodule of M, so by (1.3), S is prime submodule. But S is a proper submodule of the multiplication module M, this implies that there exists a prime ideal P of R such that S=PM [8]. Now $(0) = W \cap S = IM \cap PM = (I \cap P) M$. But M is faithful multiplication module, therefore $I \cap P = (0)$. Since every prime submodule is semi-prime submodule, and by assumption we get P=(0). But S=PM therefore S = (0).

(3.3) Proposition: Let M be a faithful multiplication module and let W be submodule of M such that W=IM. Suppose that every non-zero proper semi-prime submodule of M is primary. If I is weak essential ideal of R then W is weak essential submodule of M.

Proof: Suppose that $W\cap S = (0)$ for some non-zero proper semi-prime submodule S of M. By assumption S is a primary submodule of M. Since M is multiplication module then [S: M] is semi-prime submodule of M [5]. But S is primary submodule of M, therefore S is a prime submodule [6], this implies there exists a prime ideal P of R such that S=PM [8]. Now $(0) = W\cap S= IM\cap PM= (I\cap P)M$. But M is faithful multiplication module, therefore $I\cap P = (0)$. Since every prime submodule is semi-prime submodule, and by assumption we get P= (0). But S=PM therefore S = (0).

(3.4) Proposition: Let M be a finitely generated faithful multiplication module and let W be a submodule of M. If W is weak essential submodule of M then [W :(m)] is weak essential ideal of R for each $m\in M$. The converse is true if every non-zero proper semi-prime submodule of M is irreducible.

Proof: Assume that W is weak essential submodule of M. By (3.2), [W: M] is weak essential ideal of R. But for each $m\in M$, $[W: M] \subseteq [W :(m)]$. Since M is faithful multiplication, thus $[N: M] M \subseteq [W :(m)] M$ [8]. This implies that [W :(m)] M is a weak essential submodule of M (1.5) (2). Hence [W :(m)] is weak essential ideal of R (3.2). Conversely, assume that [W :(m)] is a weak essential ideal of R for each $m\in M$, and let S be a non-zero proper semi-prime submodule of M. Since M is a multiplication module and S is irreducible submodule, then by (1.3), S is prime submodule, so there exists a prime ideal P of R such that S=PM [8]. It is clear that P is semi-prime ideal of R, but [W :(m)] is weak essential ideal of R, therefore $[W :(m)] \cap P \neq (0)$. Since M is a faithful multiplication module, then $[W: (m)] M \cap PM \neq (0)$. Thus $W\cap S \neq (0)$ that is W is a weak essential submodule of M.

By the same way we can prove the following.

(3.5) Proposition: Let M be a finitely generated faithful multiplication module and let W be a submodule of M. If W is weak essential submodule of M then [W :(m)] is weak essential ideal of R for each $m\in M$. The converse is true if every non-zero proper semi-prime submodule of M is primary.

From the last four propositions we have the following two theorems.

(3.6) Theorem: Let M be a finitely generated faithful multiplication module, and let W be a submodule of M such that W=IM for some ideal I of R. If each non-zero proper semi-prime submodule of M is irreducible, then the following statements are equivalent.

1. W is a weak essential submodule of M.

2. I is a weak essential ideal of R.

3. [W:(m)] is a weak essential ideal of R for each $m\in M$.

Proof: (1) ⇒(2): By (3.2).

(2) ⇒(3): Assume that I is an essential ideal of R. Since M is finitely generated faithful module, then by [5], I = [IM: M]. But $[IM:M] \subseteq [IM:(m)]$ for each $m\in M$, and [IM:M] is a weak essential ideal of R, also we consider [IM:M] as an R-module, then by (1.4)(2), [M:(m)] is a weak essential submodule of R, hence we get the result.

(3) ⇒(1): By (3.5).

(3.7) Theorem: Let M be a finitely generated faithful multiplication module, and let W be a submodule of M such that W=IM for some ideal I of R. If each non-zero proper semi-prime submodule of M is primary then the following statements are equivalent.

1. W is a weak essential submodule of M.

2. I is a weak essential ideal of R.

3. [W:(m)] is a weak essential ideal of R for each $m \in M$.

Proof: By the same way of (3.6), only in the direction **(3) ⇒(1)** we depend on (3.5).

ξ 4. Weak uniform modules

Recall that a non-zero R-module M is called uniform if every non-zero submodule of M is an essential submodule [6]. Abdullah, N.K. gave in her thesis [3] a generalization of uniform modules, she name it semi-uniform module that is a module M in which every non-zero submodule is semi-essential. In this section we introduce another generalization of uniform modules in fact this class of modules lies between uniform modules and semi-uniform modules. We call it weak uniform modules. We start by the following definition.

(4.1) Definition: A non-zero module M is called weak uniform, if each non-zero submodule of M is weak essential. And a ring R is called uniform ring if it is uniform module as an R-module.

(4.2) Remarks:

1. It is clear that each uniform module is weak uniform module. However, the converse is not true in general, for example: The Z-module Z_{36} is a weak uniform. In fact the only non-zero semi-prime submodule of Z_{36} are $\overline{(2)}$, $\overline{(3)}$ & $\overline{(6)}$ and all of them have non-zero intersections with each non trivial submodule of Z_{36} which they are $\overline{(2)}$, $\overline{(3)}$, $\overline{(4)}$, $\overline{(6)}$ and $\overline{(9)}$, $\overline{(12)}$ and $\overline{(18)}$. Therefore all submodules of Z_{36} are weak essential. On the other hand $\overline{(18)} \cap \overline{(12)} = \overline{(0)}$, this mean $\overline{(18)}$ is not essential submodule of Z_{36}. Thus Z_{36} is not uniform module.

2. Also it can be easy shown that each weak uniform module is semi-uniform. The converse is not true in general. For example the submodule $\overline{(2)}$ of Z_{36} is semi-uniform since the only non-zero semi-prime submodules of $\overline{(2)}$ are $\overline{(4)}$ & $\overline{(6)}$ and the last submodules have non-zero intersections with each non trivial submodule of $\overline{(2)}$. On the other hand the submodule $\overline{(2)}$ is not weak uniform since it is contain a submodule $\overline{(18)}$ which is not weak essential because $\overline{(18)} \cap \overline{(12)} =$ (0) where $\overline{(12)}$ is semi-prime submodule of $\overline{(2)}$.

It is shown in [3] that the uniform property is hereditary. Now we show by example that the weak uniform property is not hereditary. The Z-module Z_{36} is weak uniform module (4.2) (1). But $\overline{(3)}$ is not weak uniform submodule of Z_{36} since $\overline{(12)}$ is not weak essential submodule of $\overline{(3)}$, the only non-zero semi-prime submodule of $\overline{(3)}$ are $\overline{(6)}$, $\overline{(9)}$ & $\overline{(18)}$ while $\overline{(12)} \cap \overline{(18)} = \overline{(0)}$.

Compare the following proposition with [3].

(4.3) Theorem: Let M be a finitely generated faithful and multiplication R-module. Then M is a weak uniform module if and only if R is weak uniform ring.

Proof: Assume that M is a weak uniform module, and let I be a non-zero ideal of R such $I \cap S = (0)$ for each non-zero semi-prime ideal S of R. Since M is a multiplication module, so $IM \cap SM = (0)$ [9].On the other hand because of M is multiplication and S is a semi-prime ideal of R therefore SM is semi-prime submodule of M [5]. But M is weak uniform module and IM is a submodule of M, so SM = (0). Since M is faithful module, then S = (0) and hence I is weak essential ideal of R. Conversely, let R be a weak uniform ring, and let W be a non-zero submodule of M and S be a non-zero semi-prime submodule of M such that $W \cap S = (0)$. Thus $[W: M] \cap [S: M] = (0)$. But [S: M] is semi-prime ideal of R [5], and R is a weak uniform ring, so [S: M] = (0) which implies that S = (0).That is W is weak essential submodule of M.

(4.4) Theorem: Let M be an R-module and let N be an essential submodule of M such that N does not contained in any semi-prime submodule of M. If N is a weak uniform submodule then M is weak uniform module.

Proof: Let K be any submodule of M with $K \cap S = (0)$ for each non-zero semi-prime submodule S of M. So $N \cap (K \cap S) = (0)$, and then $(N \cap K) \cap (N \cap S) = (0)$. By assumption, $N \not\subset S$ then $N \cap S$ is a semi-prime submodule of N [6]. On the other hand $N \cap K$ is a submodule of N, and N is a weak uniform, therefore $(N \cap S) = (0)$. Since N is essential submodule of M, then S= (0).

(4.5) Corollary: Let M be an R-module such that M does not contained in any semi-prime submodule of E (M). If M is a weak uniform module then E (M) is weak uniform module where E (M) is the injective hull of M.

Proof: By assumption M is an essential submodule of E (M), and by (4.4) we get the result.

References

1. Kasch, F.1982, Modules and Rings, Academic press, London.
2. Lu, C.P.1984, Prime submodules of modules, Comment.Math.Univ.St. Paul. 33:61-69.
3. Abdullah N.K. 2005, Semi-essential submodules and semi-uniform modules, M. Sc. Thesis, Univ. of Tkrit.
4. Dauns, J. and McDonald, B.R.1980, Prime modules and one-sided ideals in ring theory and algebra III , Proceedings of the third Oklahoma conference:301-344.
5. Athab, E.A.1996, Prime and semi-prime submodules, M. Sc. Thesis, Univ. of Baghdad.
6. Goodearl, K.R.1976, Ring theory, Marcel Dekker, New York.
7. Abdul-Razak, H.M.1999, Quasi-prime modules and quasi-prime submodules,M.Sc.Thesis,Univ.of Baghdad.
8. Elbast, Z. A. and Smith, P. F. 1988, Multiplication modules, Comm. Algebra Vol. 16:755-774.
9. Ahmed, A. A. 1992, on submodules of multiplication modules, M. Sc. Thesis, Univ. of Baghdad.

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المستخلص:

يقال للمقاس الجزئي الغير صفري N من M انه جوهري اذا كان$N \cap L \neq 0$ لكل مقاس غيرصفري L في M. كما يقال للمقاس الجزئي الغير صفري K في M انه شبه جوهري اذا كان $P \cap K \neq 0$ لكل موديول جزئي غير صفري اولي P في M.في هذا البحث ندرس نوعا اخر من المقاسات الجزئية الجوهرية يقع بين المقاسات الجزئية الجوهرية و المقاسات الجزئية الشبه جوهرية. نطلق على هذه المقاسات الجزئية إسم المقاسات الجزئية الجوهرية الضعيفة.

الكلمات المفتاحية: المقاسات الجزئية شبه الاولية ' المقاسات الجزئية الجوهرية ' المقاسات المنتظمة .