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Fibrewise Totally Topological Spaces

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Abstract. In this paper we define and study new concepts of fiberwise topological space over $\mathfrak{P}$ namely fiberwise totally topological space over $\mathfrak{P}$. Also we introduce the concepts of fiberwise totally closed and totally open topological space over $\mathfrak{P}$. Also, we define and study the concepts fiberwise locally sliceable and fiberwise locally section able totally topological space over $\mathfrak{P}$. Furtherom we state and prove several propositions concerning with these concepts.

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INTRODUCYION:

In order to begin the category in the classification of fibrewise (briefly. $f.w.$) sets over a given set, named the base set, which say $\mathfrak{P}$. A . $f.w.$, set over $\mathfrak{P}$ consest of function $p : \mathrm{G} \to \mathfrak{P}$, that is named the projection on the set G. The fiber over b for every point b of $\mathfrak{P}$ is the subset $\mathrm{G_b} = p^{-1}(\mathrm{b})$ of G. Since we do not require p is surjective, the fiber Perhaps, will be empty, also, for every $\mathfrak{P}^*$ subset of $\mathfrak{P}$ we considered $\mathrm{G}_{\mathfrak{P}^*} = p^{-1}(\mathfrak{P}^*)$ like a . $f.w.$,set with the projection determined by p over$\mathfrak{P}^*$, the alternative $\mathrm{G}_{\mathfrak{P}^*}$ notation is often referred to as $\mathrm{G}|\mathfrak{P}^*$. We considered for every set Z, the Cartesian product $\mathfrak{P} \times Z$ by the first projection like a $f.w.$ set $\mathfrak{P}$. If G and K are fiberwise set over $\mathfrak{P}$, with projection p_G and $p_K$respectively, function $\Gamma : G \to K$ is said to be fibrewise if $p_K \circ \Gamma = p_G$, in other words if $\Gamma(G_b) \subset K_b$ for every $\mathrm{b} \in \mathfrak{P}$. As well as, we built on some of the result in [1, 3, 5, 8,10, 11]. For other notations or notions which are not mentioned here we go behind closely I.M. James [2], R. Engelking [7] and N. Bourbaki [4]

Defintion 1.1.[6]. A function p : (G , τ_G) $\longrightarrow$ ($\mathfrak{P}$, $\mathcal{L}$) is called totally continuous if the inverse image of every open sub set of $\mathfrak{P}$ is a clopen sub set of G.

Definition 1.2. [2]. Let ($\mathfrak{P}$, $\mathcal{L}$) be a topological space. The $f.w.$ topology on a $f.w.$ set G over $\mathfrak{P}$ means any topology on for which the projection p is continuous.

Definition 1.3. [2]. A fibrewise function $\Gamma : (G , \tau_G) \longrightarrow (K , \eta)$ is called a fibrewise continuous where G and K are fibrewise topological spaces over $\mathfrak{P}$; it for each $g \in G_b$, where $b \in \mathfrak{P}$ and every open set V of $\Gamma(g)$ in K , there exists an open set U containing g in G_b such that $\Gamma(\mathrm{U}) \subseteq \mathrm{V}$.

Note. If G and K with projections p_G and p_k respectively, are $f.w.$ sets over $\mathfrak{P}$, a function $\Gamma : G \longrightarrow K$ is named $f.w.$ function if $p_k \circ \Gamma = p_G$, or $\Gamma(G_b) \subset K_b$ for every $b \in \mathfrak{P}$.

Observe that a $f.w$ function $\Gamma : G \longrightarrow K$ over $\mathfrak{P}$ limited by restriction, a $f.w.$ function $\Gamma_{\mathfrak{P}^*}: G_{\mathfrak{P}^*} \longrightarrow K_{\mathfrak{P}^*}$ over $\mathfrak{P}^*$ for every sab set $\mathfrak{P}^*$of $\mathfrak{P}$.

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Definition 1.4. [2].The $\oint.\omega$ function $\Gamma : G \longrightarrow K$, where G and K are $\oint.\omega$ topological spaces over $\mathfrak{P}$ is named:

Continuous if for every $g \in G_b$; $b \in \mathfrak{P}$, the inverse image of every open set of $\Gamma(g)$ is an open set of g .

Open if for every $g \in G_b$; $b \in \mathfrak{P}$, the direct image of every open set of g is an open set of $\Gamma(g)$.

Closed for every $g \in G_b$; $b \in \mathfrak{P}$, the direct image of every closed set of g is a closed set of $\Gamma(g)$.

Defintion 1.5. [2].The $\oint.\omega$. topological space (G , τ) over $(\mathfrak{P} , \mathcal{L})$ is named $\oint.\omega$. closed, (resp., $\oint.\omega$. open) if the projection p is closed (resp., open)

Definition 1.6. [3]. Let (G, τ_G) and $(\mathfrak{P},\mathcal{L})$ be topological space. A function $p: G \to \mathfrak{P}$ is a local homeomorphism if for every point g in G there exists an open set U containing g, such that the image is an open in $\mathfrak{P}$ and the restriction is a homeomorphism.

Definition 1.7.[2]. Let G be a space over $\mathfrak{P}$. suppose that that for each point b of $\mathfrak{P}$ and each point G_b there exists a neighboruhood V of b and a neighborhood U of G such that the projection maps U homomorphically on to V. Then G is discrete over $\mathfrak{P}$.

i.e: The condition is that the projection p_G : $(G , \tau_G) \longrightarrow (\mathfrak{P}, \mathcal{L})$ is locally a homeomorphism and open map.

Definition 1.8. [7]. Assume that we are given a topological space G, a family $\{\Gamma_s\} s\epsilon s$ of continuous functions, and a family $\{\Gamma_s\}_{s\epsilon s}$ of topological spaces where the function $\Gamma_S: G \to K_S$ that transfers $g\epsilon G$ to the point $\{\Gamma_s (g)\}\epsilon\Pi_{s\epsilon s} k_s$ is continuous, it is called the diagonal of the functions $\{\Gamma_s\}_{s\epsilon s}$ and denoted by $\gamma_{s\epsilon s}\Gamma_s \; or \Gamma_1\gamma\Gamma_2\gamma \ldots \gamma\Gamma_K$ if S = {1,2, ... , n}.

Definition 1.9. [9]. For every topological space G^* and any subspace G of G^*, the function $\phi : G \to G^*$ define by $\phi(g) = g$ is called embedding of the subspace G in the space G^* . Observe that ϕ is continuous, since ϕ^{-1} (U)=G $\cap U$, where U is open set in G^*. The embedding ϕ is closed (resp., open) iff the subspace G is closed (resp., open).

Iberwise Totally Topological Spaces

In this section we establish $\oint.\omega$. totally topological spaces. Several topological properties on this space obtained and studied.

Definition 2.1. Let $(\mathfrak{P} , \mathcal{L})$ be a topological space. The fibrewise totally topological (briefly, $\oint.\omega$.T.t.s) on a fibrewise set G over $\mathfrak{P}$ mean topological on G for which the projection p is totally continuous.

Example 2.2. Let $(\mathbb{R} , \tau_u)$ and $(\mathbb{R} , \tau_{ind})$ be a topological spaces. Define the function $p : (\mathbb{R} , \tau_u) \to (\mathbb{R} , \tau_{ind})$ such that p (x) = x $\forall$ x $\in \mathbb{R}$, then p is a totally continuous . Then $(\mathbb{R} , \tau_u)$ is. $\oint.\omega$. t. s and $\oint.\omega$. T. t. s.

Example 2.3. Let G = {1 , 2 , 3} , τ_G = {G , φ , {1} , {2 , 3}}, τ_G^c = {φ , {2 , 3} , {1} , G} , $\mathfrak{P}$ = {a , b} and $\mathcal{L}$ = {$\mathfrak{P}$, φ , {a} , {b}} . Define p:$(G , \tau_G) \to (\mathfrak{P} , \mathcal{L})$ by $p(1) = b$, $p(2) = p(3) = a$ .Then the projection function p is totally continuous. Thus (G , τ_G) is $\oint.\omega$. T. t. s.

Example 2.4. Let G = {a , b , c , d} , τ_G = {G , φ , {d} , {a,b} , {a,b ,d}} and τ_G^c = {φ , G , {a , b , c} , {c , d} , {c}} . Define the identity function p: $(G , \tau_G) \to (G , \tau_G)$ by $p(g) = g$ $\forall$ g $\in$ G , then (G , τ_G) is $\oint.\omega$. t. s.Thus p is totally continuous .Then (G , τ_G) is $\oint.\omega$.T. . t. s.

Example 2.5. Let $(\mathbb{R} , \tau_u)$, $(\mathbb{R}$, D) be two $\oint.\omega$. t. s,. Define $p : (\mathbb{R} , \tau_u) \to (\mathbb{R}$, D) ; $p(x) = x$ $\forall$ x $\in \mathbb{R}$. Then p is not continuous and not totally continuous. Then $(\mathbb{R}, \tau_u)$ is not. $\oint.\omega$. T. t. s.

Remark 2.6. Every fibrewise totally topological space is fibrewise topological space.

Proof: Clear that by Define 2.1

The convers of Remark 2.6 need not true in general .

Example 2.7. Let G = $\mathfrak{P}$ = {a , b , c} , τ_G = {G , φ , {a} , {b} , {a , b} , {a , c}} and $\mathcal{L}$ = { $\mathfrak{P}$, φ , {a} , {b , c}} . Define p : $(G , \tau_G) \to (\mathfrak{P} , \mathcal{L})$ by $p(a) = b$, $p(b) = a$, $p(c) = c$. Clearly the projection function p is continuous, then (G , τ_G) is $\oint.\omega$. t. s. But p is not totally continuous. Thus (G , τ_G) is not $\oint.\omega$. T. t. s.

Remarks 2.8.

In $\oint.\omega$. totally topology, we work over totally topological base space $(\mathfrak{P} , \mathcal{L})$, if $\mathfrak{P}$ is a point – space , the theory changes to that of ordinary topology.

A $\oint.\omega$ totally topological space over $\mathfrak{P}$ is just a topological space (G , τ_G) with a totally continuous projection p : $(G , \tau_G) \longrightarrow (\mathfrak{P} , \mathcal{L})$

The coarsest such totally topology is obtained by $\wp$, in which the clopen of (G , τ_G) is exactly the inverse image of the open set of ($\mathfrak{P}$, $\mathcal{L}$) ; called , the $\mathcal{f}.\mathcal{w}$.indiscrete totally topology.

We consider the totally topological product $\mathfrak{P}$×Z , for every topological space Z, as a $\mathcal{f}.\mathcal{w}$. totally topological space over $\mathfrak{P}$ by the first projection.

Definition 2.9. The $\boldsymbol{\mathcal{f}.\mathcal{w}}$. function $\mathbf{\Gamma}$: (G , τ_G) ⟶(K,$\boldsymbol{\eta}$) where (G , τ_G) and (K,$\boldsymbol{\eta}$) are $\boldsymbol{\mathcal{f}.\mathcal{w}}$.**T.t.s.**, over ($\boldsymbol{\mathfrak{P}}, \boldsymbol{\mathcal{L}}$) is said to be :

Totally continuous if for every point g $\boldsymbol{\epsilon G_b}$;b∈ $\boldsymbol{\mathfrak{P}}$, the inverse image of every open set $\mathbf{\Gamma}$(g) is a clopen set contain g . $\mathbf{\Gamma}$ is called totally continuous.

Totally open if for every point g ∈ G_b ; b ∈ $\mathfrak{P}$, the image of every clopen set of g is an open set of Γ(g). Γ is called totally open.

Totally closed if for every point g ∈ G_b ; b ∈ $\mathfrak{P}$, the image of every clopen set of g is a closed set of Γ(g). Γ is called totally closed.

Example 2. 10. Let G = {a , b , c } , τ_G = {G , φ , {a} , {b , c}} , K= {e, f,g} and $\eta = \{K, \varphi, \{e\}, \{f, g\}\}$ .Let $\mathfrak{P} = \{1,2,3\}, \mathcal{L} = \{\mathfrak{P}, \varphi, \{1\}, \{2,3\}\}$.Define $\wp_G: G \to \mathfrak{P}$ such that $\wp_G(a) = 1, \wp_G(b) = 2, \wp_G(c) = 3$. Define $\wp_K: K \to \mathfrak{P}$ s.t $\wp_K(e) = 1, \wp_k(f) = 2$, $\wp_k(g) = 3$. Let $\Gamma: G \to K$ $s.t$ suth that Γ(a) = e, , Γ(b) = f and Γ(c) = g . Then Γ is totally continuous, totally open and totally closed

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Example 2.11. Let G = {g_1, g_2 } , τ_G = {G , φ , {g_1} , {g_2}} , K={k_1, k_2 }and $\eta = \{K, \varphi\}$.Let $\mathfrak{P} = \{b_1, b_2\}$, $\mathcal{L} = \{\mathfrak{P}, \varphi\}$. Define $\wp_G$: G → $\mathfrak{P}$ such that$\wp_G(g_1) = b_1, \wp_G(g_2) = b_2$.Define $\wp_k$:K→ $\mathfrak{P}$ s.t $\wp_K(k_1) = b_2$,$\wp_k(k_2)$=b_1. Let $\Gamma: (G, \tau_G) \to (K, \eta)$ s.t, $\Gamma(g_1) = k_1$, $\Gamma(g_2) = k_2$.Then Γ is totally open and not totally close.

Let $\Gamma: G \to K$ be a $\mathcal{f}.\mathcal{w}$. function, G is $\mathcal{f}.\mathcal{w}$.set and K is a $\mathcal{f}.\mathcal{w}$. t.s., over $\mathfrak{P}$. We can give G the induced (resp., totally induced) topology, in the ordinary sense, and this is necessarily a $\mathcal{f}.\mathcal{w}$. t (resp., totally induced).We may refer to it, as the induced (resp., totally induced) and note following characterizations.

Proposition 2.12. Let $\mathbf{\Gamma}$: ($\mathbf{G}, \boldsymbol{\tau_G}$) ⟶($\mathbf{K}, \boldsymbol{\eta}$) be a $\boldsymbol{\mathcal{f}.\mathcal{w}}$. function, where (K , η) is a $\boldsymbol{\mathcal{f}.\mathcal{w}}$. totally topological space over ($\boldsymbol{\mathfrak{P}}$, $\boldsymbol{\mathcal{L}}$) and G has an induced $\boldsymbol{\mathcal{f}.\mathcal{w}}$. topology. Then for every $\boldsymbol{\mathcal{f}.\mathcal{w}}$. totally topological space (Z , σ) a $\boldsymbol{\mathcal{f}.\mathcal{w}}$. function Ψ : (Z , σ)⟶ (G , τ) is totally continuous iff the composition $\mathbf{\Gamma}$ ∘ Ψ : (Z , σ) ⟶ (K, η) is totally continuous.

Proof. ⟹) suppose that Ψ is totally continuous. Let z ∈ Z ; b ∈ $\boldsymbol{\mathfrak{P}}$ and let V be open set of ($\mathbf{\Gamma}$ ∘ Ψ) (z) = k ∈ K_b in K. Since $\mathbf{\Gamma}$ is totally continuous then $\mathbf{\Gamma^{-1}}$ (V) is clopen set containing Ψ(z) = g ∈ G_b in G . Since Ψ is totally continuous then $\Psi^{-1}(\mathbf{\Gamma}^{-1}(V))$ is clopen set containing z ∈ Z_b in Z and $\Psi^{-1}(\mathbf{\Gamma}^{-1}(V)) = (\mathbf{\Gamma} \circ \Psi)^{-1}$ (V) is clopen set containing z ∈ Z_b in Z .Then $\mathbf{\Gamma}$ ∘ Ψ is totally continuous .

⟸) suppose that Γ ∘ Ψ is totally continuous let z ∈ Z_b in Z ; b ∈ $\mathfrak{P}$ and U is clopen set of Ψ (z) = g ∈ G_b in G . Since Γ is open then , Γ (U) open set containing Γ(g) = Γ (Ψ (z)) = k ∈ K_b in K . since Γ∘ Ψ is totally continuous, then $(\Gamma \circ \Psi)^{-1}$ (Γ (U)) = Ψ^{-1}(U) is clopen set containing z ∈ Z_b in Z , then Ψ is totally continuous.

Proposition 2.13. Let $\mathbf{\Gamma}$: ($\mathbf{G}, \boldsymbol{\tau_G}$) ⟶($\mathbf{K}, \boldsymbol{\eta}$) be a $\boldsymbol{\mathcal{f}.\mathcal{w}}$.T. continuous function, where (K , η) is a $\boldsymbol{\mathcal{f}.\mathcal{w}}$.T. **t.s** over ($\boldsymbol{\mathfrak{P}}$, $\boldsymbol{\mathcal{L}}$) and ($\mathbf{G}, \boldsymbol{\tau_G}$) has an induced $\boldsymbol{\mathcal{f}.\mathcal{w}}$.**t**.. Then for every $\boldsymbol{\mathcal{f}.\mathcal{w}}$. T. **t.s** (Z , σ) a $\boldsymbol{\mathcal{f}.\mathcal{w}}$. function Ψ : (Z ,$\sigma$)⟶ (G , τ) is continuous iff the composition $\mathbf{\Gamma}$ ∘ Ψ : (Z , σ) ⟶ (K, η) is totally continuous.

Proof: The proof is like to previous Proposition 2.12

Proposition 2.14. Let $\mathbf{\Gamma}$: (G,$\boldsymbol{\tau_G}$) ⟶ (K,$\boldsymbol{\eta}$) be a $\boldsymbol{\mathcal{f}.\mathcal{w}}$. function where , (K , η) $\boldsymbol{\mathcal{f}.\mathcal{w}}$ totally topological space over ($\boldsymbol{\mathfrak{P}}$, $\boldsymbol{\mathcal{L}}$) and G has an induced $\boldsymbol{\mathcal{f}.\mathcal{w}}$. topology. Then for every $\boldsymbol{\mathcal{f}.\mathcal{w}}$. totally topological space (Z,σ), the surjective $\boldsymbol{\mathcal{f}.\mathcal{w}}$. function Ψ : (Z , σ) ⟶ (G , τ) is totally open iff the composition $\mathbf{\Gamma}$ ∘ Ψ : (Z , σ) ⟶ (K , η) is totally open

Proof:⟹) suppose that Ψ is totally open.Let z ∈ $\mathbf{Z_b}$; b ∈ $\boldsymbol{\mathfrak{P}}$ and U be a clopen set of Z then Ψ(U) is open set containing Ψ(z) = g ∈ G_b in G . Since **Ψ is surjective, then Γ** is open then $\mathbf{\Gamma}$ (Ψ(U)) is open set containing $\mathbf{\Gamma}$ (g) = k ∈ K_b in K . And $\mathbf{\Gamma}$ (Ψ(U)) = ($\mathbf{\Gamma}$ ∘ Ψ) (U) is open in K where U clopen in (Z , σ) , then ($\mathbf{\Gamma}$ ∘ Ψ) is totally open

⟸) suppose Γ ∘ Ψ is totally open. let z ∈ Z ; b ∈ $\mathfrak{P}$. Let U be clopen set of z in Z since Γ∘ Ψ is totally open, then Γ ∘ Ψ (U) is open set containing (Γ ∘ Ψ) (z) = k ∈ K_b in K Since Γ is continuous , then Γ^{-1}(Γ ∘ Ψ (U)) is open set of Ψ(z)= g ∈ G_b in G. But Γ^{-1} (Γ ∘ Ψ) (U) = Ψ (U) open in G, then Ψ is a totally open .

Fibrewise Totally Closed and Fibrewis Totally Open Topological Spaces

In this section we introduce the $\mathscr{f.w}$.totally closed and $\mathscr{f.w}$.totally open topological space over $\mathfrak{P}$. several topological properties on these concept are studied

Definition 3.1. The fibrewise topological (G , τ_G) over ($\mathfrak{P}$, $\mathcal{L}$) is called fibrewise totally closed (briefly, $\mathscr{f.w}.\mathbf{T}.\mathfrak{S}$) if the projection $\mathscr{p}$ is totally closed.

Example 3.2 . Let G = {1 , 2 , 3} , τ_G ={G , φ , {1} , {2 , 3}} , $\mathfrak{P}$ = {c , d , e} and $\mathcal{L}$ ={ $\mathfrak{P}$,{d} , {c , e}} . let $\mathscr{p}_G$: $(G , \tau) \longrightarrow (\mathfrak{P} , \mathcal{L})$ such that $\mathscr{p}_G$ (1) = d ,$\mathscr{p}_G$ (2) = e and $\mathscr{p}_G$ (3) = c , then $\mathscr{p}_G$ is totally closed .Then (G , τ) is $\mathscr{f.w}.\mathbf{T}.\mathfrak{S}.\mathfrak{s}$.

Example 3. 3. Let (IR , τ_u) , (IR , τ_{ind}) be topological space. Define function $\mathscr{p}$: (IR , τ_u) → (IR , τ_{ind}) ; $\mathscr{p}$(x) = x ∀ x ∈ IR . Then $\mathscr{p}$ is not totally closed since {0} is closed in (IR , τ_u) then $\mathscr{p}(\{0\})$ = {0} ∉ (IR , τ_{ind}) i.e., {0} not closed in (IR , τ_{ind}) .

Remark 3.4. Every fibrewise totally closed is fibrewise closed topological space.

Proof : Clear that by Define 3.1

The convers of Remark 3.4 need not true in general .

Example 3.5. Let G=$\{\mathbf{1},\mathbf{2},\mathbf{3}\}, \tau_G = \{\mathbf{G}, \boldsymbol{\varphi}, \{\mathbf{1}\}\}, \mathfrak{P} = \{\mathbf{a},\mathbf{b},\mathbf{c}\}$ and $\mathcal{L} = \{\mathfrak{P}, \boldsymbol{\varphi}, \{\mathbf{c}\}, \{\mathbf{b},\mathbf{c}\}\}$. Define the function $\mathscr{p}_G$: $(\mathbf{G},\tau_G) \to (\mathfrak{P},\mathcal{L}); \mathscr{p}_G(\mathbf{1}) = \mathscr{p}_G(\mathbf{2}) = \mathbf{a}, \mathscr{p}_G(\mathbf{3}) = \mathbf{b}$. Then $\mathscr{p}_G$ is closed since the family of closed in G is τ_G^c = $\{\mathbf{G},\boldsymbol{\varphi},\{\mathbf{2},\mathbf{3}\}\}$ and the family closed set in $\mathfrak{P}$ **is** $\mathcal{L}^c$=$\{\mathfrak{P},\boldsymbol{\varphi},\{\mathbf{a},\mathbf{b}\},\{\mathbf{a}\}\}$, then every closed set in G is closed set in $\mathfrak{P}$.Thus $(\mathbf{G},\tau_G)$ $\mathscr{f.w}.\mathfrak{S}.\mathfrak{t}.\mathfrak{s}$.. But $\mathscr{p}_G$ is not totally closed since $\{\mathbf{b},\mathbf{c}\}$ open set in $\mathfrak{P}$ and $\mathscr{p}_G^{-1}(\{\mathbf{b},\mathbf{c}\})$ =$\{\mathbf{3}\}$ is not open in G .Thus $(\mathbf{G},\tau_G)$ is not $\mathscr{f.w}.\mathbf{T}.\mathfrak{S}.\mathfrak{s}$.

Proposition 3.6. Let Γ: (G, τ) ⟶ (K,$\boldsymbol{\eta}$) be a $\mathfrak{S}.\mathscr{f.w}$. function, where G,**k** are $\mathscr{f.w}.\mathfrak{t}.\mathfrak{s}$ over $\mathfrak{P}$, if K is $\mathscr{f.w}.\boldsymbol{T}.\mathfrak{S}.\mathfrak{s}$., then G is $\mathscr{f.w}.\mathfrak{S}$.

Proof . Suppose Γ:(G,τ)→(K,η) be a closed $\mathscr{f.w}$. function and K is $\mathscr{f.w}.\mathbf{T}.\mathfrak{S}$.. i.e., the projection function $\mathscr{p}_k$: (k , $\boldsymbol{\eta}$) → ($\mathfrak{P}$, $\mathcal{L}$) is .T. $\mathfrak{S}$. To show that G is $\mathscr{f.w}.\mathfrak{S}$. i.e., the projection function $\mathscr{p}_G$: (G , τ) → ($\mathfrak{P}$, $\mathcal{L}$) is closed .Now, let F be a closed subset of $\mathbf{G_b}$, where b∈$\mathfrak{P}$, since Γ is T. $\mathfrak{S}$, then $\Gamma(\mathbf{F})$ is closed in subset of $\mathbf{k_b}$,where b∈$\mathfrak{P}$. Since $\mathscr{p}_K$ is T. $\mathfrak{S}$, then $\mathscr{p}_k(\Gamma(\mathbf{F}))$ is closed in ($\mathfrak{P}$, $\mathcal{L}$), but $\mathscr{p}_k(\Gamma(\mathbf{F})) = (\mathscr{p}_k \circ \Gamma)(\mathbf{F}) = \mathscr{p}_G(\mathbf{F})$ is closed in($\mathfrak{P}$, $\mathcal{L}$). Thus, $\mathscr{p}_G$ is closed and (G , τ) is $\mathscr{f.w}.\mathfrak{S}$.

Proposition 3.7. Let Γ : (G, τ) ⟶ (K,$\boldsymbol{\eta}$) be a totally closed $\mathscr{f.w}$. function, where G,$\boldsymbol{k}$ are fibrewise topological space over ($\mathfrak{P}$,$\mathcal{L}$):

(a) If K is fibrewise closed, then G is fibrewise closed . [2]

(b) If K is fibrewise totally closed, then G is fibrewise totally closed.

Proof:The proof is similar to the proof of Proposition 3.6

Proposition 3.8. Let Γ : (G,τ_G) ⟶ (K,$\boldsymbol{\eta}$) be a T. $\mathfrak{S}.\mathscr{f.w}$. function, where (G , τ_G) and (K , η) are $\mathscr{f.w}$. T.$\mathfrak{t}.\mathfrak{s}$., over ($\mathfrak{P}$, $\mathcal{L}$) Then G is $\mathscr{f.w}$. T. $\mathfrak{S}$. if K is a $\mathscr{f.w}.\mathfrak{S}$.

Proof : Assume that Γ : (G , τ_G) ⟶ (K , η) is a T. $\mathfrak{S}$., i.e., every clopen set in G is closed in K by Define 3.1.

A fiberwise function Γ and (K,η) is .$\mathscr{f.w}.\mathfrak{S}$, then the projection function $\mathscr{p}_K: (K,\eta) \to (\mathfrak{P},\mathcal{L})$ is closed . i.e., every closed set in K is closed in $\mathfrak{P}$. To prove (G,τ_G) is $\mathscr{f.w}$. T. $\mathfrak{S}$. i.e., to prove the projection function $\mathscr{p}_G$: $(G,\tau_G) \to (\mathfrak{P},\mathcal{L})$ is T. $\mathfrak{S}$. Since G,K are T.t.$\mathfrak{s}$., then ∀ open subset of K is clopen in G say F{ by define T.t }. Now , let g ∈ G_b ; b ∈ $\mathfrak{P}$ and F clopen set of g since Γ is T. $\mathfrak{S}$., then Γ (F) is closed set of Γ (g) since Γ (g) = $k \in K_b$ in K and $\mathscr{p}_K$ is closed , hence $\mathscr{p}_K$ (Γ (F)) is closed set in $\mathfrak{P}$. But ($\mathscr{p}_K \circ \Gamma$) (F) =$\mathscr{p}_G(F)$ since ($\mathscr{p}_K \circ \Gamma$) (F) is closed set in $\mathfrak{P}$, then $\mathscr{p}_G(F)$ is closed set of F . Thus , $\mathscr{p}_G$is T. $\mathfrak{S}$., then G is $\mathscr{f.w}$. T. $\mathfrak{S}$.

Proposition 3.9. Let (G,τ_G) is a $\mathscr{f.w}$.T.$\mathfrak{t}.\mathfrak{s}$. over ($\mathfrak{P}$,$\mathcal{L}$). Assume that $(\boldsymbol{G_j},\boldsymbol{\delta_j})$ is $\mathscr{f.w}.\mathfrak{S}$. for all member $(\boldsymbol{G_j},\boldsymbol{\delta_j})$ of a finite covering of (G,τ_G). Then (G,τ_G) is a $\mathscr{f.w}$.T. $\mathfrak{S}$.

Proof : Let (G,τ_G) is a $\mathscr{f.w}$. T.t.$\mathfrak{s}$. over ($\mathfrak{P}$,$\mathcal{L}$) ,then the projection function $\mathscr{p}_G: (G,\tau_G) \to (\mathfrak{P},\mathcal{L})$ exist . To show that (G,τ_G) is a $\mathscr{f.w}$.T. $\mathfrak{S}$., i.e., To show that $\mathscr{p}_G$ is T. $\mathfrak{S}$. Now, since (G_j,δ_j) is a $\mathscr{f.w}$. $\mathfrak{S}$, then the projection function $\mathscr{p}_{G_j}: (G_j,\delta_j) \to (\mathfrak{P},\mathcal{L})$ is closed for all member (G_j,δ_j) of a finite covering of (G,τ_G) . Assume that F is clopen subset of (G,τ_G) since (G,τ_G) is a $\mathscr{f.w}$. T.t.$\mathfrak{s}$. , then $\mathscr{p}_G(F) = \cup\left((G_J,\delta_J) \cap F\right)$ which a finite union of closed sets then $\mathscr{p}_G$ is T. $\mathfrak{S}$. Thus , (G,τ_G) is a $\mathscr{f.w}$.T. $\mathfrak{S}$.

Proposition 3.10. Let (G , τ_G) be a $\mathscr{f.w}$.T.$\mathfrak{t}.\mathfrak{s}$., over ($\mathfrak{P}$,$\mathcal{L}$). Then (G , τ_G) is a $\mathscr{f.w}$.T.$\mathfrak{S}$ iff for every fiber G_b , b ∈ $\mathfrak{P}$ of G and every clopen set E of G_b in G , there exists an open set O of b in $\mathfrak{P}$ such that $\mathbf{G_O} \subset \mathbf{E}$

Proof : ⇒) suppose that (G , τ_G) is a $\mathfrak{f}.\mathfrak{w}$. T.$\mathfrak{S}$ i.e. , the projection function p_G : (G , τ_G) ⟶ ($\mathfrak{P}$,$\mathcal{L}$) is T.$\mathfrak{S}$. Now , let b$\in$($\mathfrak{P}$,$\mathcal{L}$) and E be clopen set of G_b in (G , τ_G) ,G-E is clopen set in (G , τ_G) ,this implies p_G (G-E) is closed set in ($\mathfrak{P}$,$\mathcal{L}$) since the projection function is totally closed by Define 3.1 , let O= $\mathfrak{P} - p_G$ (G-E) , then O is an open set of b in ($\mathfrak{P}$,$\mathcal{L}$) and $G_O = p_G^{-1}(\mathfrak{P} - p_G(G-E))$ is a subset of E. i.e. , $G_O \subset E$

⇐) Suppose that the assumption hold and p_G : (G , τ_G) ⟶ ($\mathfrak{P}$,$\mathcal{L}$) .

To show that (G , τ_G) is $\mathfrak{f}.\mathfrak{w}$. T.$\mathfrak{S}$. Let F be clopen setin G

, the p_G (F) is closed, let b ∈ $\mathfrak{P}$ – p_G (F) is open in $\mathfrak{P}$ and every clopen set E of G_b in G. By assumption there is open O of b such that O ⊂ $\mathfrak{P}$– p_G (F). Hence , $\mathfrak{P}$ – p_G (F) is open in $\mathfrak{P}$. Hence , p_G (F) is closed in $\mathfrak{P}$ .Then the projection function p_G is T.$\mathfrak{S}$.Then (G , τ_G) is $\mathfrak{f}.\mathfrak{w}$. T.$\mathfrak{S}$. t.$\mathfrak{s}$.

Definition 3.11. The $\mathfrak{f}.\mathfrak{w}$. topological space (G , τ_G) over ($\mathfrak{P}$,$\mathcal{L}$) is called $\mathfrak{f}.\mathfrak{w}$. totally open (briefly, $\mathfrak{f}.\mathfrak{w}$. T. $\mathcal{O}$.) if the projection p is totally open.

Example 3.12. Let G = $\mathfrak{P}$ ={g_1, g_2, g_3 },τ_G = {U \ U ⊆ G }and $\mathcal{L} = \{\mathfrak{P}, \varphi, \{g_1\}\}$. Define function p_G: (G,τ_G) ⟶ ($\mathfrak{P}$, $\mathcal{L}$) such that p_G (g)= g_1 ∀ g ∈ G.Then p_G is open since ∀ U open set in G, then p_G (U)= {g_1} ∈ $\mathcal{L}$. Thus (G,τ_G) is $\mathfrak{f}.\mathfrak{w}$. open . and p_G is a T. $\mathcal{O}$. ∀ U clopen sets in G,then $p_G(U) = \{g_1\} \in \mathcal{L}$. **Thus** (G, τ_G)**is a** $\mathfrak{f}.\mathfrak{w}$. T. $\mathcal{O}$.

Example 3.13. Let (IR , τ_u) , (IR , τ_{ind}) be topological space. Define function p: (IR , τ_u) → (IR , τ_{ind}) ; p(x) = x ∀ x ∈ IR . Then p is not totally open since (0 , 1) is open subset of (IR , τ_u) then p((0 , 1)) = (0 , 1) ∉ (IR , τ_{ind}) , and $(0 , 1)^c$ = [0 , 1] in (IR , τ_u) then p([0 , 1]) = [0 , 1] ∉ (IR , τ_{ind}) .

Remark3.14. Every $\mathfrak{f}.\mathfrak{w}$. totally open is $\mathfrak{f}.\mathfrak{w}$ open topological space.

Proof : Clear that by Define 3.11

The convers of Remark 3.14 need not true in general.

Example 3.15. Let G = {g_1, g_2, g_3 }, τ_G = {G, φ, {g_1} }, $\mathfrak{P}$ = {b_1, b_2, b_3 }and $\mathcal{L} = \{\mathfrak{P}, \varphi, \{b_2\}\}$ .Define the projection function p_G: (G,τ_G) ⟶ ($\mathfrak{P}$, $\mathcal{L}$) such that p_G (g_1) = b_2, p_G (g_2) = b_3 and p_G (g_3) = b_1. the projection function p_G is open, then(G,τ_G) is $\mathfrak{f}.\mathfrak{w}.\mathcal{O}$. But the projection function p_G is not T. $\mathcal{O}$., since {g_2, g_3}closed in G,then p_G ({g_2, g_3 } = {b_1, b_3})∉ $\mathcal{L}$, i.e.; {b_1, b_3}is not open in $\mathfrak{P}$. Thus (G,τ_G) is not $\mathfrak{f}.\mathfrak{w}$. T. $\mathcal{O}$.

Proposition 3.16 let Γ : (G , τ) → (K , η) be open $\mathfrak{f}.\mathfrak{w}$. function ,and where G,K are $\mathfrak{f}.\mathfrak{w}$.T. t.$\mathfrak{s}$. over $\mathfrak{P}$.

(a): If K is $\mathfrak{f}.\mathfrak{w}.\mathcal{O}$, then G is $\mathfrak{f}.\mathfrak{w}$. $\mathcal{O}$. [2] .

(b): If If K is $\mathfrak{f}.\mathfrak{w}$., T. $\mathcal{O}$., then G is $\mathfrak{f}.\mathfrak{w}$. $\mathcal{O}$

Proof : (b)

Suppose that Γ: (G , τ_G) → (K , η) because open $\mathfrak{f}.\mathfrak{w}$. function and K is $\mathfrak{f}.\mathfrak{w}$. T. $\mathcal{O}$., i.e., the projection function p_K: (k , η) → ($\mathfrak{P}$, $\mathcal{L}$) is T. $\mathcal{O}$.To show that G is $\mathfrak{f}.\mathfrak{w}$. open i.e., the projection function p_G: (G , τ) → ($\mathfrak{P}$, $\mathcal{L}$) is open .Now, let E is open subset of k_b , since p_K is T. $\mathcal{O}$., then $p_K(\Gamma(E))$ is open in $\mathfrak{P}$, but $p_K(\Gamma(E))$ =(p_K°Γ)(E) = $p_G(E)$ is open in $\mathfrak{P}$. Thus, p_G is open and G is $\mathfrak{f}.\mathfrak{w}$ $\mathcal{O}$

Proposition 3.17. Let Γ : (G , τ_G) → (K , η) be T. $\mathcal{O}$. $\mathfrak{f}.\mathfrak{w}$. function , where (G , τ_G) and (K , η) are $\mathfrak{f}.\mathfrak{w}$. T. t.$\mathfrak{s}$. over ($\mathfrak{P}$,$\mathcal{L}$) . If (K , η) is $\mathfrak{f}.\mathfrak{w}$. $\mathcal{O}$. , then (G , τ_G) is $\mathfrak{f}.\mathfrak{w}$. T. $\mathcal{O}$.

Proof : Suppose that **Γ** : (G , τ_G) → (K , η) is a **T.** $\mathcal{O}$ $\mathfrak{f}.\mathfrak{w}$. function and (K , η) is a $\mathfrak{f}.\mathfrak{w}$. $\mathcal{O}$. t.$\mathfrak{s}$.. i.e. , then the projection function p_K : (K , η) → ($\mathfrak{P}$,$\mathcal{L}$) is open . To show that (G , τ_G) is $\mathfrak{f}.\mathfrak{w}$. T. $\mathcal{O}$. i.e. to prove the projection function p_G: (G , τ_G) → ($\mathfrak{P}$,$\mathcal{L}$) is T. $\mathcal{O}$. Now, let E b a clopen subset of G_b , b$\in$($\mathfrak{P}$,$\mathcal{L}$) . Since **Γ** is T. $\mathcal{O}$, then **Γ**(**E**) is a open subset of k_b , b$\in$($\mathfrak{P}$,$\mathcal{L}$).since p_K is open , then $p_K(\Gamma(E))$ is open subset in ($\mathfrak{P}$,$\mathcal{L}$), but($p_K \circ \Gamma$)(E) = p_G(E) is open, then p_G is T. $\mathcal{O}$. Then (G , τ_G) is $\mathfrak{f}.\mathfrak{w}$. T. $\mathcal{O}$. t.$\mathfrak{s}$.

Proposition 3.18. let **Γ**: (G , τ_G) → (K , η) be a $\mathfrak{f}.\mathfrak{w}$. function where (G , τ_G) and (K,η)are $\mathfrak{f}.\mathfrak{w}$.T.t.$\mathfrak{s}$., over ($\mathfrak{P}$,$\mathcal{L}$) . Let that the product : id_G×**Γ**: $(G, \tau_G) \times_{\mathfrak{P}} (G, \tau_G)$→$(G, \tau_G) \times_{\mathfrak{P}} (K, \eta)$. If : id_G × **Γ** is totally open and (G , τ_G) is $\mathfrak{f}.\mathfrak{w}$ T. $\mathcal{O}$. Then **Γ** it self T. $\mathcal{O}$.

Proof: Consider the following Diagram

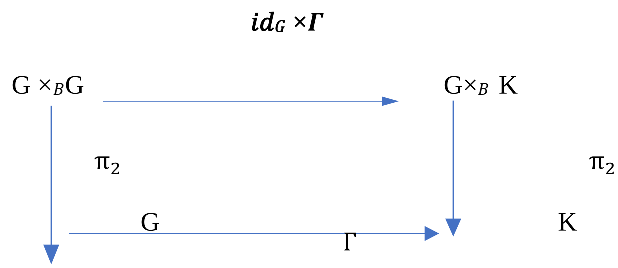

FIGURE 1. Diagram of Proposition 3.18

Proof: Let $\pi_2: G \times_{\mathfrak{P}} K \to K$ be the projection function surjective and open , since (K, η) is $f.w.\mathcal{O}.$ but the projection function $\pi_2: (G \times_{\mathfrak{P}} G, \tau_G \times_{\mathfrak{P}} \tau_G) \to (G, \tau_G)$ is T.$\mathcal{O}$. since(G, τ_G) is $f.w.$T.$\mathcal{O}$ t.s. And the product function id G $\times \Gamma: G \times_{\mathfrak{P}} G \to G \times_{\mathfrak{P}} K$ is an open. Then the composition function is T.$\mathcal{O}$. Then $\Gamma: (G, \tau) \to (K, \eta)$ is T.$\mathcal{O}$, by the Proposition (3.16) .

Proposition 3.19. (a) Let $\{\mathbf{G_i}\}$ be a finite family $\boldsymbol{f.w.\mathcal{O}}$. **s.**, over $\mathfrak{P}$.Then the $\boldsymbol{f.w.}$ **t.**, product G $=\boldsymbol{\Pi_{\mathfrak{P}} G_i}$ is also open [**2**].

(b): Let $\{G_i\}$ be a finite family of $f.w.$ T.$\mathcal{O}$., space over $\mathfrak{P}$.Then $f.w.$t . product G $=\Pi_{\mathfrak{P}} G_i$ is also T.$\mathcal{O}$.

Proof :(b)

Let $\{G_i\}$ be a finite family $f.w.$T.$\mathcal{O}$. Suppose that G $=\Pi_{\mathfrak{P}} G_i$ is a $f.w.$ t.s., $\mathfrak{P}$, then p: G $=\Pi_{\mathfrak{P}} G_i \to \mathfrak{P}$ is exist. To show that p is T.$\mathcal{O}$., . Now since $\{G_i\}$ be a finite family of $f.w.$ T.$\mathcal{O}$., then the project $p_i: G_i \to \mathfrak{P}$ is T.$\mathcal{O}$., for each i . Let E be a clopen subset of G, then $p(E) = p\left(\Pi_{\mathfrak{P}}(G_i \cap E)\right) = \Pi_{\mathfrak{P}} p_i(G_i \cap E)$ which is afinte product of open sets and hence p is T.$\mathcal{O}$. Thus the fibrewise topological product G $=\Pi_{\mathfrak{P}} G_i$ is a $f.w$ T.$\mathcal{O}$.

Proposition 3.20. Let $\Gamma : (G, \tau_G) \to (K, \eta)$ be a surjection $f.w.$ continuous where (G, τ_G) and (K, η) are $f.w.$T. t.s., over $(\mathfrak{P}, \mathcal{L})$ .Then (K, η) is $f.w.$ T.$\mathfrak{S}$. (resp., T.$\mathcal{O}$) if (G, τ_G) is $f.w$ T.$\mathfrak{S}$. (resp., T.$\mathcal{O}$.). Proof : Suppose that $\Gamma : (G, \tau_G) \to (K, \eta)$ is continuous fibrewise surjection and (G, τ_G) is $f.w.$ T.$\mathfrak{S}$. (resp., T.$\mathcal{O}$.) i.e., the projection function $p_G: (G, \tau_G) \to (\mathfrak{P}, \mathcal{L})$ is T.$\mathfrak{S}$. (resp., T.$\mathcal{O}$.). To show that (K, η) is $f.w.$ T.$\mathfrak{S}$. (resp., T.$\mathcal{O}$.) i.e., the projection $p_K: (K, \eta) \to (\mathfrak{P}, \mathcal{L})$ is T.$\mathfrak{S}$. (resp., T.$\mathcal{O}$.) . Let E be a clopen subset of K_b ;b$\epsilon\mathfrak{P}$ since Γ is continuous fibrewise surjection , then $\Gamma^{-1}(E)$ is closed (resp., open) subset of G_b ;b$\epsilon\mathfrak{P}$.Since (G, τ_G) is T.$\mathfrak{S}$. (resp., T.$\mathcal{O}$.) , then P_G is T.$\mathfrak{S}$. (resp., T.$\mathcal{O}$.) , then $p_G(\Gamma(E))$ is closed (resp., open) in $(\mathfrak{P}, \mathcal{L})$.

But $p_G(\Gamma(E)) = (p_G \circ \Gamma)(E)$

$p_G(F(E)) = p_K(E)$

Where $p_K(E)$ is closed (resp., open) in $(\mathfrak{P}, \mathcal{L})$. Then the projection function $p_K: (K, \eta) \to (\mathfrak{P}, \mathcal{L})$ is T.$\mathfrak{S}$., (resp., T.$\mathcal{O}$.) . Thus (K, η) is $f.w$ .T.$\mathfrak{S}$. (resp., T.$\mathcal{O}$.)

Proposition 3.21. If (G, τ) is a $f.w.$ t.s. over $(\mathfrak{P}, \mathcal{L})$.Also (G, τ) is $f.w.$ T.$\mathfrak{S}$. (resp., T.$\mathcal{O}$.) over $(\mathfrak{P}, \mathcal{L})$. Then$(G^*_B, \tau^*)$ is a $f.w.$T .$\mathfrak{S}$., (resp., T.$\mathcal{O}$.) for every subspace $\mathfrak{P}^*$of $\mathfrak{P}$.

Proof: Suppose that (G, τ) is a $f.w.$T. t.s.over $(\mathfrak{P}, \mathcal{L})$.Also (G, τ) is $f.w.$. T.$\mathfrak{S}$., (resp., T.$\mathcal{O}$.) .i.e., the projection function $p_G: (G, \tau) \to (\mathfrak{P}, \mathcal{L})$ is T.$\mathfrak{S}$., (resp., T.$\mathcal{O}$.).To show that $(G_{\mathfrak{P}^*}, \tau_{\mathfrak{P}^*})$ is $f.w.$ T.$\mathfrak{S}$. (resp.T.$\mathcal{O}$.) over $(\mathfrak{P}^*, \mathcal{L}^*)$ i.e., the projection finction $p_{G\mathfrak{P}^*}: (G_{\mathfrak{P}^*}, \tau_{\mathfrak{P}^*}) \to (\mathfrak{P}^*, \mathcal{L}^*)$ is T.$\mathfrak{S}$., (resp. T.$\mathcal{O}$.). Now, let E be a clopen subset of (G,τ)where $p_G(E)$ is closed (resp., open) by Define 3.1 and 3.11 , then

E$\cap G_{\mathfrak{P}^*}$ is clopen in $(G_{\mathfrak{P}^*}, \tau_{\mathfrak{P}^*})$ and

$p_{G\mathfrak{P}^*}(E \cap G_{\mathfrak{P}^*}) = p_G(E \cap G_{\mathfrak{P}^*})$

$p_{G\mathfrak{P}^*}(E \cap G_{\mathfrak{P}^*}) = p_G(E) \cap P_G(G^*_{\mathfrak{P}})$

$p_{G\mathfrak{P}^*}(E \cap G_{\mathfrak{P}^*}) = p_G(E) \cap \mathfrak{P}^*$

Then $p_G(E) \cap \mathfrak{P}^*$ is closed (resp., open) set in $\mathfrak{P}^*$, then $p_{G\mathfrak{P}^*}$ is T.$\mathfrak{S}$., (resp., T.$\mathcal{O}$.): Then $(G_{\mathfrak{P}^*}, \tau_{\mathfrak{P}^*})$ is T.$\mathfrak{S}$., (resp., T.$\mathcal{O}$.)

Proposition 3.22. Let(G, τ) be a $f.w.$t.s. over $(\mathfrak{P}, \mathcal{L})$. Also $(G_{\mathfrak{P}j}, \tau_{\mathfrak{P}j})$ is $f.w.$. T.$\mathfrak{S}$. t.s.over $(\mathfrak{P}_j, \mathcal{L}_{\mathfrak{P}j})$ for every member of a open covering of $\mathfrak{P}$. Then G is a $f.w.$ T.$\mathfrak{S}$., (resp., T.$\mathcal{O}$.) . t.s. over $(\mathfrak{P}, \mathcal{L})$.

Proof: Let (G, τ) be a $f.w.$ t.s. over $(\mathfrak{P}, \mathcal{L})$ then, the projection $p_G : (G, \tau) \to (\mathfrak{P}, \mathcal{L})$ exist.To prove that p is T.$\mathfrak{S}$. (resp., T.$\mathcal{O}$.). Since $G_{\mathfrak{P}j}$ is $f.w.$ T.$\mathfrak{S}$., (resp., T.$\mathcal{O}$.) over $\mathfrak{P}_j$ for every member open covered of $\mathfrak{P}$,then the projection $\mathcal{p}_{\mathfrak{P}j}$: $G_{\mathfrak{P}j} \to \mathfrak{P}_j$ is T.$\mathfrak{S}$., (resp T.$\mathcal{O}$.). Now, let E be clopen set of G_b, b∈$\mathfrak{P}$, $\mathcal{p}(E) = \cup\, \mathcal{p}_{\mathfrak{P}j}(E \cap G_{\mathfrak{P}j})$ which is a finite union of closed set (resp., open set) of $(\mathfrak{P}, \mathcal{L})$. Thus , $\mathcal{p}_G$ is T.$\mathfrak{S}$., (resp., T.$\mathcal{O}$.) and (G, τ) is T.$\mathfrak{S}$., (resp T.$\mathcal{O}$.) $\mathcal{f}.\mathcal{w}$. t.$\mathfrak{s}$. over $(\mathfrak{P}, \mathcal{L})$.

Fibrewise Locally Sliceable and Fibrewise Locally Section Able Totally Topological Space

In this section, we generalize $\mathcal{f}.\mathcal{w}$. locally sliceable and $\mathcal{f}.\mathcal{w}$ locally section able totally topological space over $(\mathfrak{P}, \mathcal{L})$. Some topological properties related to these concepts are studied.

Definition 4.1. The $\mathcal{f}.\mathcal{w}$.T.t.$\mathfrak{s}$., (G , τ_G) over $(\mathfrak{P},\mathcal{L})$ is called locally sliceable (briefly, $\ell.\mathbb{S}$.) if for every point $g \in G_b$, b∈ $\mathfrak{P}$, , there exist open set $\mathbb{W}$ of b and section $\mathcal{S} : \mathbb{W} \longrightarrow G_{\mathbb{W}}$ such that $\mathcal{S}$ (b) = g .

The condition implies that $\mathcal{p}$ is totally open since if U is a clopen set of g in G then $\mathcal{S}^{-1}(G_{\mathbb{W}} \cap U) \subseteq \mathcal{p}(U)$ is open set of b in $\mathbb{W}$ and thus in $\mathfrak{P}$

Example 4.2. Let $G = \{1,2,3\}, \tau_G = \{G, \varphi, \{1\}, \{2\}, \{3\}, \{1,2\}\}, \{1,3\}, \{2,3\}$ $\mathfrak{P} = \{x, y, z\}$ and $\mathcal{L} = \{\mathfrak{P}, \varphi, \{x\}, \{y, z\}\}$. The function $\mathcal{p}: (G, \tau_G) \to (\mathfrak{P}, \mathcal{L})$ such that $\mathcal{p}(1) = z$, $\mathcal{p}(2) = x$, $\mathcal{p}(3) = y$. Then $\mathcal{p}$ is totally continuous,thus (G, τ_G) is $\mathcal{f}.\mathcal{w}$.T.t.$\mathfrak{s}$.,. Let $G_x = \{2\}, G_y = \{3\}$, $G_z = \{1\}$ and let $\mathbb{W}$ open sub set of $\mathfrak{P}$ and section $\mathcal{S} : \mathbb{W} \to G_{\mathbb{W}}$ such that $\mathcal{S}(x) = 1, \mathcal{S}(y) = 2$ and $\mathcal{S}(z) = 3$. Then (G, τ_G) is $\ell.\mathbb{S}$.

Remark 4.3. Every locally sliceable fibrewis totally topological spaces are fibrewis totally open.

Proof: Clear that by Define 4.1

The converse of Remark 4.3 need not true in general .

Example 4. 4. A function $\mathcal{p} : (\mathbb{R}, I) \to (\mathbb{R}, \tau_u)$; $\mathcal{p}(x) = x\ \forall x \in \mathbb{R}$, then $\mathcal{p}$ is T.$\mathfrak{S}$.(resp., T.$\mathcal{O}$.), sinces every clopen set in $(\mathbb{R}, I)$ is a closed (resp., open) set in $(\mathbb{R}, \tau_u)$. But $\mathcal{p}$ is not totally continuous, since every open set in $(\mathbb{R}, \tau_u)$ is not a clopen set in $(\mathbb{R}, I)$. Thus $(\mathbb{R}, I)$ is not $\mathcal{f}.\mathcal{w}$.T.t.$\mathfrak{s}$., and not $\ell.\mathbb{S}$.

The class of $\ell.\mathbb{S}$.T.t.$\mathfrak{s}$., is finitely multiplicative as stated in .

Proposition 4.5. Let $\{(G_j , \tau_j)\}_{j=1}^{n}$ be a finite family of $\ell.\mathbb{S}$. $\mathcal{f}.\mathcal{w}$.T.t.$\mathfrak{s}$, over $(\mathfrak{P}, \mathcal{L})$ .The $\mathcal{f}.\mathcal{w}$.T.t., product $G = \Pi_{\mathfrak{P}}\, G_j$ is $\ell.\mathbb{S}$..

Proof : Let $g = (g_j)$ be a point of G_b , $b \in \mathfrak{P}$, so that $g_j = \pi_j(g)$ for every index j . Since G_j is $\ell.\mathbb{S}$. T. t. $\mathfrak{s}$., there is an open set $\mathbb{W}_j$ of b and section $\mathcal{S}_j : \mathbb{W}_j \longrightarrow G_j|\mathbb{W}_j$ where $\mathcal{S}_j(b) = g_j$. Then the intersection $\mathbb{W} = \mathbb{W}_1 \cap \mathbb{W}_2 \cap \ldots \cap \mathbb{W}_n$ is an open set of b and section $\mathcal{S}: \mathbb{W} \longrightarrow G_{\mathbb{W}}$ is given by $(\pi_j \circ \mathcal{S})(\mathbb{W}) = \mathcal{S}_j(\mathbb{W})$ for every index j and every point $w \in \mathbb{W}$, then (G , τ_G) is $\ell.\mathbb{S}$. T. t. $\mathfrak{s}$.

Proposition 4.6. Let $\Gamma : (G , \tau_G) \longrightarrow (K , \eta)$ be a continuous $\mathcal{f}.\mathcal{w}$.surjection, where (G , τ_G) and (K ,η) are $\mathcal{f}.\mathcal{w}$.T.t.$\mathfrak{s}$., over $(\mathfrak{P},\mathcal{L})$. If (G , τ) is $\ell.\mathbb{S}$., then (K , η) is $\ell.\mathbb{S}$.

Proof : Let $k \in K_b$; $b \in \mathfrak{P}$. Then $k = \mathcal{p}(g)$, for some $g \in G_b$, If G is $\ell.\mathbb{S}$., then there exists an open set $\mathbb{W}$ of b and a section $\mathcal{S} : \mathbb{W} \longrightarrow G_{\mathbb{W}}$ such that $\mathcal{S}$ (b) =g. Then $\Gamma \circ \mathcal{S} : \mathbb{W} \longrightarrow K_{\mathbb{W}}$ is a section such that $\mathcal{S}$ (b) = k . Then (K , η) is $\ell.\mathbb{S}$.

Definition 4.7. Let $\mathcal{f}.\mathcal{w}$.T.t.$\mathfrak{s}$., (G , τ) over $(\mathfrak{P},\mathcal{L})$ is called $\mathcal{f}.\mathcal{w}$., discrete (briefly, $\mathcal{f}.\mathcal{w}.\mathfrak{D}$.) if the projection $\mathcal{p}$ is totally local homeomorphism.

i.e: The projection : $G \longrightarrow \mathfrak{P}$ is totally local a homeomorphism and totally open map.

Example 4.8. Let $G = \{g_1, g_2\}, \tau_G = \{G, \varphi, \{g_1\}\}, \tau_G^c = \{\varphi, G, \{g_2\}\}, \mathfrak{P} = \{1,2\}$ and $\mathcal{L} = \{\mathfrak{P}, \varphi, \{1\}\}$. Define the function $\mathcal{p}: (G, \tau_G) \to (\mathfrak{P}, \mathcal{L})$ where $\mathcal{p}(g_1) = 1, \mathcal{p}(g_2) = 2$· We have $G_1 = \{g_1\}, G_2 = \{g_2\}$ · Let $\mathcal{S}_1: \{1\} \to \{g_1\}$ such that $\mathcal{S}_1(1) = g_1, \mathcal{S}_2: \{2\} \to \{g_2\}$ such that $\mathcal{S}_2(2) = g_2$. Then $\mathcal{p}$ is totally locally homomorphism and thus (G, τ_G) is $\mathcal{f}.\mathcal{w}.\mathfrak{D}$. t. $\mathfrak{s}$.

Remark 4.9. Let (G , τ) be the $\mathcal{f}.\mathcal{w}$.T.t.$\mathfrak{s}$., over $(\mathfrak{P},\mathcal{L})$. If (G , τ) is the $\mathcal{f}.\mathcal{w}.\mathfrak{D}$.T.t.$\mathfrak{s}$., then (G , τ) is locally sliceable and totally open

Proof: Clear that forever point b of $\mathfrak{P}$ and every g of G_b there is clopen set U of g in G and open set $\mathbb{W}$ of b in $\mathfrak{P}$ where $\mathcal{p}$ maps U homomorphically onto of $\mathbb{W}$ where $\mathbb{W}$ is every covered by U. Then the $\mathcal{f}.\mathcal{w}.\mathfrak{D}$.T.t.$\mathfrak{s}$., are locally sliceable there for is $\mathcal{f}.\mathcal{w}$.totally open.

The class of $\mathcal{f}.\mathcal{w}.\mathfrak{D}$.T.t.$\mathfrak{s}$., are finitely multiplicative .

Proposition 4.10. Let $\{(G_j , \tau_j)\}_{j=1}^{n}$ be afinity of $\mathcal{f}.\mathcal{w}.\mathfrak{D}$.T.t.$\mathfrak{s}$., over $(\mathfrak{P},\mathcal{L})$. Then the $\mathcal{f}.\mathcal{w}$.T.t.$\mathfrak{s}$., product $G = \Pi_{\mathfrak{P}}\, G_j\, \tau_j$ is $\mathcal{f}.\mathcal{w}$. discrete .

Proof :Let g a point in G where $g \in G_b$; $b \in \mathfrak{P}$, then there is for every index j a clopen set U_j of π_j (g) in G_j , where the projection $\mathcal{p}_j = \mathcal{p}_o\, \pi_j$ maps U_j homomorphically onto the open $\mathcal{p}_j(U_j) = \mathbb{W}_j$ of b . Then , the clopen $\Pi_{\mathfrak{P}}$

U_j of g is mapped homomorphically onto the intersection $\mathbb{W} = \cap\ \mathbb{W}_j$ which is open of b. Then $G = (\Pi_{\mathfrak{P}} G_j, \tau_j)$ is the 𝒻.𝓌.𝔇.T.𝔱.𝔰.

Proposition 4.11. Let Γ : (G , τ) ⟶ (K , η) be a function over 𝔓, where (G , τ) is the 𝒻.𝓌.𝔇.T.𝔱.𝔰., over (𝔓,ℒ)and (K , η) is totally open over (𝔓,ℒ).Then Γ is totally continuous.

Proof:Let k ∈ K be open set in K and let g be a point of $\Gamma^{-1}(K)$. Then there exist U set clopen in G i.e. ,U is open and closed of g , then U is neighboruhood of g and a neighboruhood V of 𝓅 (g) by define (4.7).

There for U ∩ $𝓅^{-1}(\mathbb{W})$ is a neighboruhood of g contained in $\Gamma^{-1}(\mathbb{W})$. Thus Γ is totally continuous.

Proposition 4.12. If (G , τ) is 𝒻.𝓌.T.𝔱.𝔰., over (𝔓,ℒ), then (G , τ) is 𝒻.𝓌.𝔇., iff (G , τ) is 𝒻.𝓌.T.𝒪. and the diagonal embedding

$\gamma : G \longrightarrow G \times_{\mathfrak{P}} G$ is totally open.

Proof :⟹) suppose that (G , τ) is 𝒻.𝓌.𝔇., then (G , τ) is a 𝒻.𝓌.T., open {by remark (4.9). To prove that γ is totally open, i.e., to show that γ(G) is open in $G \times_{\mathfrak{P}} G$. So, let g ∈ G_b ; b ∈ 𝔓 , and let E be a clopen set of g in G , where 𝕎 = 𝓅 (E) is open set of b in 𝔓 and 𝓅 maps E totally homomorphically onto 𝕎 . Then , $E \times_{\mathfrak{P}} E$ is contained in γ (G) since if not , then there exist distinct $e, e^* \in G_W$, where w ∈ 𝕎 and e, e^* ∈ E contradiction. Then γ(G) is open set , hence γ is totally open.

⟸) Suppose that (G , τ) is 𝒻.𝓌. totally open and The diagonal embedding $\gamma : G \longrightarrow G \times_{\mathfrak{P}} G$ let g ∈ G_b ; b ∈ 𝔓 , then γ(g) = (g , g) such that τ × τ clopen set in $G \times_{\mathfrak{P}} G$ which is contained in γ(G). we claim τ × τ clopen set is of the from $E \times_{\mathfrak{P}} E$, where E is a clopen set of g in G . Then 𝓅|E is totally homeomorphism. Therefore, (G , τ) is 𝒻.𝓌.𝔇.T.𝔱.𝔰.

Open subset of 𝒻.𝓌.𝔇.T.𝔱.𝔰., are also 𝒻.𝓌.discrete .In fact we have

Proposition 4.13. A function Γ : (G , τ) ⟶ (K , η) is a totally continuous 𝒻.𝓌.,.injection where (G , τ) and (K , η) are 𝒻.𝓌.T.𝒪 𝔱.𝔰., over (𝔓,ℒ). If (K , η) is 𝒻.𝓌.𝔇., then (G , τ) is so.

Proof: Consider the diagram shown below

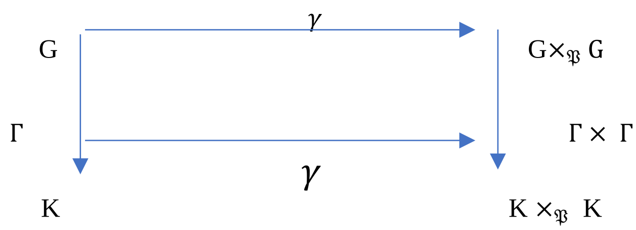

FIGURE 2. Diagram of Proposition 4.13

Since Γ is totally continuous then Γ × Γ is totally continuous . Now γ(K) is η × η totally open in $K \times_{\mathfrak{P}} K$, by Proposition (4.9), since K is a 𝒻.𝓌.𝔇., so $\gamma(G) = \gamma(\Gamma^{-1}(k)) = (\Gamma \times \Gamma)^{-1}(\gamma(k))$ is a τ × τ clopen in $G \times_{\mathfrak{P}} G$. Thus , the conclusion follows from Proposition (4.11). Then $\gamma : G \longrightarrow G \times_{\mathfrak{P}} G$ is totally open.

Proposition 4.14. Let Γ : (G , τ) ⟶ (K , η) be an a T.𝒪.𝒻.𝓌., surjection function, where (G , τ) and (K , η) are 𝒻.𝓌.𝒪.T 𝔱.𝔰., over (𝔓,ℒ). If G is a 𝒻.𝓌.𝔇., then K is 𝒻.𝓌.𝔇.

Proof : From figure (4.1) , with , if G is a 𝒻.𝓌.𝔇., then Δ(G) is an τ × τ totally open in $G \times_{\mathfrak{P}} G$, by proposition (4.12). Hance

$\gamma(K) = \gamma(\Gamma(G))$

$\gamma(K) = (\Gamma \times \Gamma)(\gamma(G))$

Then γ(K) is an η × η totally open in $K \times_{\mathfrak{P}} K$. Thus the conclusion follows again Proposition (4.12).

Proposition 4.15. If ℰ : (G , τ) ⟶ (K , η) and Γ : (G , τ) ⟶ (K , η) are totally continuous 𝒻.𝓌.function , where (G , τ) is a 𝒻.𝓌.T.𝔱., and (K , η) is a 𝒻.𝓌.𝔇.T 𝔱.𝔰., over (𝔓,ℒ). Then the coincidence set K (ℰ , Γ) of ℰ and Γ is clopen G .

Proof: The coincidence set is precisely $\gamma^{-1}(\mathcal{E} \times \Gamma)^{-1}(\gamma(K))$, where

$$G \xrightarrow{\gamma} G \times_{\mathfrak{P}} G \xrightarrow{\mathcal{E} \times \Gamma} K \times_{\mathfrak{P}} K \xleftarrow{\gamma} K$$

FIGURE 3. Diagram of Proposition 4.15

Then the result by proposition (4.12). Such that , take K , $\mathcal{E} = id_G$, and Γ = 𝒮op where 𝒮 is a section. We conclude that 𝒮 is an totally open embedding when G is a 𝒻.𝓌.𝔇 T 𝔱.𝔰.

Proposition 4.16. If Γ : (G , τ) ⟶ (k , η) is a continuous $\mathscr{f}.\mathscr{w}.$ functions , where (G , τ) is $\mathscr{f}.\mathscr{w}.$T $\mathcal{O}$., and (K , η) is a $\mathscr{f}.\mathscr{w}.\mathfrak{D}$.T.t.𝔰., over (𝔓,$\mathcal{L}$). Then, the $\mathscr{f}.\mathscr{w}.$, graph $\mathcal{E}$: (G , τ) ⟶ (G , τ) $\times_{\mathfrak{P}}$ (K , η) of Γ is an totally open embedding.

Proof: The $\mathscr{f}.\mathscr{w}.$ graph is defined in the same way as the ordinary graph , but with values in the $\mathscr{f}.\mathscr{w}.$T.t., product. Therefore , the diagram shown below is commutative .

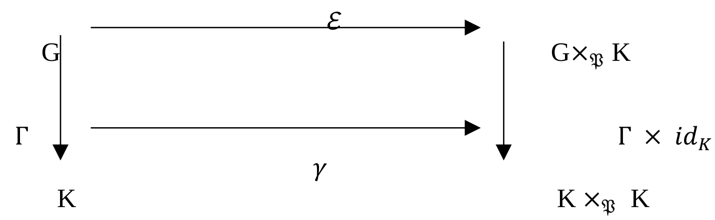

FIGURE 3. Diagram of Proposition 4.16

Since γ(K) is an η × η – totally open in K $\times_{\mathfrak{P}}$ K , by proposition (4.11) , $\mathcal{E}$ (G) = $(\Gamma \times \mathrm{id}_K)^{-1}\left(\gamma\,(K)\right)$ is an τ × η – totally open in G $\times_{\mathfrak{P}}$ K , then $\mathcal{E}$ is totally open embedding.

Proposition 4.17. If (G , τ) is $\mathscr{f}.\mathscr{w}.\mathfrak{D}$.T.t.𝔰.,over (𝔓,$\mathcal{L}$), then for every point g ∈ $G_{\mathfrak{P}}$; b ∈ 𝔓, there is an open set 𝕨 of b such that a unique section $\mathcal{S}$: 𝕨 ⟶ $G_{𝕨}$ exist satisfying $\mathcal{S}$ (b) = g .We may refer to $\mathcal{S}$ as the section through g .

Definition 4.18. The $\mathscr{f}.\mathscr{w}.$T.t.𝔰., (G , τ) over (𝔓,$\mathcal{L}$)is called locally section able (briefly,$\ell.\delta.$) if every point b ∈ 𝔓 , admitsa open set 𝕨 and a section $\mathcal{S}$: 𝕨 ⟶ $G_{𝕨}$.

Example 4.19. Let G = { g_1 , g_2 } , $\tau_G = \{G , \varphi , \{g_1\}\}$. let 𝔓 , = {3 , 4} , $\mathcal{L} = \{\mathfrak{P} ,, \varphi , \{3\}\}$. let $\wp$: (G , τ_G) ⟶ (𝔓 , $\mathcal{L}$) . where $\wp(g_1)$ = 3, $\wp(g_2)$ = 4 . We have $G_3 = \{g_1\}$, $G_4 = \{g_2\}$. Let $\mathcal{S}_1 : \{3\} \longrightarrow \{g_1\}$ where $\mathcal{S}_1(3) = g_1$, $\mathcal{S}_2$: $\{4\} \longrightarrow \{g_2\}$ where $\mathcal{S}_2(4) = g_2$. then (G , τ_G) is $\ell.\delta.$

Remark 4.20. The $\mathscr{f}.\mathscr{w}.$ non – empty locally sliceable totally topological spaces are locally section.

Proof: Clear that by define (4.1) and by define (4.18)

The converse of Remark 4.20 in not true, since the locally section able totally topological are not necessarily fibrewise open.

Example 4.21. G = (-1 , 1] ⊂ ℝ with (G , τ) , the natural projection onto 𝔓 = ℝ | ℤ ; (𝔓,$\mathcal{L}$) then $\wp$: G ⟶ 𝔓 is not totally open , hence (G , τ) is ont $\mathscr{f}.\mathscr{w}.$T.$\mathcal{O}$., then (G , τ) is $\ell.\delta.$ but not $\ell.\mathbb{S}.$

The class of locally section able totally topological space is finitely multiplicative as we show next.

Proposition 4.22. Let { (G_j τ_j)} be a finite family of locally section able totally topological space over (𝔓,$\mathcal{L}$) .Then the $\mathscr{f}.\mathscr{w}.$T.t.𝔰.,product G = $\Pi_{\mathfrak{P}}\, G_j$ is a locally section able.

Proof: Given a point b of 𝔓 , there exist an copen set $\mathbb{W}_j$ of b and a section $\mathcal{S}_j : \mathbb{W}_j \longrightarrow G_j \mid \mathbb{W}_j$ for every index j . Since there are only a finite number of indices , the intersection W of the open sets $\mathbb{W}_j$ is also an open set of b , and a section $\mathcal{S} : \mathbb{W} \longrightarrow (\Pi_{\mathfrak{P}} G_j)_{\mathbb{W}}$ is given by $\pi_j \circ \mathcal{S}(\mathbb{W}) = \mathcal{S}_j\,(\mathbb{W})$, for w ∈ 𝕎 , then G = $\Pi_{\mathfrak{P}}\, G_j$ is a locally section able.

Our last two results apply equally well to each of above three properties.

Proposition 4.23. Let (G , τ) be a $\mathscr{f}.\mathscr{w}.$T.t.𝔰., over (𝔓,$\mathcal{L}$). Suppose that (G , τ) is locally slice able , $\mathscr{f}.\mathscr{w}.$ discrete or locally section able over (𝔓,$\mathcal{L}$). Then so is $G_{\mathfrak{P}^*}$ over $\mathfrak{P}^*$ for every open set $\mathfrak{P}^*$ of 𝔓

Proof: clear that

Proposition 4.24. Let (G , τ) be a $\mathscr{f}.\mathscr{w}.$T.t.𝔰., over (𝔓,$\mathcal{L}$). Assume that $G_{\mathfrak{P}_j}$ is a locally sliceable $\mathscr{f}.\mathscr{w}.\mathfrak{D}.$, or locally section able over $\mathfrak{P}_j$ for every member $\mathfrak{P}_j$ of an open covering of 𝔓. Than so is G over 𝔓 .

Proof: Clear that .

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