

SOLUTION OF TWO DIMENSIONAL FRACTIONAL ORDER VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract

In this paper, our aim is to study variational formulation and solutions of 2-dimensional integro-differential equations of fractional order.

We will give a summary of representation to the variational formulation of linear non-homogenous 2-dimensional Volterra integro-differential equations of the second kind with fractional order.

An example will be discussed and solved by using the MathCAD software package when it is needed.

1. Introduction

The integral equation is an equation in which the unknown generally any function of one or more variables, occurs under an integral sign, [6]. Also, the one-dimensional integral equation is an integral in which the unknown function depends only on one independent variable, [5]. While the multi-dimensional integral equation is an extension of the one-dimensional integral equations. Many researchs studied the multi-dimensional partial integro-differential equations, say Becker in 1999 also Vander and Sommeijer in 1996, [2] and [12].

This paper consists of four sections, in section one, introduction about integral equations. In section two, we give some definitions of the partial fractional derivatives and fractional integration. In section three, we deal with the variational method to solve linear problem. In section four, we give the variational formulation to solve the two-dimensional fractional integro-differential equations with example explain this approach.

2. The Variation Approach

The main problem in calculus of variation is to find the maximum or minimum values of a given functional $F(u)$, this necessary condition is called the Euler-Lagrange equation and the solution of this problem is called the direct problem of calculus of variation, [1], [3] and [8].

Some important definitions and concepts which are needed to understand the variational approach will be given in [7] and [11].

The most important difficulty of the subject of calculus of variation is to find the variational formulation, which corresponds to the linear operator equation

$$L u = f \dots\dots\dots (1)$$

Where f denotes a scalar-vector valued functional and L denotes a linear operator.

Theorem (2.1):

If the given linear operator L is symmetric with respect to the chosen non-degenerate bilinear form $\langle u, v \rangle$. But if the linear operator L is not symmetric with respect to the chosen bilinear form $\langle u, v \rangle$, and the problem is to find the variational formulation, then this could be done the transformation:

$$(u, v) = \langle u, Lv \rangle, \text{ where } v \in V \text{ and } u \in D(L) \dots\dots\dots (2)$$

The bilinear form (2) makes the given linear operator symmetric since:

$$(Lu_1, u_2) = \langle Lu_1, Lu_2 \rangle = \langle Lu_2, Lu_1 \rangle = (Lu_2, u_1)$$

Therefore, in general we will use the bilinear form (2) to find a variational formulation because of the symmetry of L . Since L is symmetric and by using theorem (2.1), the solution of equation (1) is a critical point to the functional:

$$F(u) = \frac{1}{2} (Lu, u) - (f, u)$$

$$= \frac{1}{2} \langle Lu, Lu \rangle - \langle f, Lu \rangle \dots\dots\dots(3)$$

The functional (3) is a variational formulation for the linear equation $Lu = f$, [9].

3. Fractional Calculus

3.1 Introduction to Fractional Calculus

In recent years, there has been a growing interest in the field of fractional calculus. Equation of fractional order has appeared more and more in different research fields and engineering applications.

Fractional calculus is the field of mathematical analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order.

The fractional calculus may be considered as old and yet a novel topic, actually, it is an old topic since starting from some spectrum of Leibniz (1695 – 1697) and Euler (1730) who said "when n is an integer, the ratio $d^n p$, p is a function of x , to dx^n can always be expressed algebraically.

Now, it is asked: what kind of ration can be made if n is fraction ? ", it has been developed up to nowadays.

In fact, the idea of generalizing the notion of derivative to non-integer order, in particular to the order of $\frac{1}{2}$ (which is called semi-integerl or semi-derivative) is found in the correspondence of Leibniz and Bernoulli, L'Hopital and Wallis. Euler took the first step by observing that the result of the derivative evaluation of the power function has a meaning for non-integer order thanks to his Gamma function, [4].

3.2 Riemann – Liouville Partial Fractional Derivatives

The fractional derivative of order q of $f(x,y)$ is

$$\partial_{x-q}^q f(x, y) = \frac{1}{\Gamma(-q)} \int_a^x (x-t)^{-(q+1)} f(t, y) dt \dots(4)$$

and

$$\partial_{y-q}^q f(x, y) = \frac{1}{\Gamma(-q)} \int_a^y (y-t)^{-(q+1)} f(x, t) dt \dots(5)$$

respectively in case q is a negative fractional number, and:

$$\partial_{x-q}^q f(x, y) = \frac{d^n}{dx^n} \left[\frac{1}{\Gamma(n-q)} \int_a^x (x-t)^{n-q-1} f(t, y) dt \right] \dots\dots\dots(6)$$

$$\partial_{y-q}^q f(x, y) = \frac{d^n}{dx^n} \left[\frac{1}{\Gamma(n-q)} \int_a^x (y-t)^{n-q-1} f(x, t) dt \right] \dots\dots\dots(7)$$

respectively in case q is a positive fractional number, [9].

3.3 Gamma Function $\Gamma(x)$

The Gamma function $\Gamma(x)$ plays an important role in the theory of differentiation. The definition for the $\Gamma(x)$ is given by

$$\Gamma(x) = \int_0^\infty y^{x-1} \exp(-y) dy, x > 0$$

It is more useful than other definitions of $\Gamma(x)$, [10].

4. Variational Formulation for Fractional Calculus

Consider the 2-dimensional linear fractional order integro-differential equation of Volterra type:

$$\partial_x^q f(x, y) = g(x, y) + \lambda \int_c^{d(y)} \int_a^{b(x)} k(x, y, z, m) f(z, m) dz dm \dots\dots\dots(8)$$

where $a \leq x \leq b(x)$ and $c \leq y \leq d(y)$. which have integral operator of the form

$$L = \partial_x^q - \lambda \int_c^{d(y)} \int_a^{b(x)} k(x, y, z, m) dz dm$$

where ∂_x^q is a partial fractional operator of x , q is a fraction number and $g(x,y)$ is given function, $0 < q < 1$.

The operator L is linear since it is easily seen that:

$$\begin{aligned} L(w_1 f_1 + w_2 f_2) &= w_1 \partial_x^q f_1(x, y) - \\ &w_1 \lambda \int_c^{d(y)} \int_a^{b(x)} k(x, y, z, m) f_1(z, m) dz dm + \\ &w_2 \partial_x^q f_2(x, y) - \\ &w_2 \lambda \int_c^{d(y)} \int_a^{b(x)} k(x, y, z, m) f_2(z, m) dz dm \\ &= w_1 Lf_1 + w_2 Lf_2 \end{aligned}$$

Therefore, the operator L is a linear.

Now, define the bilinear form

$$\langle f_1, f_2 \rangle = \int_m^{T_2} \int_p^{T_1} f_1(x, y) f_2(x, y) dx dy \dots\dots\dots(9)$$

where $p \leq x \leq T_1$, $m \leq y \leq T_2$.

Thus, the variational formulation of the given linear operator L could be found as follows

$$F(f) = \frac{1}{2} \langle Lf, Lf \rangle - \langle g, Lf \rangle$$

$$= \frac{1}{2} \int_m^{T_2} \int_p^{T_1} (Lf(x, y))^2 dx dy - \int_m^{T_2} \int_p^{T_1} g(x, y) Lf(x, y) dx dy$$

$$= \frac{1}{2} \int_m^{T_2} \int_p^{T_1} [\partial_x^q f(x, y) -$$

$$\lambda \int_c^{d(y)} \int_a^{b(x)} k(x, y, z, m) f(z, m) dz dm]^2 dx dy -$$

$$\int_m^{T_2} \int_p^{T_1} [g(x, y) [\partial_x^q f(x, y) -$$

$$\lambda \int_c^{d(y)} \int_a^{b(x)} k(x, y, z, m) f(z, m) dz dm] dx dy$$

.....(10)

To solve the above variational formulation, one must approximate the solution f of equation (8) as a linear combination of n linearly independent functions $\{Q_i(x, y)\}_{i=1}^n$, (the direct Ritz method) such that;

$$f(x, y) = \sum_{i=1}^n a_i Q_i(x, y) \dots\dots\dots(11)$$

where $\{a_i\}_{i=1}^n$ are the unknown parameters that must be determined. Then, after substituting this approximated solution into the functional given by equation (10), one can get:

$$F(f) = \frac{1}{2} \int_m^{T_2} \int_p^{T_1} [\partial_x^q f_a(x, y) -$$

$$\lambda \int_c^{d(y)} \int_a^{b(x)} k(x, y, z, m) f_a(z, m) dz dm]^2 dx dy -$$

$$\int_m^{T_2} \int_p^{T_1} g(x, y) [\partial_x^q f_a(x, y) -$$

$$\lambda \int_c^{d(y)} \int_a^{b(x)} k(x, y, z, m) f_a(z, m) dz dm] dx dy$$

.....(12)

where f_e is the exact results associated with the problem, while f_a is the approximate results of this problem.

The problem here is to find the critical point of the above functional F(f) which could be found by using the method of variation.

Finally, the values of $\{a_i\}_{i=1}^n$ are obtained by solving the system of linear equations which can be obtained by setting $\frac{\partial F}{\partial a_i} = 0$,

$$i = 1, 2, \dots, n.$$

Then one can solve this system by using the Math CAD softwear Backage for equation (12), we get tabulated results.

To illustrate this method, consider the following example.

Example:

Consider the following Volterra fractional integro-differential equation

$$\partial_x^{0.5} f(x, y) = g(x, y) + \int_1^y \int_1^x (x^2 y + z) f(z, m) dz dm$$

where $g(x, y) = 4y\sqrt{\frac{x}{\pi}} - \frac{1}{2}x^4 y^3 - \frac{1}{3}x^3 y^2$

the analytical solution of this problem is given by:

$$f_e(x, y) = 2xy$$

By using the direct Ritz method and Riemann-Liouville partial fractional derivatives in equation (6), then

$$f_a(x, y) = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 y^2 + a_5 xy \dots(13)$$

Hence

$$\partial_x^{0.5} f_a(x, y) = \frac{a_0}{\sqrt{\pi x}} + 2a_1 \sqrt{\frac{x}{\pi}} + \frac{a_2 y}{\sqrt{\pi x}} +$$

$$\frac{2a_3}{7} \sqrt{\frac{x^7}{\pi}} + \frac{a_4 y}{\sqrt{\pi x}} + 2a_5 y \sqrt{\frac{x}{\pi}}$$

.....(14)

The operator is given by:

$$L = \partial_x^{0.5} - \int_1^y \int_1^x (x^2 y + z) dz dm$$

Assume

$$R_a(x, y) = \partial_x^{0.5} f_a(x, y) - \int_1^y \int_1^x (x^2 y + z) f_a(z, m) dz dm$$

.....(15)

Hence, the variational formulation with (13), (14), and (15) is given by

$$F(f) = \frac{1}{2} \int_1^3 \int_1^2 (R_a(x, y))^2 dx dy - \int_1^3 \int_1^2 g(x, y) R_a(x, y) dx dy$$

.....(16)

Finally, the values of $\{a_i\}_{i=1}^n$ are obtained by solving the system of linear equations which can be obtained by setting $\frac{\partial F}{\partial a_i} = 0$, $i = 1, 2, \dots, n$.

Then one can solve this system by using the Math CAD softwear Backage for equation (16), we get tabulated results.

The approximated solutions by using the variational method.

Basis functions	Coefficients of the Basis	Approximated Solutions
1	a_0	-3.159×10^{-4}
x	a_1	9.94×10^{-3}
y	a_2	2.471×10^{-4}
x^2	a_3	2.787×10^{-3}
y^2	a_4	-3.13×10^{-3}
xy	a_5	-8.729×10^{-3}

Another case, when a partial fractional for y: Consider the linear fractional order integro-differential equation of Volterra type:

$$\partial_y^q f(x, y) = g(x, y) + \lambda \int_c^{d(y)} \int_a^{b(x)} k(x, y, z, m) f(z, m) dz dm \dots\dots\dots(17)$$

where $a \leq x \leq b(x)$ and $c \leq y \leq d(y)$. which have integral operator of the form

$$L = \partial_y^q - \lambda \int_c^{d(y)} \int_a^{b(x)} k(x, y, z, m) dz dm$$

where ∂_y^q is a partial fractional operator of y, q is a fraction number and g(x,y) is given function, $0 < q < 1$.

We can prove similarly as in the first case the linearly and symmetry of the operator L. Therefore, the variational formulation is given by

$$F(f) = \frac{1}{2} \int_m^{T2} \int_p^{T1} [\partial_y^q f_a(x, y) - \lambda \int_c^{d(y)} \int_a^{b(x)} k(x, y, z, m) f_a(z, m) dz dm]^2 dx dy - \int_m^{T2} \int_p^{T1} g(x, y) [\partial_y^q f_a(x, y) - \lambda \int_c^{d(y)} \int_a^{b(x)} k(x, y, z, m) f_a(z, m) dz dm] dx dy \dots\dots\dots(18)$$

We use the similar approach in case one, to find the critical point of the above functional F(f).

Conclusions

1. The 2-dimensional fractional integro-differential equations are so difficult, in most cases, to be solved analytically, therefore numerical methods are required.
2. When we solve the fractional integral equations using variational approach, the absolute error often approaches zero between the exact and approximate results.

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الخلاصة

ان الهدف من هذا البحث دراسة الصيغة التفاضلية للمعادلات التفاضلية التكاملية ذات رتب كسرية وبعدين وحلولها، واعطيت تمثيلات مختصرة للصياغة التفاضلية للمعادلات الخطية ذات بعدين التي هي معادلات تفاضلية تكاملية ذات رتب كسرية ومن النوع الثاني ومن صنف (Volterra). كما اعطي مثال ونوقش حله باستخدام برنامج (MathCAD 2001 Software Package).