JOURNAL OF COMPUTING, VOLUME 4, ISSUE 5, MAY 2012, ISSN 2151-9617
https://sites.google.com/site/journalofcomputing
WWW.JOURNALOFCOMPUTING.ORG

Fibrewise Near Topological Spaces

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Abstract – In this paper we define and study new concepts of fibrewise topological spaces over B namely, fibrewise near topological spaces over B. Also, we introduce the concepts of fibrewise near closed and near open topological spaces over B; Furthermore we state and prove several Propositions concerning with these concepts.

Key Words – Fibrewise topological spaces, Fibrewise near topological spaces, Fibrewise near closed and near open topological spaces.
Math. Subject Classification 2010: Primary 55R70, 55P05, Secondary 54C05, 54C08, 54C10.

1 Introduction and Preliminaries

TO being with we work in the category fibrewise sets over a given set, called the base set. If the base set is denoted by B then a fibrewise set over B consists of a set X together with a function $p : X \to B$, called the projection. For each point b of B the fibre over b is the subset $X_b = p^{-1}(b)$ of X; fibres may be empty since we do not require p to be surjective, also for each subset B^* of B we regard $X_{B^*} = p^{-1}(B^*)$ as a fibrewise set over B^* with the projection determined by p.

Definition 1.1. [8] Let X and Y are fibrewise sets over B, with projections $p_X : X \to B$ and $p_Y : Y \to B$, respectively, a function $\phi : X \to Y$ is said to be fibrewise if $p_Y o\phi = p_X$, in other words if $\phi(X_b) \subset Y_b$ for each point b of B.

Note that a fibrewise function $\phi : X \to Y$ over B determines by restriction, a fibrewise function $\phi_{B^*} : X_{B^*} \to Y_{B^*}$ over B* for each B* of B.

Given an indexed family $\{X_r\}$ of fibrewise sets over B the fibrewise product $\prod_B X_r$ is defined, as a fibrewise set over B, and comes equipped with the family of fibrewise projection $\pi_r : \prod_B X_r \to X_r$, specifically the fibrewise product is defined as the subset of the ordinary product $\prod X_r$ in which the fibres are the products of the corresponding fibers of the factors X_r. The fibrewise product is characterized by the following Cartesian property: for each fibrewise set X over B the fibrewise functions $\phi : X \to \prod_B X_r$ correspond precisely to the families of fibrewise functions $\{\phi_r\}$, with $\phi_r = \pi_r o\phi : X \to X_r$. For example if $X_r = X$ for each index r the diagonal $\Delta : X \to \prod_B X$ is defined so that $\pi_r o\Delta = id_X$ for each r. If $\{X_r\}$ is as before, the fibrewise coproduct $\coprod_B X_r$ is also defined, as a fibrewise set over B, and comes equipped with the family of fibrewise insertions $\sigma_r : X_r \to \coprod_B X_r$, specifically the fibrewise coproduct coincides, as a set, with the ordinary coproduct (disjoint union) the fibres being the coproducts of the corresponding fibers of the summands X_r. The fibrewise coproduct is characterized by the following co-Cartesian property: for each fibrewise set X over B the fibrewise functions $\psi : \coprod_B X_r \to X$ correspond precisely to the families of fibrewise functions $\{\psi_r\}$, where $\psi_r = \psi o\sigma_r : X_r \to X$. For example if $X_r = X$ for each index r the codiagonal $\nabla : \coprod_B X \to X$ is defined so that $\nabla o\sigma_r = id_X$ for each r. The notation $X \times_B Y$ is used for the fibrewise product in the case of the family {X, Y} of two fibrewise sets and similarly for finite families generally. For other notions or notations which are not defined here we follow closely James [8], Engelking [6] and Bourbaki [4].

Definition 1.2. [8] Suppose that B is a topological space, the fibrewise topology on a fibrewise set X over B, mean any topology on X for which the projection p is continuous.

Remark 1.2. [8]
(a) The coarsest such topology is the topology induced by p, in which the open sets of X are precisely the inverse image of the open sets of B; this is called the fibrewise indiscrete topology.
(b) The fibrewise topological space over B is defined to be a fibrewise set over B with a fibrewise topology.

Definition 1.3. [8] The fibrewise function $\phi : X \to Y$, where X and Y are fibrewise topological spaces over B is called:
(a) Continuous if for each point $x \in X_b$, where $b \in B$, the inverse image of each open set of $\phi(x)$ is an open set of x.
(b) Open if for each point $x \in X_b$, where $b \in B$, the direct image of each open set of x is an open set of $\phi(x)$.

Definition 1.4. [8] The fibrewise topological space X over B is called fibrewise closed (resp., open) if the projection p is closed (resp., open) function.

Definition 1.5. A subset A of a topological space (X, τ) is called:
(a) Regular open [14] (briefly R-open) if A = Int(Cl(A)).

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(b) Pre-open [10] (briefly P-open) if $A \subseteq Int(Cl(A))$.
(c) Semi-open [9] (briefly S-open) if $A \subseteq Cl(Int(A))$.
(d) γ-open [5] (= b-open [2]) (briefly γ-open) if $A \subseteq Cl(Int(A)) \cup Int(Cl(A))$.
(e) α-open [12] (briefly α-open) if $A \subseteq Int(Cl(Int(A)))$.
(f) β-open [1] (= semi-pre-open set [3]) (briefly β-open) if $A \subseteq Cl(Int(Cl(A)))$.

The complement of a R-open (resp., P-open, S-open, γ-open, α-open, β-open) is called R-closed (resp., P-closed, S-closed, γ-closed, α-closed, β-closed). The family of all R-open (resp., P-open, S-open, γ-open, α-open, β-open) are larger than τ (except R-open) and closed under forming arbitrary union.

Definition 1.6. A function $\varphi : X \to Y$ is said to be strongly δ-continuous [7] (briefly R-continuous) (resp., P-continuous [10], S-continuous [9], γ-continuous [5], α-continuous [11], β-continuous [1]) if the inverse image of each open set in Y is R-open (resp., P-open, S-open, γ-open, α-open, β-open) in X.

Definition 1.7. A function $\varphi : X \to Y$ is said to be R-open [13] (resp., P-open [10], S-open [9], γ-open [5], α-open [11], β-open [1]) if the image of each open set in X is R-open (resp., P-open, S-open, γ-open, α-open, β-open) in Y.

2 Fibrewise Near Topological Spaces

In this section, we introduce the concepts of fibrewise near topology, several topological property on the obtained fibrewise near topology are studies.

Definition 2.1. Suppose that B is a topological space, the fibrewise near topology (briefly fibrewise j-topology) on a fibrewise set X over B, mean any j-topology on X for which the projection p is j-continuous where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.

The fibrewise j-topological space over B is defined to be a fibrewise set over B with a fibrewise j-topology, where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.

For example the topological product $B \times T$, for any topological (resp., j-topological) space T, can be regarded as a fibrewise topological (resp., j-topological) spaces over B, using the first projection, and similarly for any subspace of $B \times T$.

Remark 2.1. In fibrewise topology we work over a topological base space B, say. When B is a point-space the theory reduces to that of ordinary topology. A fibrewise topological (resp., j-topological) space over B is just a topological (resp., j-topological) space X together with a continuous (resp., j-continuous) projection $p : X \to B$. So the implication between fibrewise topological spaces and the families of fibrewise j-topological spaces are given in the following diagram:

$$\begin{array}{ccccccc} \text{fw. R-top.} & \Rightarrow & \text{fw. top.} & \Rightarrow & \text{fw. } \alpha\text{-top.} & \Rightarrow & \text{fw. S-top.} \\ & & & & \Downarrow & & \Downarrow \\ & & & & \text{fw. P-top.} \Rightarrow & \text{fw. } \gamma\text{-top.} & \Rightarrow \text{fw. } \beta\text{-top.} \end{array}$$

Where, fw. ≡ fibrewise and top. ≡ topology

The following example show that these implications are not reversible.

Example 2.1.

(a) Let X = B = {a, b, c, d}. Let $\tau_X = \{X, \phi, \{a\}, \{b, c\}\}$ and τ_B = discrete topology. Define an identity projection $p : (X, \tau_X) \to (B, \tau_B)$; p(x) = x for each $x \in X$. Then X is fibrewise β-topology, but not fibrewise γ-topology.
(b) Let X = B = {a, b, c}. Let $\tau_X = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\tau_B = \{B, \phi, \{a\}, \{a, b\}\}$. Define a projection $p : (X, \tau_X) \to (B, \tau_B)$; p(a) = a, p(b) = c, p(c) = b. Then X is fibrewise γ-topology, but not fibrewise P-topology.
(c) Define the identity projection from an indiscrete space (X, τ_{ind}) onto a discrete space (B, τ_{dis}). Then X is fibrewise γ-topology, but not fibrewise S-topology.
(d) Define the injective projection from an indiscrete space (X, τ_{ind}) into a discrete space (B, τ_{dis}). Then X is fibrewise P-topology, but not fibrewise α-topology.
(e) Let X = B = {a, b, c}. Let $\tau_X = \{X, \phi, \{a\}, \{c\}, \{a, c\}\}$ and τ_B = discrete topology. Define the projection $p : (X, \tau_X) \to (B, \tau_B)$; p(a) = a, p(b) = b, p(c) = b. Then X is fibrewise S-topology, but not fibrewise α-topology.
(f) Let X = B = {a, b, c}. Let $\tau_X = \{X, \phi, \{a\}\}$ and τ_B = discrete topology. Define the projection $p : (X, \tau_X) \to (B, \tau_B)$; p(a) = p(b) = a, p(c) = c. Then X is fibrewise α-topology, but not fibrewise topology.
(g) Let X = B = {a, b, c, d}. Let $\tau_X = \tau_B = \{X, \phi, \{d\}, \{a, b\}, \{a, b, d\}\}$. Define an identity projection $p : (X, \tau_X) \to (B, \tau_B)$; p(x) = x for each $x \in X$. Then X is fibrewise topology, but not fibrewise R-topology.

Definition 2.2. A fibrewise function $\phi : X \to Y$, where X and Y are fibrewise topological spaces over B is called:

(a) j-continuous if for each point $x \in X_b$, where $b \in B$, the inverse image of each open set of $\phi(x)$ is a j-open set of x, where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.
(b) j-open if for each point $x \in X_b$, where $b \in B$, the direct image of each open set of x is a j-open set of $\phi(x)$, where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.
(c) j-irresolute if for each point $x \in X_b$, where $b \in B$, the inverse image of each j-open set of $\phi(x)$ is a j-open set of x, where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.

Let $\phi : X \to Y$ be a fibrewise function where X is a fibrewise set and Y is a fibrewise topological space over B. We can give X the induced (resp., j-induced) topology, in the ordinary sense, and this is necessarily a fibrewise topology (resp., j-topology). We may refer to it, as the induced (resp., j-induced) fibrewise topology, where $j \in \{R, P, S, \gamma, \alpha, \beta\}$, and note the following characterizations.

Proposition 2.1. [8] Let $\phi : X \to Y$ be a fibrewise function, where Y is a fibrewise topological space over B and X is a fibrwise set has the induced fibrewise topology. Then for each fibrewise topological space Z, a fibrewise function $\psi : Z \to X$ is continuous if and only if the composition $\phi o \psi : Z \to Y$ is continuous.

Proposition 2.2. Let $\phi : X \to Y$ be a fibrewise function, where Y is a fibrewise topological space over B and X is a fibrwise set has the induced fibrewise topology. If for each fibrewise topological space Z, a fibrewise function $\psi : Z \to X$ is j-continuous if and only if the composition $\phi o \psi : Z \to Y$ is j-continuous, where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.
Proof. ($\Rightarrow$) Suppose that ψ is j-continuous. Let $z \in Z_b$, where $b \in B$ and V open set of $(\phi o \psi)(z) = y \in Y_b$ in Y. Since ϕ is continuous, $\phi^{-1}(V)$ is an open set containing $\psi(z) = x \in X_b$ in X. Since ψ is j-continuous, then $\psi^{-1}(\phi^{-1}(V))$ is a j-open set containing $z \in Z_b$ in Z and $\psi^{-1}(\phi^{-1}(V)) = (\psi o \phi)^{-1}(V)$ is a j-open set containing $z \in Z_b$ in Z, where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.
($\Leftarrow$) Suppose that $\phi o \psi$ is j-continuous. Let $z \in Z_b$, where $b \in B$ and U open set of $\psi(z) = x \in X_b$ in X. Since ϕ is open, $\phi(U)$ is an open set containing $\phi(x) = \phi(\psi(z)) = y \in Y_b$ in Y. Since $\phi o \psi$ is j-continuous, then $(\phi o \psi)^{-1}(\phi(U)) = \psi^{-1}(U)$ is a j-open set containing $z \in Z_b$ in Z, where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.

Proposition 2.3. Let $\phi : X \to Y$ be a fibrewise function, where Y is a fibrewise topological space over B and X is a fibrwise set has the j-induced fibrewise topology. If for each fibrewise topological space Z, a fibrewise function $\psi : Z \to X$ is j-irresolute if and only if the composition $\phi o \psi : Z \to Y$ is j-continuous, where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.
Proof. Clear.

Proposition 2.4. Let $\phi : X \to Y$ be a fibrewise function, where Y is a fibrewise topological space over B and X is a fibrwise set has the induced fibrewise topology. If for each fibrewise topological space Z, a fibrewise function $\psi : Z \to X$ is open, surjection if and only if the composition $\phi o \psi : Z \to Y$ is open.
Proof. Clear.

Corollary 2.1. (of Proposition 2.2)
(a) Let $\phi : X \to Y$ be a fibrewise function, where Y is a fibrewise topological space over B and X is a fibrwise set has the induced fibrewise topology. If for each fibrewise topological space Z, a fibrewise function $\psi : Z \to X$ is R-continuous, then the composition $\phi o \psi : Z \to Y$ is j-continuous, where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.
(b) Let $\phi : X \to Y$ be a fibrewise continuous function, where Y is a fibrewise topological space over B and X is a fibrwise set has the induced fibrewise topology. If for each fibrewise topological space Z, a fibrewise function $\psi : Z \to X$ is α-continuous, then the composition $\phi o \psi : Z \to Y$ is j-continuous, where $j \in \{P, S, \gamma, \alpha, \beta\}$.
(c) Let $\phi : X \to Y$ be a fibrewise continuous function, where Y is a fibrewise topological space over B and X is a fibrwise set has the induced fibrewise topology. If for each fibrewise topological space Z, a fibrewise function $\psi : Z \to X$ is P-continuous, then the composition $\phi o \psi : Z \to Y$ is j-continuous, where $j \in \{P, \gamma, \beta\}$.
(d) Let $\phi : X \to Y$ be a fibrewise continuous function, where Y is a fibrewise topological space over B and X is a fibrwise set has the induced fibrewise topology. If for each fibrewise topological space Z, a fibrewise function $\psi : Z \to X$ is S-continuous, then the composition $\phi o \psi : Z \to Y$ is j-continuous, where $j \in \{S, \gamma, \beta\}$.
(e) Let $\phi : X \to Y$ be a fibrewise continuous function, where Y is a fibrewise topological space over B and X is a fibrwise set has the induced fibrewise topology. If for each fibrewise topological space Z, a fibrewise function $\psi : Z \to X$ is γ-continuous, then the composition $\phi o \psi : Z \to Y$ is j-continuous, where $j \in \{\gamma, \beta\}$.

Corollary 2.2. (of Proposition 2.3)
(a) Let $\phi : X \to Y$ be a fibrewise function, where Y is a fibrewise topological space over B and X is a fibrwise set has the R-induced fibrewise topology. If for each fibrewise topological space Z, a fibrewise function $\psi : Z \to X$ is R-irresolute, then the composition $\phi o \psi : Z \to Y$ is j-continuous, where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.
(b) Let $\phi : X \to Y$ be a fibrewise function, where Y is a fibrewise topological space over B and X is a fibrwise set has the α-induced fibrewise topology. If for each fibrewise topological space Z, a fibrewise function $\psi : Z \to X$ is α-irresolute, then the composition $\phi o \psi : Z \to Y$ is j-continuous, where $j \in \{P, S, \gamma, \alpha, \beta\}$.
(c) Let $\phi : X \to Y$ be a fibrewise function, where Y is a fibrewise topological space over B and X is a fibrwise set has the P-induced fibrewise topology. If for each fibrewise topological space Z, a fibrewise function $\psi : Z \to X$ is P-irresolute, then the composition $\phi o \psi : Z \to Y$ is j-continuous, where $j \in \{P, \gamma, \beta\}$.
(d) Let $\phi : X \to Y$ be a fibrewise function, where Y is a fibrewise topological space over B and X is a fibrwise set has the S-induced fibrewise topology. If for each fibrewise topological space Z, a fibrewise function $\psi : Z \to X$ is S-irresolute, then the composition $\phi o \psi : Z \to Y$ is j-continuous, where $j \in \{S, \gamma, \beta\}$.
(e) Let $\phi : X \to Y$ be a fibrewise function, where Y is a fibrewise topological space over B and X is a fibrwise set has the γ-induced fibrewise topology. If for each fibrewise topological space Z, a fibrewise function $\psi : Z \to X$ is γ-irresolute, then the composition $\phi o \psi : Z \to Y$ is j-continuous, where $j \in \{\gamma, \beta\}$.

Let us pass of general cases of Propositions (2.2) and (2.3) as follows:
Similarly in the case of families $\{\phi_r\}$ of fibrewise functions, where $\phi_r : X \to Y_r$ with Y_r fibrewise topological spaces over B for each r. In particular, given a family $\{X_r\}$ of fibrewise topological spaces over B, the fibrewise

topological product $\prod_B X_r$ is defined to be the fibrewise product with the fibrewise topology induced (resp., j-induced) by the family of projections $\pi_r : \prod_B X_r \to X_r$. Then for each fibrewise topological space Z over B a fibrewise function $\theta : Z \to \prod_B X_r$ is j-continuous (resp., j-irresolute) if and only if each of the fibrewise function $\pi_r o\theta : Z \to X_r$ is j-continuous (resp., j-continuous). For example when $X_r = X$ for each index r we see that the diagonal $\Delta : X \to \prod_B X$ is j-continuous (resp., j-irresolute) if and only if the composition $\pi_r o\Delta = id_X$ is j-continuous (resp., j-continuous), where j∈{R, P, S, γ, α, β}.

Again if $\{X_r\}$ is a family of fibrewise topological spaces over B and $\psi : \coprod_B X_r \to X$ is a fibrewise function where X a fibrewise topology over B and $\coprod_B X_r$ is fibrewise topological coproduct at the set-theoretic level with the ordinary coproduct topology (resp., j-topology), also for each fibrewise topology X_r with the family of fibrewise insertions $\sigma_r : X_r \to \coprod_B X_r$ is j-continuous (resp., j-irresolute), if and only if the composition $\psi_r = \psi o\sigma_r : X_r \to X$ is j-continuous (resp., j-continuous). For example when $X_r = X$ for each index r we see that the codiagonal $\nabla : \coprod_B X \to X$ is continuous (resp., j-continuous), where j∈{R, P, S, γ, α, β}.

3 Fibrewise Near Closed and Near Open Topological Spaces

In this section, we introduce the concepts of fibrewise near closed, near coclosed, near biclosed, near open, near coopen and near biopen (briefly j-closed, j-coclosed, j-biclosed, j-open, j-coopen and j-biopen) topological spaces, where j∈{R, P, S, γ, α, β}, several topological property on the obtained concepts are studies.

Definition 3.1. A function φ : X → Y is called j-coclosed (resp., j-biclosed) function where X and Y are topological spaces if it is maps j-closed sets onto closed (resp., j-closed) sets where j∈{R, P, S, γ, α, β}.

Definition 3.2. A function φ : X → Y is called j-coopen (resp., j-biopen) function where X and Y are topological spaces if it is maps j-open sets onto open (resp., j-open) sets where j∈{R, P, S, γ, α, β}.

Definition 3.3. A fibrewise topological space X over B is called fibrewise j-closed (resp., j-coclosed, j-biclosed) if the projection p is j-closed (resp., j-coclosed, j-biclosed) where j∈{R, P, S, γ, α, β}.

Proposition 3.1. Let ϕ : X → Y be a closed fibrewise function, where X and Y are fibrewise topological spaces over B.
(a) If Y is fibrewise closed, then X is fibrewise closed [8].
(b) If Y is fibrewise j-closed, then X is fibrewise j-closed, where j∈{R, P, S, γ, α, β}.
(c) If Y is fibrewise j-coclosed, then X is fibrewise closed, where j∈{P, S, γ, α, β}.
(d) If Y is fibrewise j-coclosed, then X is fibrewise j-closed, where j∈{P, S, γ, α, β}.
(e) If Y is fibrewise j-biclosed, then X is fibrewise j-closed, where j∈{P, S, γ, α, β}.

Proof. The proofs of the five facts are similar; so, we will only prove the fact (b): Suppose that ϕ : X → Y is closed fibrewise function and Y is fibrewise j-closed i.e., the projection $p_Y : Y \to B$ is j-closed. To show that X is fibrewise j-closed i.e., the projection $p_X : X \to B$ is j-closed. Now let F be a closed subset of X_b, where b ∈ B, since ϕ is closed, then ϕ(F) is closed subset of Y_b. Since p_Y is j-closed, then $p_Y(\phi(F))$ is j-closed in B, but $p_Y(\phi(F)) = (p_Y o\phi)(F) = p_X(F)$ is j-closed in B. Thus p_X is j-closed and X is fibrewise j-closed where j∈{R, P, S, γ, α, β}.

Proposition 3.2. Let ϕ : X → Y be a j-closed fibrewise function, where X and Y are fibrewise topological spaces over B.
(a) If Y is fibrewise j-coclosed, then X is fibrewise closed, where j∈{R, P, S, γ, α, β}.
(b) If Y is fibrewise j-coclosed, then X is fibrewise j-closed, where j∈{P, S, γ, α, β}.
(c) If Y is fibrewise j-biclosed, then X is fibrewise j-closed, where j∈{R, P, S, γ, α, β}.

Proof. The proof is similar to the proof of Proposition (3.1), so it is omitted.

Proposition 3.3. Let ϕ : X → Y be a j-coclosed fibrewise function, where X and Y are fibrewise topological spaces over B.
(a) If Y is fibrewise closed, then X is fibrewise j-coclosed, where j∈{R, P, S, γ, α, β}.
(b) If Y is fibrewise j-closed, then X is fibrewise j-biclosed, where j∈{R, P, S, γ, α, β}.
(c) If Y is fibrewise j-coclosed, then X is fibrewise j-coclosed, where j∈{P, S, γ, α, β}.
(d) If Y is fibrewise j-biclosed, then X is fibrewise j-biclosed, where j∈{P, S, γ, α, β}.

Proof. The proof is similar to the proof of Proposition (3.1), so it is omitted.

Proposition 3.4. Let ϕ : X → Y be a j-biclosed fibrewise function, where X and Y are fibrewise topological spaces over B.
(a) If Y is fibrewise j-coclosed, then X is fibrewise j-coclosed, where j∈{R, P, S, γ, α, β}.
(b) If Y is fibrewise j-coclosed, then X is fibrewise j-biclosed, where j∈{P, S, γ, α, β}.
(c) If Y is fibrewise j-biclosed, then X is fibrewise j-biclosed, where j∈{R, P, S, γ, α, β}.

Proof. The proof is similar to the proof of Proposition (3.1), so it is omitted.

Remark 3.1. Proposition 3.1.(c), (d), (e), Proposition 3.2.(b), Proposition 3.3.(c), (d) and Proposition 3.4.(b) are not true when j = R since, every closed sets not necessity R-closed sets.

Proposition 3.5. Let X be a fibrewise topological space over B.

(a) Suppose that X_i is fibrewise closed for each member X_i of a finite covering of X. Then X is fibrewise closed [8].

(b) Suppose that X_i is fibrewise j-coclosed for each member X_i of a finite covering of X. Then X is fibrewise j-coclosed, where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.

Proof. The proofs of the two facts are similar; so, we will only prove the fact (b): Let X be a fibrewise topological space over B, then the projection $p : X \to B$ exists. To show that p is j-coclosed. Now, since X_i is fibrewise j-coclosed, then the projection $p_i : X_i \to B$ is j-coclosed for each member X_i of a finite covering of X. Let F be a j-closed subset of X, then $p(F) = \cup p_i(X_i \cap F)$ which is a finite union of closed sets and hence p is j-coclosed. Thus, X is fibrewise j-coclosed where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.

Remark 3.2. Proposition 3.5.(b) is not true in the cases fibrewise j-closed and j-biclosed topological space since a finite union of j-closed sets not necessity j-closed set, where $j \in \{R, P, S, \gamma, \beta\}$.

Corollary 3.1. Let X be a fibrewise topological space over B. Suppose that X_i is fibrewise α-closed (resp., α-biclosed) for each member X_i of a finite covering of X. Then X is fibrewise α-closed (resp., α-biclosed).

Proposition 3.6. Let X be a fibrewise topological space over B. Then

(a) X is fibrewise closed if and only if for each fibre X_b of X and each open set U of X_b in X, there exists an open set O of b such that $X_O \subset U$ [8].

(b) X is fibrewise j-closed if and only if for each fibre X_b of X and each open set U of X_b in X, there exists a j-open set O of b such that $X_O \subset U$, where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.

(c) X is fibrewise j-coclosed if and only if for each fibre X_b of X and each j-open set U of X_b in X, there exists an open set O of b such that $X_O \subset U$, where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.

(d) X is fibrewise j-biclosed if and only if for each fibre X_b of X and each j-open set U of X_b in X, there exists a j-open set O of b such that $X_O \subset U$, where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.

Proof. The proofs of the four facts are similar; so, we will only prove the fact (b): ($\Rightarrow$) Suppose that X is fibrewise j-closed i.e., the projection $p : X \to B$ is j-closed. Now, let $b \in B$ and U open set of X_b in X, then X–U is closed in X, this implies p(X–U) is j-closed in B, let O = B–p(X–U), then O a j-open set of b in B and $X_O = p^{-1}(O) = X - p^{-1}(p(X-U)) \subset U$ where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.

($\Leftarrow$) Suppose that the assumption hold and $p : X \to B$. Now, let F be a closed subset of X and $b \in B-p(F)$ and each open set U of fibre X_b in X. By assumption there exists a j-open O of b such that $X_O \subset U$. It is easy to show that $O \subset B-p(F)$, hence B–p(F) is j-open in B and this implies p(F) is j-closed in B and p is j-closed. Thus X is fibrewise j-closed where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.

Definition 3.4. A fibrewise topological space X over B is called fibrewise j-open (resp., j-coopen, j-biopen) if the projection p is j-open (resp., j-coopen, j-biopen) where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.

Proposition 3.7. Let $\phi : X \to Y$ be an open fibrewise function, where X and Y are fibrewise topological spaces over B.

(a) If Y is fibrewise open, then X is fibrewise open [8].

(b) If Y is fibrewise j-open, then X is fibrewise j-open, where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.

(c) If Y is fibrewise j-coopen, then X is fibrewise open, where $j \in \{P, S, \gamma, \alpha, \beta\}$.

(d) If Y is fibrewise j-coopen, then X is fibrewise j-open, where $j \in \{P, S, \gamma, \alpha, \beta\}$.

(e) If Y is fibrewise j-biopen, then X is fibrewise j-open, where $j \in \{P, S, \gamma, \alpha, \beta\}$.

Proof. The proofs of the five facts are similar; so, we will only prove the fact (b): Suppose that $\phi : X \to Y$ is open fibrewise function and Y is fibrewise j-open i.e., the projection $p_Y : Y \to B$ is j-open. To show that X is fibrewise j-open i.e., the projection $p_X : X \to B$ is j-open. Now let O is open subset of X_b, where $b \in B$, since ϕ is open, then $\phi(O)$ is open subset of Y_b, since p_Y is j-open, then $p_Y(\phi(O))$ is j-open in B, but $p_Y(\phi(O)) = (p_Y o\phi)(O) = p_X(O)$ is j-open in B. Thus p_X is j-open and X is fibrewise j-open, where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.

Proposition 3.8. Let $\phi : X \to Y$ be a j-open fibrewise function, where X and Y are fibrewise topological spaces over B.

(a) If Y is fibrewise j-coopen, then X is fibrewise open, where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.

(b) If Y is fibrewise j-coopen, then X is fibrewise j-open, where $j \in \{P, S, \gamma, \alpha, \beta\}$.

(c) If Y is fibrewise j-biopen, then X is fibrewise j-open, where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.

Proof. The proof is similar to the proof of Proposition (3.7), and therefore is omitted.

Proposition 3.9. Let $\phi : X \to Y$ be a j-coopen fibrewise function, where X and Y are fibrewise topological spaces over B.

(a) If Y is fibrewise open, then X is fibrewise j-coopen, where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.

(b) If Y is fibrewise j-open, then X is fibrewise j-biopen, where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.

(c) If Y is fibrewise j-coopen, then X is fibrewise j-coopen, where $j \in \{P, S, \gamma, \alpha, \beta\}$.

(d) If Y is fibrewise j-biopen, then X is fibrewise j-biopen, where $j \in \{P, S, \gamma, \alpha, \beta\}$.

Proof. The proof is similar to the proof of Proposition (3.7), and therefore is omitted.

Proposition 3.10. Let $\phi : X \to Y$ be a j-biopen fibrewise function, where X and Y are fibrewise topological spaces over B.

(a) If Y is fibrewise j-coopen, then X is fibrewise j-coopen, where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.
(b) If Y is fibrewise j-coopen, then X is fibrewise j-biopen, where $j \in \{P, S, \gamma, \alpha, \beta\}$.
(c) If Y is fibrewise j-biopen, then X is fibrewise j-biopen, where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.

Proof. The proof is similar to the proof of Proposition (3.7), and hence is omitted.

Remark 3.3. Proposition 3.7.(c), (d), (e), Proposition 3.8.(b), Proposition 3.9.(c), (d) and Proposition 3.10.(b) are not true when j = R since, every open sets not necessity R-open sets.

Proposition 3.11.

(a) Let $\{X_r\}$ be a finite family of fibrewise open spaces over B. Then the fibrewise topological product $X = \prod_B X_r$ is also open [8].
(b) Let $\{X_r\}$ be a finite family of fibrewise j-open (resp., j-coopen, j-biopen) spaces over B. Then the fibrewise topological product $X = \prod_B X_r$ is also j-open (resp., j-coopen, j-biopen), where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.

Proof. The proofs of the four facts are similar; so, we will only prove the case when $\{X_r\}$ be a finite family of fibrewise j-open: Suppose that $X = \prod_B X_r$ is a fibrewise topological space over B, then $p : X = \prod_B X_r \to B$ is exists. To show that p is j-open. Now, since $\{X_r\}$ be a finite family of fibrewise j-open spaces over B, then the projection $p_r : X_r \to B$ is j-open for each r. Let O be an open subset of X, then $p(O) = p(\prod_B (X_r \cap O)) = \prod_B p_r(X_r \cap O)$ which is a finite product of j-open sets and hence p is j-open. Thus, the fibrewise topological product $X = \prod_B X_r$ is a fibrewise j-open, where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.

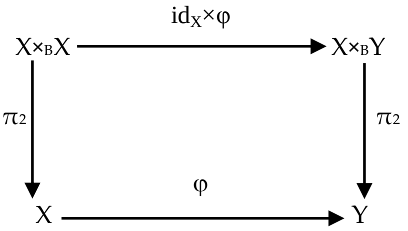
In other words the class of fibrewise open (resp., j-open, j-coopen, j-biopen) spaces is finitely multiplicative. In fact Proposition (3.11) remains true for infinite families provided each member of the family is fibrewise nonempty in the sense that the projection is surjective.

Remark 3.4. If X is fibrewise open (resp., j-open, j-coopen, j-biopen) then the second projection $\pi_2 : X \times_B Y \to Y$ is open (resp., j-open, j-coopen, j-biopen) for all fibrewise topological spaces Y. because for every non-empty open (resp., open, j-open, j-open) set $W_1 \times_B W_2 \subset X \times_B Y$, we have $\pi_2(W_1 \times_B W_2) = W_2$ is open (resp., j-open, open, j-open), where $j \in \{R, P, S, \gamma, \alpha, \beta\}$. We use this in the proof of the following results.

Proposition 3.12. Let $\phi : X \to Y$ be a fibrewise function, where X and Y are fibrewise topological spaces over B. Let $id_X \times \phi : X \times_B X \to X \times_B Y$.

(a) If $id_X \times \phi$ is open and that X is fibrewise open. Then ϕ itself is open [8].
(b) If $id_X \times \phi$ is open and that X is fibrewise open, Y is fibrewise j-open. Then ϕ itself is j-open, where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.
(c) If $id_X \times \phi$ is j-open and that X is fibrewise open, Y is fibrewise j-coopen. Then ϕ itself is open, where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.
(d) If $id_X \times \phi$ is j-open and that X is fibrewise open, Y is fibrewise j-biopen. Then ϕ itself is j-open, where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.
(e) If $id_X \times \phi$ is j-open and that X is fibrewise j-open, Y is fibrewise j-biopen. Then ϕ itself is j-biopen, where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.
(f) If $id_X \times \phi$ is j-coopen and that X is fibrewise j-coopen, Y is fibrewise open. Then ϕ it self is open, where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.
(g) If $id_X \times \phi$ is j-biopen and that X is fibrewise j-biopen, Y is fibrewise j-biopen. Then ϕ itself is j-biopen, where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.

Proof. The proofs of the seven facts are similar; so, we will only prove the fact (b): Consider the following commutative diagram:



The projection on the left is surjective and j-open, since Y is fibrewise j-open, while the projection on the right is open, since X is fibrewise open. Therefore $\pi_2 o(id_X \times \phi) = \phi o \pi_2$ is j-open, and so ϕ is j-open, by Proposition 3.7.(b) as asserted, where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.

Our next three results apply equally to fibrewise closed (resp., j-closed, j-coclosed, j-biclosed) and the fibrewise open (resp., j-open, j-coopen, j-biopen) spaces, where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.

Proposition 3.13. Let $\phi : X \to Y$ be a continuous fibrewise surjection, where X and Y are fibrewise topological spaces over B.

(a) If X is fibrewise closed (resp., open), then Y is fibrewise closed (resp., open) [8].
(b) If X is fibrewise j-closed (resp., j-open), then Y is fibrewise j-closed (resp., j-open), where $j \in \{R, P, S, \gamma, \alpha, \beta\}$.

Proof. The proofs of the two facts are similar; so, we will only prove the fact (b): Suppose that $\phi : X \to Y$ is continuous fibrewise surjection and X is fibrewise j-closed (resp., j-open) i.e., the projection $p_X : X \to B$ is j-closed (resp., j-open). To show that Y is fibrewise j-closed (resp., j-open) i.e., the projection $p_Y : Y \to B$ is j-closed (resp., j-open). Let G be a closed (resp., open) subset of Y_b, where b

$\in$ B. Since ϕ is continuous fibrewise, then $\phi^{-1}(G)$ is closed (resp., open) subset of X_b. Since p_X is j-closed (resp., j-open), then $p_X(\phi^{-1}(G))$ is j-closed (resp., j-open) in B, but $p_X(\phi^{-1}(G)) = (p_X o \phi^{-1})(G) = p_Y(G)$ is j-closed (resp., j-open) in B. Thus p_Y is j-closed (resp., j-open) and Y is fibrewise j-closed (resp., j-open), where j$\in$\{R, P, S, γ, α, β\}.

Corollary 3.2. Let $\phi : X \to Y$ be a j-continuous fibrewise surjection, where X and Y are fibrewise topological spaces over B.

(a) If X is fibrewise j-coclosed (resp., j-coopen), then Y is fibrewise closed (resp., open), where j$\in$\{R, P, S, γ, α, β\}.

(b) If X is fibrewise j-biclosed (resp., j-biopen), then Y is fibrewise j-closed (resp., j-open), where j$\in$\{R, P, S, γ, α, β\}.

Corollary 3.3. Let $\phi : X \to Y$ be a j-irresolute fibrewise surjection, where X and Y are fibrewise topological spaces over B.

(a) If X is fibrewise j-coclosed (resp., j-coopen), then Y is fibrewise j-coclosed (resp., j-coopen), where j$\in$\{R, P, S, γ, α, β\}.

(b) If X is fibrewise j-biclosed (resp., j-biopen), then Y is fibrewise j-biclosed (resp., j-biopen), where j$\in$\{R, P, S, γ, α, β\}.

Proposition 3.14. Let X be a fibrewise topological space over B.

(a) Suppose that X is fibrewise closed (resp., open) over B. Then X_{B^*} is fibrewise closed (resp., open) over B* for each subspace B* of B [8].

(b) Suppose that X is fibrewise j-closed (resp., j-open) over B. Then X_{B^*} is fibrewise j-closed (resp., j-open) over B* for each subspace B* of B, where j$\in$\{R, P, S, γ, α, β\}.

(c) Suppose that X is fibrewise j-coclosed (resp., j-coopen) over B. Then X_{B^*} is fibrewise j-coclosed (resp., j-coopen) over B* for each subspace B* of B, where j$\in$\{R, P, S, γ, α, β\}.

(d) Suppose that X is fibrewise j-biclosed (resp., j-biopen) over B. Then X_{B^*} is fibrewise j-biclosed (resp., j-biopen) over B* for each subspace B* of B, where j$\in$\{R, P, S, γ, α, β\}.

Proof. The proofs of the four facts are similar; so, we will only prove the fact (b): Suppose that X is a fibrewise j-closed (resp., j-open) i.e., the projection $p : X \to B$ is j-closed (resp., j-open). To show that X_{B^*} is fibrewise j-closed (resp., j-open) over B* i.e., the projection $p_{B^*} : X_{B^*} \to B^*$ is j-closed (resp., j-open). Now, let G be a closed (resp., open) subset of X, then $G \cap X_{B^*}$ is closed (resp., open) in subspace X_{B^*} and $p_{B^*}(G \cap X_{B^*}) = p(G \cap X_{B^*}) = p(G) \cap B^*$ which is j-closed (resp., j-open) set in B*. Thus p_{B^*} is j-closed (resp., j-open) and X_{B^*} is fibrewise j-closed (resp., j-open) over B*, where j$\in$\{R, P, S, γ, α, β\}.

Proposition 3.15. Let X be a fibrewise topological space over B.

(a) Suppose that X_{B_i} is fibrewise closed (resp., open) over B_i for each member B_i of an open covering of B. Then X is fibrewise closed (resp., open) over B [8].

(b) Suppose that X_{B_i} is fibrewise j-open over B_i for each member B_i of an open covering of B. Then X is fibrewise j-open over B, where j$\in$\{R, P, S, γ, α, β\}.

(c) Suppose that X_{B_i} is fibrewise j-coclosed (resp., j-coopen) over B_i for each member B_i of an open covering of B. Then X is fibrewise j-coclosed (resp., j-coopen) over B, where j$\in$\{R, P, S, γ, α, β\}.

(d) Suppose that X_{B_i} is fibrewise j-biopen over B_i for each member B_i of an open covering of B. Then X is fibrewise j-biopen over B, where j$\in$\{R, P, S, γ, α, β\}.

Proof. The proofs of the four facts are similar; so, we will only prove the fact (b): Suppose that X is a fibrewise topological space over B, then the projection $p : X \to B$ exists. To show that p is j-open. Now, since X_{B_i} is fibrewise j-open over B_i, then the projection $p_{B_i} : X_{B_i} \to B_i$ is j-open for each member B_i of an open covering of B. Let G be an open subset of X, then we have $p(G) = \cup p_B(X_{B_i} \cap G)$ which is a union of j-open sets and hence p is j-open. Thus, X is fibrewise j-open over B, where j$\in$\{R, P, S, γ, α, β\}.

Remark 3.5. In the Proposition 3.15. above if we put in the case (b) j-closed and in the case (d) j-biclosed, the Proposition is not true since the union of j-closed sets not necessity j-closed set, where j$\in$\{R, P, S, γ, β\}.

Corollary 3.4. Let X be a fibrewise topological space over B.

(a) Suppose that X_{B_i} is fibrewise α-closed over B_i for each member B_i of an open covering of B. Then X is fibrewise α-closed over B.

(b) Suppose that X_{B_i} is fibrewise α-biclosed over B_i for each member B_i of an open covering of B. Then X is fibrewise α-biclosed over B.

In fact the last Proposition is also true for locally finite closed coverings by using theorem (1.1.11) and corollary (1.1.12) in [6].

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