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A CERTAIN SUBCLASS OF MULTIVALENT HARMONIC FUNCTIONS DEFINED BY RUSCHEWEYH DERIVATIVES

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A CERTAIN SUBCLASS OF MULTIVALENT HARMONIC FUNCTIONS DEFINED BY RUSCHEWEYH DERIVATIVES

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Abstract. We introduce a new class of harmonici multivalent functions define by generalized Rucheweyh derivative operator .We also obtain several interesting propertiesi such as sharp coefficienit estimates , distortioni bound , extreme points , Hadamardi product and other several results. Derivative; extreme points.

Keywords: Multivalent Functions, Harmonici Functions Defined By Ruscheweyh Derivatives

1. Introduction

continuous function $f = u + iv$ is complex-valued harmonic function in a domain $D \subset \mathbb{C}$ if both u and v are real harmonic in D , we can write $f = h + \bar{g}$, where h and g analytic in D .We call h and g analytic part and analytic part of f. *The* Harmonic functions $f = h + \bar{g}$ that are multivalent and sense preserving in the open disk $U = \{s : |s| < 1\}$ .So $f = h + \bar{g}$ is sense preserving and locally one-to-one in D if $|h'(s)| > |g'(s)|$ in D. See Clunie and Shell-Small [2](see also[4],[3]and[1]).

Let $\mathcal{H}$ be a class of all harmonic functions $f = h + \bar{g}$ that are multivalent and sense preserving in the open disk U where

$$h(s) = s^w + \sum_{l=w+1}^{\infty} a_n s^l \ , \ \ g(s) = \sum_{l=w}^{\infty} b_n s^l \qquad (1)$$

Normalized by normalized by $f\ (0) = h(0) = f_s(0) - 1 = 0$ with $f_Z(0)$ denotes partial derivative of f(s) at $s = 0$ and we call h and g analytic part and co-analytic part of f respectively.

Now, we introduce a new class $AJ_{\mathcal{H}}(\lambda, \alpha, k, \gamma)$ of function $f \in \mathcal{H}$ where h and g of the form

$$h(s) = s^w - \sum_{l=w+1}^{\infty} a_n s^l \ , \ g(s) = -\sum_{l=w}^{\infty} b_n s^l \qquad (2)$$

Satisfying

$$\mathcal{R}e\left\{(1+ke^{i\gamma})\frac{\left(s^{q+1}\left(D^{\lambda+w-1}h(s)\right)^{q+1}\right)+\overline{\left(s^{q+1}\left(D^{\lambda+w-1}g(s)\right)^{q+1}\right)}+\overline{\left(2s^{q}\left(D^{\lambda+w-1}g(s)\right)^{q}\right)}}{s^{q}\left(D^{\lambda+w-1}f(s)\right)^{q}-\overline{z^{q}\left(D^{\lambda+w-1}g(s)\right)^{q}}}\right\} \geq \alpha \qquad (3)$$

Where

Where $s = re^{i\theta}, \gamma, \xi, \alpha$ and θ are real such that $0 \leq \gamma < 1, 0 \leq \alpha < 1, 0 \leq \xi < 1, \alpha < k \leq 1, 0 \leq r < 1.$ and $D^{\lambda+w-1}f(s)$ is the Ruscheweyh derivative of f and is defined by

$$D^{\lambda+w-1}f(s) = \frac{s^w}{(1-s)^{\lambda+w}} * f(s)$$

$$= s^w + \sum_{l=w+1}^{\infty} B_n(\lambda) a_n s^l, \lambda > -w$$

$$B_n(\lambda) = \frac{\Gamma(\lambda+l)}{\Gamma(\lambda+w)(l-w)!}$$

Also

$$D^{\lambda+w-1}f(s) = D^{\lambda+w-1}h(s) + \overline{D^{\lambda+w-1}g(s)} \qquad (4)$$

Where

$\left(D^{\lambda+w-1}h(s)\right)^q = \frac{w!}{(w-q)!}s^w + \sum_{l=w+1}^{\infty} \frac{l!}{(l-q)!} B_l(\lambda) a_l s^l$

$\left(D^{\lambda+w-1}g(s)\right)^q = \sum_{l=w}^{\infty} \frac{l!}{(l-q)!} B_l(\lambda) b_l s^l$

Further, let $\overline{\mathcal{H}}$ be the subfamily of $\mathcal{H}$ consisting of harmonic functions $f = h + \bar{g}$ where $h(s) = s^w - \sum_{l=w+1}^{\infty} |a_l| s^l$ And $g(s) = -\sum_{l=w}^{\infty} |b_l| s^l, a_l \geq 0, b_l \geq 0$ (5)

Let $\overline{AJ_{\mathcal{H}}}(\lambda, \alpha, k, \gamma)$ be the subclass of functions $f \in \overline{\mathcal{H}}$. To attempt the various properties of Harmonic convex functions of from $f = h + \bar{g}$ and (5), we need the following sufficient condition studied by J.M.Jahangiri[3]

2. Main Results

In the first theorem, we introduce a sufficient coefficient bound for harmonic functions in the class $AJ_{\mathcal{H}}(\lambda, \alpha, k, \gamma)$.

THEOREM 1:

Let $f = h + \bar{g} \in \mathcal{H}$ with

$$\sum_{l=w+1}^{\infty}\left(\frac{\frac{l!}{(l-q)!}\{(l-q)(k+1)+(1-\alpha)\}}{\frac{w!}{(w-q)!}(1-\alpha)}|a_l| - \sum_{l=w}^{\infty}\frac{\frac{l!}{(l-q)!}\{(l-q)(k+1)+(1+2k+\alpha)\}}{\frac{w!}{(w-q)!}(1-\alpha)}|b_l|\right)B_l(\lambda) \leq 1$$

Where $a_1 = 1, 0 \leq \alpha < 1, \lambda > -1, 0 \leq k < \infty$. Then f is harmonic w −valent In U and $f \in \overline{AJ_{\mathcal{H}}}(\lambda, \alpha, k, \gamma)$

PROOF:

We have

$$\mathcal{R}e\left\{\frac{(1+ke^{i\gamma})\left\{\begin{matrix}\left(s^{q+1}\left(D^{\lambda+w-1}h(s)\right)^{q+1}\right)+\overline{\left(s^{q+1}\left(D^{\lambda+w-1}g(s)\right)^{q+1}\right)}\\+\overline{\left(2s^{q}\left(D^{\lambda+w-1}g(s)\right)^{q}\right)}\end{matrix}\right\}+\left\{\begin{matrix}s^{q}\left(D^{\lambda+w-1}f(s)\right)^{q}\\-\overline{s^{q}\left(D^{\lambda+w-1}g(s)\right)^{q}}\end{matrix}\right\}}{s^{q}\left(D^{\lambda+w-1}h(s)\right)^{q}-\overline{s^{q}\left(D^{\lambda+w-1}g(s)\right)^{q}}}\right\}$$

$$=\mathcal{R}e\left(\frac{A(s)}{B(s)}\right)\geq\alpha$$

Where$a_1 = 1, 0 \leq \alpha < 1, \lambda > -1, \alpha < k \leq 1$. Then f is harmonic multivalent in U and $f \in AJ_{\mathcal{H}}$ $(\lambda, \alpha, k, \gamma)$.

Here, we let

$$A(\lambda,s)=(1+ke^{i\gamma})\left\{s^{q+1}\left(D^{\lambda+w-1}h(s)\right)^{q+1}+\overline{s^{q+1}\left(D^{\lambda+w-1}g(s)\right)^{q+1}}+\overline{2s^{q}\left(D^{\lambda+w-1}g(s)\right)^{q}}\right\}+\left(s^{q}\left(D^{\lambda+w-1}h(\mathcal{Z})\right)^{q}-\overline{s^{q}\left(D^{\lambda+w-1}g(s)\right)^{q}}\right)$$

And

$$B(\lambda,\mathcal{Z})=s^{q}\left(D^{\lambda+w-1}f(s)\right)^{q}-\overline{s^{q}\left(D^{\lambda+w-1}g(s)\right)^{q}}$$

We want to show that

$$|A(\lambda,s)+(1-\alpha)B(\lambda,s)|-|A(\lambda,s)-(1+\alpha)B(\lambda,s)|\geq 0.$$

But

$$|A(\lambda,\mathcal{Z})+(1-\alpha)B(\lambda,s)|-|A(\lambda,s)-(1+\alpha)B(\lambda,s)|$$

$$=\left|(1+ke^{i\gamma})\left\{\left(s^{q+1}\left(D^{\lambda+w-1}h(\mathcal{Z})\right)^{q+1}\right)+\overline{s^{q+1}\left(D^{\lambda+w-1}g(s)\right)^{q+1}}+\overline{2s^{q}\left(D^{\lambda+w-1}g(s)\right)^{q}}\right\}+\left(s^{q}\left(D^{\lambda+w-1}h(\mathcal{Z})\right)^{q}-\overline{s^{q}\left(D^{\lambda+w-1}g(s)\right)^{q}}\right)+(1-\alpha)\left(s^{q}\left(D^{\lambda+w-1}h(s)\right)^{q}-\overline{s^{q}\left(D^{\lambda+w-1}g(s)\right)^{q}}\right)\right|-\left|(1+ke^{i\gamma})\left\{s^{q+1}\left(D^{\lambda+w-1}h(s)\right)^{q+1}+\overline{s^{q+1}\left(D^{\lambda+w-1}g(\mathcal{Z})\right)^{q+1}}+\overline{2s^{q}\left(D^{\lambda+w-1}g(s)\right)^{q}}\right\}-(1+\alpha)\left(s^{q}\left(D^{\lambda+w-1}h(s)\right)^{q}-\overline{s^{q}\left(D^{\lambda+w-1}g(s)\right)^{q}}\right)\right|$$

$$=\left|(1+ke^{i\gamma})\left(s^{q+1}\left(D^{\lambda+w-1}h(s)\right)^{q+1}\right)+(2-\alpha)\left(s^{q}\left(D^{\lambda+w-1}h(\mathcal{Z})\right)^{q}\right)+(1+ke^{i\gamma})\left(\overline{\mathcal{Z}^{q+1}\left(D^{\lambda+w-1}g(s)\right)^{q+1}}\right)+(2ke^{i\gamma}+\alpha)\left(\overline{s^{q}\left(D^{\lambda+w-1}g(s)\right)^{q}}\right)\right|-\left|(1+ke^{i\gamma})\left(s^{q+1}\left(D^{\lambda+w-1}h(s)\right)^{q+1}\right)+(-\alpha)\left(\mathcal{Z}^{q}\left(D^{\lambda+w-1}h(s)\right)^{q}\right)+(1+ke^{i\gamma})\left(\overline{s^{q+1}\left(D^{\lambda+w-1}g(s)\right)^{q+1}}\right)+(2+2ke^{i\gamma}+\alpha)\left(\overline{s^{q}\left(D^{\lambda+w-1}g(\mathcal{Z})\right)^{q}}\right)\right|$$

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$$= \Bigg|\left(1+ke^{i\gamma}\right)\Bigg(\left(\frac{w!}{(w-q-1)!}s^{w}+\sum_{l=w+1}^{\infty}\frac{l!}{(l-q-1)!}B_{l}(\lambda)a_{l}s^{l}\right)$$
$$+(2-\alpha)\left(\frac{w!}{(w-q)!}s^{w}+\sum_{l=w+1}^{\infty}\frac{l!}{(l-q)!}B_{l}(\lambda)a_{l}s^{l}\right)$$
$$+\left(1+ke^{i\gamma}\right)\left(\sum_{l=w}^{\infty}\frac{l!}{(l-q-1)!}B_{l}(\lambda)b_{l}s^{l}\right)$$
$$+\left(2ke^{i\gamma}+\alpha\right)\left(\sum_{n=w}^{\infty}\frac{n!}{(n-q)!}B_{l}(\lambda)b_{l}s^{l}\right)\Bigg)\Bigg|$$
$$-\Bigg|\left(1+ke^{i\gamma}\right)\Bigg(\left(\frac{w!}{(w-q-1)!}s^{p}+\sum_{l=w+1}^{\infty}\frac{l!}{(l-q-1)!}B_{l}(\lambda)a_{l}s^{l}\right)$$
$$+(-\alpha)\left(\frac{w!}{(w-q)!}s^{p}+\sum_{l=w+1}^{\infty}\frac{l!}{(l-q)!}B_{l}(\lambda)a_{l}s^{l}\right)$$
$$+\left(1+ke^{i\gamma}\right)\left(\overline{\sum_{l=p}^{\infty}\frac{l!}{(l-q-1)!}B_{l}(\lambda)b_{l}s^{l}}\right)$$
$$+\left(2+2ke^{i\gamma}+\alpha\right)\left(\overline{\sum_{l=w}^{\infty}\frac{l!}{(l-q)!}B_{l}(\lambda)b_{l}s^{l}}\right)\Bigg)\Bigg|$$

$\geq \Big|\frac{w!}{(w-q)!}\left(\left(1+ke^{i\gamma}\right)(w-q)+(2-\alpha)\right)|s|^{w}-\sum_{l=w+1}^{\infty}\frac{l!}{(l-q)!}\Big(\left(l-q\right)\left(1+ke^{i\gamma}\right)+ (2-\alpha)\Big)B_{l}(\lambda)a_{l}s^{l}-\sum_{l=w}^{\infty}\frac{l!}{(l-q)!}\left((l-q)\left(1+ke^{i\gamma}\right)+\left(2ke^{i\gamma}+\alpha\right)\right)B_{l}(\lambda)b_{l}s^{l}\Big|- \Big|\frac{p!}{(p-q)!}\left(\left(1+ke^{i\gamma}\right)(p-q)+\alpha\right)s^{p}-\sum_{n=w+1}^{\infty}\frac{l!}{(l-q)!}\left((l-q)\left(1+ke^{i\gamma}\right)+(-\alpha)\right)B_{l}(\lambda)a_{l}s^{l}- \sum_{l=w}^{\infty}\frac{l!}{(l-q)!}\left((l-q)\left(1+ke^{i\gamma}\right)+\left(2+2ke^{i\gamma}+\alpha\right)\right)B_{l}(\lambda)b_{l}s^{l}\Big|$

$$\geq \frac{2w!}{(p-q)!}(1-\alpha)|s|^{p}\left(1-\sum_{l=w+1}^{\infty}\frac{\frac{l!}{(l-q)!}\left((l-q)(1+k)+(1-\alpha)\right)}{\frac{w!}{(w-q)!}(1-\alpha)}B_{l}(\lambda)|a_{l}||s|^{l-p}\right.$$
$$\left.-\sum_{l=w}^{\infty}\frac{\frac{l!}{(l-q)!}\left((l-q)(1+k)+(1+2k+\alpha)\right)}{\frac{w!}{(w-q)!}(1-\alpha)}B_{l}(\lambda)|b_{l}||s|^{l-w}\right)\geq 0$$

This completes the proof of theorem.
The harmonic function

$$f(s) = s^w + \sum_{l=w+1}^{\infty} \frac{\frac{w!}{(w-q)!}(1-\alpha)}{\frac{l!}{(l-q)!}\big((l-q)(1+k)+(1-\alpha)\big)} T_l s^l + \sum_{l=w}^{\infty} \frac{\frac{w!}{(w-q)!}(1-\alpha)}{\frac{l!}{(l-q)!}\big((l-q)(1+k)+(1+2k+\alpha)\big)} \bar{S}_l \bar{s}^l$$

Where$\sum_{l=2}^{\infty}|T_l| + \sum_{l=1}^{\infty}|S_l| = 1$.
Putting q=1, in the above theorem, to obtain
COROLLARY 1:
Let $f = h + \bar{g} \in \mathcal{H}$ with

$$\sum_{l=2}^{\infty} \frac{l(l-\alpha)}{(1-\alpha)}|a_l| + \sum_{l=1}^{\infty} \frac{l(l+\alpha)}{(1-\alpha)}|b_l| \le \sum_{l=2}^{\infty}\left(\frac{l\{l(k+1)-k-\alpha\}}{(1-\alpha)}|a_l| + \frac{l\{l(k+1)+k+\alpha\}}{(1-\alpha)}|b_l|\right) B_l(\lambda)$$

Where $a_1 = 1$, $0 \le \alpha < 1$, $\lambda > -1$, $0 \le k < \infty$. Then f is harmonic p-valent in U and $f \in \overline{AJ_{\mathcal{H}}}(\lambda, \alpha, k, \gamma)$.
We note that the result obtained by [5].

THEOREM 2:

Let $f = h + \bar{g}$ defined by (5). Then the necessary and sufficient condition for the function f to be in the class $\overline{AJ_{\mathcal{H}}}(\lambda, \alpha, k, \gamma)$ is the subclass of$AJ_{\mathcal{H}}(\lambda, \alpha, k, \gamma)$, is that

$$\left(\sum_{l=w+1}^{\infty} \frac{\frac{l!}{(l-q)!}\big((l-q)(1+k)+(1-\alpha)\big)}{\frac{w!}{(w-q)!}(1-\alpha)}|a_l| - \sum_{l=w}^{\infty} \frac{\frac{l!}{(l-q)!}\big((l-q)(1+k)+(1+2k+\alpha)\big)}{\frac{w!}{(w-q)!}(1-\alpha)}|b_l|\right) B_l(\lambda) \le 1$$

(6)

Where $\alpha_1 = 1, 0 \le \alpha < 1, \alpha < k \le 1, \lambda > -1, 0 \le \gamma < 1$ and

$$B_l(\lambda) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+w)(l-w)!}$$

PROOF:

Since $\overline{AJ_{\mathcal{H}}}(\lambda, \alpha, k, \gamma) \subset AJ_{\mathcal{H}}(\lambda, \alpha, k, \gamma)$, then the "necessary" part follows from Theorem (1) for the "sufficient" part, we show that (6) does not hold good implies that $f \notin \overline{AJ_{\mathcal{H}}}(\lambda, \alpha, k, \gamma)$. Now, a function $f \in \overline{AJ_{\mathcal{H}}}(\lambda, \alpha, k, \gamma)$if and only if

$$\mathcal{R}e\left\{\left(1+ke^{i\gamma}\right)\frac{s\left(D^{\lambda+w-1}f(s)\right)^{q+1}}{\left(D^{\lambda+w-1}f(s)\right)^{q}} + 1\right\} \ge \alpha$$

Therefore,

$$\mathcal{R}e\left(\frac{\left\{\begin{array}{c}\frac{w!}{(w-q)}(1-\alpha)|s|^{w}-\sum_{l=w+1}^{\infty}\frac{l!}{(l-q)!}\begin{pmatrix}(l-q)(1+k)\\+(1-\alpha)\end{pmatrix}|a_l|B_l(\lambda)|s|^{l}\\ -\sum_{l=w}^{\infty}\frac{l!}{(l-q)!}\begin{pmatrix}(l-q)(1+k)\\+(1+2k+\alpha)\end{pmatrix}|b_l|B_l(\lambda)|s|^{l}\end{array}\right\}}{\left\{\mathcal{Z}-\sum_{l=w+1}^{\infty}\frac{l!}{(l-q)!}|a_l|B_l(\lambda)|s|^{l}+\sum_{l=w}^{\infty}\frac{l!}{(l-q)!}|b_l|B_l(\lambda)|\bar{s}|^{l}\right\}}\right)\geq 0$$

The last inequality must hold for all s, $|s| = r < 1$. Choosing the values of $\mathcal{Z}$ on the positive real axis where $0 < |s| = r < 1$ we must have,

$$=\left(\frac{\left\{\begin{array}{c}\frac{w!}{(w-q)!}(1-\alpha)-\sum_{l=w+1}^{\infty}\frac{l!}{(l-q)!}\begin{pmatrix}(l-q)(1+k)\\+(1-\alpha)\end{pmatrix}|a_l|B_l(\lambda)r^{l-w}\\ -\sum_{l=w}^{\infty}\frac{l!}{(l-q)!}\begin{pmatrix}l-q)(1+k)\\+(1+2k+\alpha)\end{pmatrix}|b_l|B_l(\lambda)r^{l-w}\end{array}\right\}}{\left\{1-\sum_{l=w+1}^{\infty}\frac{l!}{(l-q)!}|a_l|B_l(\lambda)r^{l-1}+\sum_{l=w}^{\infty}\frac{l!}{(l-q)!}|b_l|(B_l(\lambda)\mathcal{Z})r^{l-1}\right\}}\right)\geq 0 \qquad (7)$$

We note that if the condition (6) does not hold, then the numerator in (7) when r goes to 1 is negative.

This is a contradiction for $f(\mathcal{Z}) \in \overline{AJ_{\mathcal{H}}}(\lambda,\alpha,k,\gamma)$ and the proof is complete.

In the next theorem we obtain the extreme points of the closed convex hulls of $\overline{AJ_{\mathcal{H}}}(\lambda,\alpha,k,\gamma)$ denoted by $clco\,\overline{AJ_{\mathcal{H}}}(\lambda,\alpha,k,\gamma)$.

THEOREM 3:

The function $f(\mathcal{Z}) \in$ clco $\overline{AJ_{\mathcal{H}}}(\lambda,\alpha,k,\gamma)$ if and only if

$$f(s)=\sum_{n=w}^{\infty}\big(T_n h_n(s)+S_n g_n(s)\big) \qquad (8)$$

Where

$$h_w(s)=s^w,\quad h_l(s)=s^w-\frac{\frac{w!}{(w-q)!}(1-\alpha)}{\frac{l!}{(l-q)!}(l-q)(1+k)+(1+\alpha)}s^l,\qquad l=w+1,w+2,....$$

$$g_l(s)=s^w-\frac{\frac{w!}{(w-q)!}(1-\alpha)}{\frac{l!}{(l-q)!}\big(l-q)(1+k)+(1+2k+\alpha)\big)}s^l,\qquad l=w,w+1,....$$

$$\sum_{l=w}^{\infty}(T_l+S_l)=1,\quad T_l\geq 0\quad and\quad S_l\geq 0$$

In particular the extreme points of $\overline{AJ_{\mathcal{H}}}(\lambda,\alpha,k,\gamma)$ are $\{h_l\} and\ \{g_l\}$.

PROOF:

Let f be written as (8). Then we have

$$f(s)=\sum_{n=w}^{\infty}(T_n+S_n)\mathcal{Z}^w=\sum_{n=w+1}^{\infty}\frac{\frac{w!}{(w-q)!}(1-\alpha)}{\frac{n!}{(n-q)!}\big((n-q)(1+k)+(1-\alpha)\big)B_n(\lambda)}T_n s^n$$
$$-\sum_{n=w}^{\infty}\frac{\frac{w!}{(w-q)!}(1-\alpha)}{\frac{n!}{(n-q)!}\big((n-q)(1+k)+(1+2k+\alpha)\big)B_n(\lambda)}S_n s^n$$

Then,

$$\sum_{l=w+1}^{\infty} \frac{\frac{l!}{(l-q)!}\big((l-q)(1+k)+(1-\alpha)\big)B_n(\lambda)}{\frac{p!}{(p-q)!}(1-\alpha)}|a_n| + \sum_{n=w}^{\infty} \frac{\frac{n!}{(n-q)!}\big((n-q)(1+k)+(1+2k+\alpha)\big)B_n(\lambda)}{\frac{w!}{(p-q)!}(1-\alpha)}|b_n|$$

$$= \sum_{l=w+1}^{\infty} T_l + \sum_{l=w}^{\infty} S_l = 1 - T_w \le 1.$$

Then $f \in$ clco $\overline{AJ_{\mathcal{H}}}(\lambda,\alpha,k,\gamma)$

Conversely,assume that $f \in$ clco $\overline{AJ_{\mathcal{H}}}(\lambda,\alpha,k,\gamma)$. Putting

$$T_l = \frac{\frac{l!}{(l-q)!}\big((l-q)(1+k)+(1-\alpha)\big)B_l(\lambda)}{\frac{w!}{(w-q)!}(1-\alpha)}|a_l|, \quad l = w+1, w+2, \ldots$$

$$S_l = \frac{\frac{l!}{(l-q)!}\big((l-q)(1+k)+(1+2k+\alpha)\big)}{\frac{w!}{(w-q)!}(1-\alpha)}|b_l|, \quad l = w, w+1, \ldots$$

$$T_1 = 1 - \sum_{l=w+1}^{\infty} T_l + \sum_{l=w}^{\infty} S_l .$$

$then$ $\sum_{l=w}^{\infty}(T_l + S_l) = 1$,$0 \le T_l \le 1$ $(l = w+1, w+2, \ldots)$, $0 \le S_l \le 1$ $(l = w, w+1, \cdots)$.

Thus, by simple calculations we get

$$f(s) = \sum_{l=w}^{\infty} (T_l h_l(\mathcal{Z}) + S_l g_l(s))$$

And the proof is complete

THEOREM 4:

Let $f \in \overline{AJ_{\mathcal{H}}}(\lambda,\alpha,k,\gamma)$ .Then

$$|f(s)| \le (1+|b_w|)r^w + \frac{\frac{w!}{(w-q)!}(1-\alpha)}{B_{w+1}(\lambda)\frac{(w+1)!}{(p+1-q)!}[(p+1-q)(1+k)+(1-\alpha)]} \times \left(1 - \frac{\frac{w!}{(w-q)!}\big((w-q)(1+k)+(1+2k+\alpha)\big)}{\frac{w!}{(w-q)}(1-\alpha)}|b_w|\right) r^{w+1}$$

, $|s| = r < 1$ (9)

and

$$|f(s)| \ge (1+|b_w|)r^w + \frac{\frac{w!}{(w-q)!}(1-\alpha)}{B_{w+1}(\lambda)\frac{(w+1)!}{(w+1-q)!}[(w+1-q)(1+k)+(1-\alpha)]} \times \left(1 - \frac{\frac{w!}{(w-q)!}\big((w-q)(1+k)+(1+2k+\alpha)\big)}{\frac{w!}{(w-q)}(1-\alpha)}|b_w|\right) r^{w+1}$$

, $|s| = r < 1$

PROOF: We have

$$|f(s)| \leq (1+|b_w|)r^p + \sum_{l=2}^{\infty}(|a_l|+|b_l|)r^l$$
$$\leq (1+|b_w|)r^w + \sum_{l=2}^{\infty}(|a_l|+|b_l|)r^{w+1}$$
$$= (1+|b_w|)r^w$$
$$+ \frac{\frac{w!}{(w-q)!}(1-\alpha)}{B_{w+1}(\lambda)\frac{(w+1)!}{(w+1-q)!}[(w+1-q)(1+k)+(1-\alpha)]}\sum_{n=w+1}^{\infty}\left(\frac{\frac{(w+1)!}{(w+1-q)!}[(w+1-q)(1+k)+(1-\alpha)]}{\frac{w!}{(w-q)!}(1-\alpha)}|a_l|\right.$$
$$\left.+\frac{\frac{(w+1)!}{(w+1-q)!}[(w+1-q)(1+k)+(1-\alpha)]}{\frac{w!}{(w-q)!}(1-\alpha)}|b_n|\right)B_l(\lambda)\,r^{w+1}$$

$$\leq (1+|b_w|)r^w + \frac{\frac{w!}{(w-q)!}(1-\alpha)}{B_{w+1}(\lambda)\frac{(w+1)!}{(w+1-q)!}[(w+1-q)(1+k)+(1-\alpha)]}$$
$$\times\sum_{l=w+1}^{\infty}\left(\frac{\frac{l!}{(l-q)!}\big((l-q)(1+k)+(1-\alpha)\big)}{\frac{w!}{(w-q)}(1-\alpha)}|a_n|\right.$$
$$\left.+\frac{\frac{l!}{(l-q)!}\big((l-q)(1+k)+(1+2k+\alpha)\big)}{\frac{w!}{(w-q)}(1-\alpha)}|b_l|\right)B_l(\lambda)\,r^{w+1}$$

$$\leq (1+|b_w|)r^w + \frac{\frac{w!}{(w-q)!}(1-\alpha)}{B_{w+1}(\lambda)\frac{(w+1)!}{(w+1-q)!}[(w+1-q)(1+k)+(1-\alpha)]}$$
$$\times\left(1-\frac{\frac{w!}{(w-q)!}\big((w-q)(1+k)+(1+2k+\alpha)\big)}{\frac{w!}{(w-q)}(1-\alpha)}|b_w|\right)r^{w+1}$$
$$\leq (1+|b_w|)r^w$$
$$+\frac{\frac{w!}{(w-q)}}{\frac{(w+1)!}{(w+1-q)!}B_{w+1}(\lambda)}\left(\frac{(1-\alpha)}{(w+1-q)(1+k)+(1-\alpha)}\right.$$
$$\left.-\frac{\big((w-q)(1+k)+(1+2k+\alpha)\big)}{(w+1-q)(1+k)+(1-\alpha)}|b_w|\right)r^{w+1}$$

The next inequality can be proved by using similar arguments. This completes the proof of theorem.
Now we define the convolution of two harmonic functions. If $f(z)$ and $g(z)$ be given by
$f(s) = s^w - \sum_{l=w+1}^{\infty}|a_n|s^l - \sum_{l=w}^{\infty}|b_n|\bar{s}^l$,$g(s) = s^w - \sum_{l=w+1}^{\infty}|c_l|s^l - \sum_{=w}^{\infty}|d_l|\bar{s}^l$
Then the Hadamard product of $f(\mathcal{Z})$ and $g(\mathcal{Z})$ defined by

$$(f*g)(s) = f(s)*g(s) = s^w - \sum_{l=w+1}^{\infty}|a_l||c_l|s^l - \sum_{l=w}^{\infty}|b_l||d_l|\bar{s}^l \qquad (10)$$

THEOREM 5:

Let $f(s) \in \overline{AJ_{\mathcal{H}}}(\lambda,\alpha,k,\gamma)$ and $g(s) \in \overline{AJ_{\mathcal{H}}}(\lambda,\alpha,k,\gamma)$ Then for$0 \le \beta \le \alpha < 1$, we have $(f*g)(s) \in \overline{AJ_{\mathcal{H}}}(\lambda,\alpha,k,\gamma) \subset \overline{AJ_{\mathcal{H}}}(\lambda,\alpha,k,\gamma)$.

PROOF:

we have

$$(f*g)(s) = \mathcal{Z}^w - \sum_{l=w+1}^{\infty} |a_l||c_l|s^l - \sum_{l=w}^{\infty} |b_n||d_n|s^n$$

By noting that $|c_n| \le 1$ and $|d_n| \le 1$, the theorem follows easily by using the condition (6).The proof of this theorem is complete.

In the next theorem we show that TS_H (λ, α, k, γ) is closed under convex combination of its members.

THEOREM 6:

Let for $i = w, w+1, ..$ the function

$$f_i(s) = s^w - \sum_{n=w+1}^{\infty} |a_{i,l}| s^l + \sum_{l=w}^{\infty} |b_{i,l}| \bar{s}^l,$$

Belong to $\overline{AJ_{\mathcal{H}}}(\lambda,\alpha,k,\gamma)$. Then $\sum_{n=1}^{\infty} \mu_i f_i(s)$belongs to $\overline{AJ_{\mathcal{H}}}(\lambda,\alpha,k,\gamma)$, where $\sum_{n=1}^{\infty} \mu_i = 1$, $0 \le \mu_i < 1$, i = 1, 2, · · ·.

PROOF:

Since $f_i \in \overline{AJ_{\mathcal{H}}}(\lambda,\alpha,k,\gamma)$. (i = 1, 2, · · ·), then from (6), we have

$$\left(\sum_{l=p+1}^{\infty} \frac{\frac{l!}{(l-q)!}\big((l-q)(1+k)+(1-\alpha)\big)}{\frac{w!}{(w-q)!}(1-\alpha)} |a_{i,l}| + \sum_{l=w}^{\infty} \frac{\frac{l!}{(l-q)!}\big((l-q)(1+k)+(1+2k+\alpha)\big)}{\frac{w!}{(w-q)!}(1-\alpha)} |b_{i,n}| \right) B_l(\lambda) \le 1$$

For$\sum_{i=1}^{\infty} \mu i = 1,\ 0 \le \mu i \le 1$, we may write the convex combination of f_i as

$$\sum_{i=1}^{\infty} \mu i\, f_i = s^w - \sum_{l=w+1}^{\infty} \left(\sum_{i=1}^{\infty} \mu i\ |a_{i,l}| \right) s^l - \sum_{l=w}^{\infty} \left(\sum_{i=1}^{\infty} \mu i\ |b_{i,l}| \right) s^l$$

Thus

$$\sum_{l=w+1}^{\infty} \frac{\frac{l!}{(l-q)!}\big((l-q)(1+k)+(1-\alpha)\big)}{\frac{w!}{(w-q)!}(1-\alpha)} B_l(\lambda) \left(\sum_{i=1}^{\infty} \mu i\, |a_{i,l}| \right) + \sum_{l=w}^{\infty} \frac{\frac{l!}{(l-q)!}\big((l-q)(1+k)+(1+2k+\alpha)\big)}{\frac{w!}{(w-q)!}(1-\alpha)} B_l(\lambda) \left(\sum_{i=1}^{\infty} \mu i\, |b_{i,l}| \right)$$

$$= \sum_{i=1}^{\infty} \mu i \left\{ \sum_{l=w+1}^{\infty} \frac{\frac{l!}{(l-q)!}\big((l-q)(1+k)+(1-\alpha)\big)}{\frac{p!}{(p-q)!}(1-\alpha)} B_l(\lambda)|a_{i,l}| + + \sum_{l=w}^{\infty} \frac{\frac{l!}{(l-q)!}\big((l-q)(1+k)+(1+2k+\alpha)\big)}{\frac{w!}{(w-q)!}(1-\alpha)} B_l(\lambda)|b_{i,l}| \right\} \le \sum_{i=1}^{\infty} \mu i = 1$$

Then $\sum_{i=1}^{\infty} \mu_i f_i \in \overline{AJ_{\mathcal{H}}}(\lambda,\alpha,k,\gamma)$ The proof is complete.

THEOREM 7:

The class $\overline{AJ_{\mathcal{H}}}$ (λ,α, k,γ) is convex set.

PROOF:

Let f_1, f_2 be the arbitrary elements of $\overline{AJ_{\mathcal{H}}}(\lambda,\alpha,k,\gamma)$.Then for every $t(0 < t < 1)$, we show that $(1-t)f_1 + tf_2 \in \overline{AJ_{\mathcal{H}}}(\lambda,\alpha,k,\gamma)$,thus we have

$$(1-t)f_1(s) + tf_2 = s^w - \sum_{n=w+1}^{\infty} \big[(1-t)|a_{n,1}| + t|a_{n,2}|\big] s^n - \sum_{n=w}^{\infty} \big[(1-t)|b_{n,1}| + t|b_{n,2}|\big] (\bar{s})^n$$

And

$$\sum_{l=w+1}^{\infty} \frac{\frac{l!}{(l-q)!}(l-q)(1+k)+(1-\alpha)}{\frac{w!}{(w-q)!}(1-\alpha)} B_l(\lambda)\left[(1-t)|a_{l,1}|+t|a_{l,2}|\right] + \sum_{l=w}^{\infty} \frac{\frac{l!}{(l-q)!}(l-q)(1+k)+(1+2k+\alpha)}{\frac{w!}{(w-q)!}(1-\alpha)} B_l(\lambda)\left[(1-t)|b_{l,1}|+t|b_{l,2}|\right]$$

$$= (1-t)\left\{ \sum_{l=w+1}^{\infty} \left(\frac{\frac{l!}{(l-q)!}(l-q)(1+k)+(1-\alpha)}{\frac{w!}{(w-q)!}(1-\alpha)} \right) B_l(\lambda)|a_{l,1}| + \sum_{l=w}^{\infty} \frac{\frac{l!}{(l-q)!}(l-q)(1+k)+(1+2k+\alpha)}{\frac{p!}{(p-q)!}(1-\alpha)} B_l(\lambda)|b_{l,1}| \right\}$$
$$+ t\left\{ \sum_{l=w+1}^{\infty} \left(\frac{\frac{l!}{(l-q)!}(l-q)(1+k)+(1-\alpha)}{\frac{w!}{(w-q)!}(1-\alpha)} \right) B_l(\lambda)|a_{l,2}| + \sum_{l=w}^{\infty} \frac{\frac{l!}{(l-q)!}(l-q)(1+k)+(1+2k+\alpha)}{\frac{w!}{(w-q)!}\left((w-q)\left((1-\alpha)1+k\right)+(k-\alpha)\right)} B_l(\lambda)|b_{l,2}| \right\} \leq (1-t)+t=1$$

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