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Minimal wave speed and traveling wave in nonlocal dispersion SIS epidemic model with delay

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Abstract

This study examines traveling wave solutions of the SIS epidemic model with nonlocal dispersion and delay. The research shows that a key factor in determining whether traveling waves exist is the basic reproduction number R_0. In particular, the system permits nontrivial traveling wave solutions for $\sigma \geq \sigma^*$ for $R_0 > 1$, whereas there are no such solutions for $\sigma < \sigma^*$. This is because there is a minimal wave speed $\sigma^* > 0$. On the other hand, there are no traveling wave solutions when $R_0 \leq 1$. In conclusion, we provide several numerical simulations that illustrate the existence of TWS.

Mathematics Subject Classification: 35K57; 37N25; 92D30

Keywords: SIS model; Minimal wave speed; Traveling waves; Basic reproduction number

1 Introduction

In recent years, the study of traveling wave solutions in epidemic models has been a major focus due to their significant contributions to disease dynamics [1, 2]. More precisely, the minimal wave speed has a significant role to play in modeling the spatial propagation of an infectious disease in structured populations. In this article, we explore the minimal wave speed and traveling wave solutions in a nonlocal dispersion SIS epidemic model with delay. Our study is founded on the existing theoretical framework for wave propagation in physical and biological systems. In the same vein, previous studies on wave behavior in complicated media have been informative as far as transmission dynamics are concerned. For example, Seema and Singhal [3] investigated SH wave transmission in magneto-electro-elastic structures, while their study on Love-type wave velocity in bedded piezo-structures [4] explained the impact of rheological models and flexoelectric effects. Along a similar line, research on surface and interface phenomena in quasicrystals [5] has indicated the intricate interactions of material characteristics on wave phenomena (see also [6]). Drawing insights from such modeling approaches, we extend these principles to epidemiological contexts by adding nonlocal interaction and delay dynamics, thereby presenting a more integrated view of epidemic wavefront evolution.

Such wave-like solutions describe the invasion of disease into new regions at a constant speed and provide insight into critical thresholds for outbreaks, disease persistence, and control strategies [7–13]. Among these, traveling wave solutions for the delayed SIS model with nonlocal dispersion and time delays provide an efficient framework for understanding spatial-temporal dynamics in disease consideration.

The SIS model is widely used in the study of diseases where individuals, after recovery, become susceptible again. The dispersion of such models is nonlocal because long-range interactions between individuals or populations are considered due to human mobility, migration, or environmental causes [14]. Such a concept generalizes the classical model of diffusion and is very applicable to the study of contemporary epidemics influenced by globalization and fast transportation [15, 16].

Another important element of epidemic modeling is time delays. Delays stand for the time that is spent for processes such as incubation periods, immune response, or behavioral changes, and may significantly alter the stability and propagation of disease waves [17–26]. In particular, delays offer a realistic way to model the variability in such processes; for instance, see studies by Naji et al. (2022) [27] and Tian et al. (2023) [28], where delays influenced wave speeds and dynamics in epidemic models.

Traveling waves in delayed SIS models with nonlocal dispersion and delays extend the classical reaction-diffusion framework of Fisher (1937) and Kolmogorov, Petrovsky, and Piskunov (1937) [29, 30]. These models explicitly include not only the spatial movement of individuals but also the effects due to long-range interactions and/or time lags, hence inducing richer wave dynamics. For example, the works by Fang and Zhao (2018) [31] and Li et al. (2016) [32] point out how such factors influence minimal wave speed, stability, and wave shape for traveling wave solutions.

This paper focuses on the SIS epidemic model with nonlocal dispersion and delays, aiming at a more complete understanding of the conditions under which traveling wave solutions exist and propagate. In particular, we will explore the following problems:

The existence and uniqueness of traveling wave solutions. The minimal wave speed (c^*) is required for disease invasion. The impact of some key parameters, including the basic reproduction number (R_0) and delay terms, on wave propagation.

The advance of the mathematical theory of traveling waves in epidemic models, in which the approach is based on theoretical analysis supported by numerical simulations, and realistic insights into the control of infectious diseases are discussed in this paper. These findings have special relevance to understanding the new epidemics, in which long-range interactions and time delays may play a decisive role in shaping the disease dynamics [33, 34].

In this paper, a saturation incidence rate, constant recruitment, and cooperation with a delayed diffusive SIR model are used to study the TWS connecting the equilibria, EES, and IFES. We also utilized the Schauder fixed point theorem to prove that the truncated problem admits a fixed point, and then it is used to show the convergence of the solution to the equilibria toward $x = \pm\infty$. In fact, it was demonstrated that when $R_0 > 1$, there is $\sigma^* > 0$, in such a way, when $\sigma \geq \sigma^*$, the system admits a TWS with speed σ.

The SIS model has a lengthy history [35]. It describes the transmission of human viruses like influenza. The SIS model with a constant population is particularly useful for describing bacterial agent disorders, including gonorrhea, meningitis, and streptococcal sore throat. SIS is a model without immunity in which the individual who has recovered from the virus returns to the class of susceptibles. Such mobility can be modeled by considering a nonlocal dispersion operator, which can be defined as follows:

$$\begin{aligned}\mathfrak{J}[\Phi](x) &:= J * \Phi(x) - \Phi(x) = \int_{\mathbb{R}} J(x-y)\Phi(y)dy - \Phi(x)\\ &= \int_{\mathbb{R}} J(y)\Phi(x-y)dy - \Phi(x), \quad \Phi \in C(\mathbb{R}),\end{aligned}$$

Then, we omit the following nonlocal dispersal SIS epidemic model with delay

$$\begin{cases} S_t = d_1(J * S(x,t) - S(x,t)) + \chi - \mu S(x,t)) - \lambda S(x,t)I(x,t) + \rho I(x,t), \\ I_t = d_2(J * I(x,t) - I(x,t)) + \lambda S(x,t-\varsigma)I(x,t-\varsigma) - (\mu+\rho)I(x,t), \end{cases} \tag{1.1}$$

with $t > 0$ and $x \in \mathbb{R}$. $S(x,t)$ and $I(x,t)$ denote the densities of susceptible, infective individuals at time t and location x in mathematical epidemiology, respectively; d_1, d_2 are positive describe the spatial motility of each class; μ is positive parameters represent the death rate of each class; $\rho > 0$ is the recovery rate of the infective individuals. $\varsigma > 0$ represents the duration of the delay. This SIS model with nonlocal diffusion and time delay is essential for the realistic modeling of disease dynamics where mobility and latency are significant. It enhances our understanding and provides effective tools for prediction and control in both human and animal epidemiology. We take the following assumptions

(**B**) d_j are positive, and $\mu, \varsigma, \rho > 0$ for $j = 1,2$.

(**P**) $J \in C^1(\mathbb{R}), J(0) > 0, J(x) = J(-x) \geq 0 \; \forall x \in \mathbb{R}, \int_{\mathbb{R}} J(x)dx = 1, \lim_{\Lambda \to +\infty} \frac{1}{\Lambda} \int_{\mathbb{R}} J(z)e^{-\Lambda z}dz = +\infty$.

2 Minimal wave speed σ^*

Finding the constant equilibria of (1.1) is necessary to demonstrate whether or not the traveling wave solutions for (1.1) exist. $E_0 = (\frac{\chi}{\mu}, 0)$, which is frequently referred to as the DFE of (1.1) is obviously always an equilibrium. To get a positive equilibrium, it is similar to considering the following ODE system.

$$\begin{cases} S_t = \chi - \mu S(t) - \lambda S(t)I(t) + \rho I(t), \\ I_t = \lambda S(t-\varsigma)I(t-\varsigma)) - (\mu+\rho)I(t). \end{cases} \tag{2.1}$$

The following is the appropriate basic reproduction number,

$$R_0 = \frac{\lambda s^0}{\mu + \rho}.$$

Notably, (2.1) permits a unique endemic equilibrium $E^* = (s^*, i^*)$ if $R_0 > 1$, with
$s^* = \frac{(\mu+\rho)}{\lambda}$, and $i^* = \frac{\lambda\chi - \mu(\mu+\rho)}{\lambda\mu} = \frac{\mu(\mu+\rho)(R_0-1)}{\lambda\mu}$.

We shall always assume that $R_0 > 1$ in the following. Two equilibria, E_0 and E^*, are admitted by the system (2.1) in this instance. Finding TWS of (1.1) that connect with E_0 and E^* is of primary importance to us. A unique solution of the form (1.1) is a TWS.

$$(s(\epsilon), i(\epsilon)), \quad \epsilon = x + \sigma t \in \mathbb{R}. \tag{2.2}$$

We replace (2.2) in (1.1), we get the wave form equations as

$$\begin{cases} \sigma s'(\epsilon) = d_1(J * s(\epsilon) - s(\epsilon)) + \chi - \mu s(\epsilon) - \lambda s(\epsilon) i(\epsilon) + \rho i(\epsilon), \\ \sigma i'(\epsilon) = d_2(J * i(\epsilon) - i(\epsilon)) + \lambda s(\epsilon - \sigma\varsigma) i(\epsilon - \sigma\varsigma) - (\mu + \rho) i(\epsilon), \end{cases} \tag{2.3}$$

with the boundary conditions

$$(s,i)(-\infty) = (\frac{\chi}{\mu}, 0), \qquad (s,i)(+\infty) = (s^*, i^*). \tag{2.4}$$

Our goal is to find a positive solution of (2.3) that meets (2.4) boundary conditions. By the second equation of (2.3), it may be linearized at $E_0 = (\frac{\chi}{\mu}, 0)$ to get

$$-\sigma i'(\epsilon) + d_2(J_* i(\epsilon)) + \lambda s^0 i(\epsilon - \sigma\varsigma) - (\mu + \rho + d_2) i(\epsilon) = 0. \tag{2.5}$$

Entering $i(\epsilon) = \mathrm{e}^{\Lambda\epsilon}$ in (2.5) yields the following characteristic equation:

$$F(\Lambda, \sigma) := -\sigma\Lambda + d_2 \int_{-\infty}^{+\infty} J(y) \mathrm{e}^{-\Lambda y} dy + \lambda s^0 \mathrm{e}^{-\sigma\varsigma\Lambda} - (\mu + \rho + d_2) = 0. \tag{2.6}$$

Consequently, the following outcomes are obtained by examining the characteristic equation (2.6).

Lemma 2.1 *Suppose* $R_0 > 1$, $\exists \sigma^* > 0$ *and* $\Lambda^* > 0$ *such that*

$$\frac{\partial F(\Lambda,\sigma)}{\partial \Lambda^2}\bigg|_{(\Lambda^*,\sigma^*)} = 0 \;\; and \;\; F(\Lambda^*, \sigma^*) = 0.$$

Additionally, the following options are valid:

(i) $F(\Lambda,\sigma) > 0$ *for every* $\Lambda \in (0, \Lambda_\sigma)$ *and* $0 < \sigma < \sigma^*$, *with* $\Lambda_\sigma \in [0, +\infty[$.

(ii) *Two positive distinct real roots* $\Lambda_1(\sigma) < \Lambda_2(\sigma)$ *that fulfil* $F(\Lambda;\sigma) = 0$ *exist if* $\sigma > \sigma^*$.

$$F(\Lambda,\sigma) \begin{cases} > 0 & \Lambda \in (0, \Lambda_1(\sigma)) \;\cup\; (\Lambda_2(\sigma), \infty), \\ < 0 & \Lambda \in (\Lambda_1(\sigma), \Lambda_2(\sigma)), \end{cases}$$

where

$$\sigma^* = \sup\{\sigma > 0 | F(\Lambda,\sigma) > 0, \forall \Lambda \in \mathbb{R}\}$$

exists and is constructive for the demonstration of this outcome simple.

We now examine the following sections to discuss whether a traveling wave solution exists.

3 The absence of traveling waves solution

Next theorem illustrates the situation in which a traveling waves solution is not admitted by the system (2.1).

Theorem 3.1 *If* $R_0 > 1$ *and* $0 < \sigma < \sigma^*$, *hence*, (2.3) *has no TWS of the form* $(s(\epsilon), i(\epsilon))$ *that satisfies* (2.4).

Proof Assume that for some $0 < \sigma < \sigma^*$, there exists a TWS noted by $(s^*(\epsilon), i^*(\epsilon))$ of system (2.3) that satisfies requirements (2.4). For each $\epsilon > 0$, we have some $M_\chi > 0$ big where $s^0 - \chi \leq s^*(\epsilon) < s^0$ for all $\epsilon \leq -M_\chi$, according to (2.4) and $R_0 > 1$. By combining the second equation of system (2.3), we get

$$\begin{aligned} \sigma i^*(\epsilon) &= d_2(J * i^*(\epsilon) - i^*(\epsilon)) + \lambda s(\epsilon - \sigma\varsigma) i(\epsilon - \sigma\varsigma) - (\mu + \rho) i^*(\epsilon), \\ &\geq d_2(J * i^*(\epsilon) - i^*(\epsilon)) + \lambda(s^0 - \epsilon) i(\epsilon - \sigma\varsigma)) - (\mu + \rho) i^*(\epsilon), \end{aligned} \tag{3.1}$$

regarding $\epsilon < -M_\epsilon$. Observing that traveling waves have asymptotic boundary conditions (2.4) and continuity, there are positive constants δ and M^0 such that, for every $\epsilon \in \mathbb{R}$, $s^*(\epsilon) \geq \delta$ and $i^*(\epsilon) \leq M^0$. By (**H**), we obtain that

$$\begin{aligned} &\frac{\lambda(s^0 - \epsilon) i^*(\epsilon - \sigma\varsigma)))}{\lambda s * (\epsilon - \sigma\varsigma) i^*(\epsilon - \sigma\varsigma)} \leq \frac{\lambda(s^0 - \epsilon) i^*(\epsilon - \sigma\varsigma)}{\lambda\delta i^*(\epsilon - \sigma\varsigma))} = \frac{\lambda(s^0 - \epsilon) i^*(\epsilon - \sigma\varsigma)) i^*(\epsilon - \sigma\varsigma)}{\lambda\delta i^*(\epsilon - \sigma\varsigma) i^*(\epsilon - \sigma\varsigma)}, \\ &\leq \frac{M^0}{\lambda\delta M^0} \lambda s^0 < \infty, \quad \epsilon > -M_\epsilon. \end{aligned}$$

A positive constant $i^-(\epsilon) > 0$ exists such that $i^*(\epsilon) \geq i^-$ for all $\epsilon \geq -M_\epsilon$, given that $i^*(\epsilon) > 0$ for $\epsilon \in \mathbb{R}$ and $i^*(+\infty) = i^* > 0$. As a result, we can select a constant $a > 1$ so that

$$\frac{\lambda(s^0 - \chi) i^*(\epsilon - \sigma\varsigma)}{(1 + i^*(\epsilon - \sigma\varsigma))^h} \leq \lambda s^*(\epsilon - \sigma\varsigma) i^*(\epsilon - \sigma\varsigma) \quad \text{for} \quad \epsilon > -M_\chi.$$

Then, for $\epsilon > -M_\chi$, the following inequality holds:

$$\sigma i^*(\epsilon) \geq d_2(J * i^*(\epsilon) - i^*(\epsilon)) + \frac{\lambda s^*(\epsilon - \sigma\varsigma) i^*(\epsilon - \sigma\varsigma)}{(1 + i(\epsilon - \sigma\varsigma))^a} - (\mu + \rho) i^*(\epsilon). \tag{3.2}$$

By combining (3.1) and (3.2), we obtain

$$\sigma i^*(\epsilon) \geq d_2(J * i^*(\epsilon) - i^*(\epsilon)) + \frac{\lambda(s^0 - \chi i^*(\epsilon - \sigma\varsigma)}{(1 + i^*(\epsilon - \sigma\varsigma))^a} - (\mu + \rho) i^*(\epsilon), \quad \epsilon \in \mathbb{R}. \tag{3.3}$$

Let $\Phi(x,t) = i^*(x + \sigma\varsigma)$ and $b(\Phi) = \inf_{\Phi \leq \varphi \leq M^0} \frac{\lambda(s^0 - \chi)\varphi}{(1+\varphi)^a}$. From (3.3), it is evident that

$$\begin{cases} \frac{\partial \Phi(x,t)}{\partial t} \geq d_2(J * \Phi(x,t) - \Phi(x,t)) + b(\Phi(x, t - \varsigma)) - (\rho + \mu)\Phi(x, t - \varsigma), \\ \Phi(x,t) = i^*(x + \sigma\varsigma), \quad x \in \mathbb{R}, t > 0. \end{cases}$$

Using the comparison principle [36], we get

$$\Phi(x,t) \geq \varphi(x,t), \quad x \in \mathbb{R}, t \geq 0, \tag{3.4}$$

when the solution of the equation is $\varphi(x,t)$

$$\begin{cases} \frac{\partial \varphi(x,t)}{\partial t} = d_2(J * \varphi(x,t) - \varphi(x,t)) + b(\varphi(x, t - \varsigma)) - (\rho + \mu)\varphi(x, t - \varsigma), \\ \varphi(x,t) = i^*(x + \sigma\varsigma), \quad x \in \mathbb{R}, t > 0. \end{cases} \tag{3.5}$$

We then demonstrate that for every $\hat{\sigma} \in (0, \sigma^*)$,

$$\lim_{t \to \infty} \inf_{|x| \leq \hat{\sigma} t} \varphi(x,t) > 0. \tag{3.6}$$

According to the asymptotic spreading concept [16]. We know that the operation $J_2 * \cdot$ [37, 38] can result in a C_0-semigroup. The only two equilibria that system (3.5) allows are $\varphi = 0$ and $b(\varphi^*) - (\mu + \rho)\varphi^* = 0$, each is a positive equilibrium $\varphi = \varphi^*$, proving that it is an ODE system equation. We write $C := C(\mathbb{R} \times [-\varsigma, 0])$ and $C_{\varphi^*} := \{\varphi \in C : 0 \leq \varphi \leq \varphi^*\}$. According to the semigroup theory [37, 38], the system (3.5) generates a monotone semi-flow $Q^t : C_{\varphi^*} \to C_{\varphi^*}$:

$$Q^t(\psi)(x) = \varphi(x + t, \varsigma), \quad x \in \mathbb{R}, t, \varsigma \geq 0, \psi \in C_{\varphi_i^*},$$

where the starting value is $\varphi(x, t - \varsigma) = \psi$, and $\varphi(x, t)$ is the sell solution of (3.5).

Indicate $C([-\varsigma, 0]) = \tilde{C}$ the formula is $\tilde{C}_{\varphi^*} = \{\varphi \in \tilde{C} : 0 \leq \varphi \leq \varphi^*\}$. The following delayed differential equation results in a solution semi-flow that is $\tilde{Q}^t : \tilde{C}_{\varphi^*} \to \tilde{C}_{\varphi^*}$.

$$\left\{ \frac{d\varphi(t)}{dt} = b(\varphi(t - \varsigma)) - (\rho + \mu)\varphi(t - \varsigma), \quad t > 0, \right.$$

where $\varphi^t = \varphi(t - \varsigma)$, and the starting value is $\varphi^0 = \psi^0 \in \tilde{C}_{\varphi^*}$. $\tilde{Q}_t$ is eventually strongly monotone on $\tilde{C}_{\varphi^*}$, according to Corollary 5.3.5 in [39]. Moreover, we derive that $\tilde{Q}_t$ is a highly monotone full orbit linking 0 to φ_i^* using the Dancer-Hess connecting orbit lemma [14]. Thus, hypothesis (A5) in [36] is true. Indeed, it is evident that $\tilde{Q}_t$ meets all of the assumptions (A1)–(A5) in [36] for every $t > 0$. It is evident that equation (3.5) is likewise satisfied by $\tilde{Q}_t$. The restriction of Q_t to $\tilde{C}_{\varphi^*}$ is thus also $\tilde{Q}_t$. This suggests that Theorem 2.17 in [36] can be used. As a result, we ultimately determine that (3.6) is true.

By selecting $\sigma_0 \in (\sigma, \sigma^*)$ and allowing $x = -\sigma_0 t$, (3.4) and (3.6) indicate that

$$\liminf_{t \to \infty} \Phi(x, t) \geq \liminf_{t \to \infty, |x| \leq \sigma_0 t} \nu(x, t) > 0. \tag{3.7}$$

The ultimate result is $\epsilon = x + \sigma t = (\sigma - \sigma_0)t \to -\infty$ as $t \to \infty$.

$$\lim_{t \to \infty} \Phi(x, t) = \lim_{t \to \infty} i^*(x + \sigma t) = \lim_{t \to \infty} i^*((\sigma - \sigma_0)t) = \lim_{\epsilon \to -\infty} i^*(\epsilon) = 0.$$

This is not consistent with (3.7). This completes the evidence. □

4 Noncritical TWS

We assume in this section that $\sigma > \sigma^*$. The following subsection are used to discuss if a traveling wave solution exists.

4.1 Upper and lower solution

Using an iterative process, we construct a pair of super- and subsolutions of (2.3) for $\sigma > \sigma^*$. The idea underlying such a structure is

Definition 4.1 (s^-, i^-) and (s^+, i^+) also represent pair of supper- and subsolutions of (2.3), respectively, and they both fulfil

$$-\sigma(s^+)'(\epsilon) + d_1(J * s(\epsilon) - s(\epsilon)) + \chi - \mu(s^+)(\epsilon) - \lambda(s^+)(\epsilon)(i^-)(\epsilon) + \ i^+(\epsilon) \leq 0, \tag{4.1}$$

$$-\sigma(s^-)'(\epsilon) + d_1(J * s(\epsilon) - s(\epsilon)) + \chi - \mu(s^-)(\epsilon) - \lambda(s^-)(\epsilon)(i^+)(\epsilon) \geq 0, \tag{4.2}$$

$$-\sigma (i^+)'(\epsilon) + d_2(J * i(\epsilon) - i(\epsilon)) + \lambda s^+(\epsilon - \sigma\varsigma)i^+(\epsilon - \sigma\varsigma) - (\mu + \rho)(i^+)(\epsilon) \leq 0, \tag{4.3}$$

$$-\sigma (i^-)'(\epsilon) + d_2(J * i(\epsilon) - i(\epsilon)) + \lambda s^-(\epsilon -_c \varsigma)i^-(\epsilon - \sigma\varsigma) - (\mu + \rho)(i^-)(\epsilon) \geq 0, \tag{4.4}$$

except for finite points of $\epsilon \in \mathbb{R}$.

Lemma 4.2 *Suppose that* $R_0 > 1$, *and* $\sigma > \sigma^*$. *Let*

$$s^+(\epsilon) = s^0, \qquad i^+(\epsilon) = \mathrm{e}^{\Lambda_1 \epsilon},$$
$$s^-(\epsilon) = \max\left\{s^0 - M\mathrm{e}^{\kappa\epsilon}, 0\right\}, \; i^-(\epsilon) = \max\{e^{\Lambda_1\epsilon}(1 - Le^{\eta\epsilon}), 0\},$$

for some positive constants κ, L, *then* (4.1)–(4.4) *are satisfied.*

Proof The following points are used to establish the evidence.

(i): Clearly $s^+(\epsilon) = s^0$ satisfies

$$-\sigma (s^+)'(\epsilon) + d_1(J * s(\epsilon) - s(\epsilon)) + \chi - \mu s^+(\epsilon) - \lambda s^+(\epsilon)i^-(\epsilon) + \rho i^-(\epsilon) \leq 0, \tag{4.5}$$

then, (4.1) is satisfied.

(ii) Clearly, $i^+(\epsilon) = \mathrm{e}^{\Lambda_1\epsilon}$, we prove that $i^+(\epsilon)$ fulfils (4.3). It is simple to verify that

$$\begin{aligned}
&d_2(J * i^+(\epsilon) - i^+(\epsilon)) + \lambda(s^+)(\epsilon - \sigma\varsigma)i^+(\epsilon - \sigma\varsigma) - (\mu + \rho)(i^+)(\epsilon) - \sigma (i^+)'(\epsilon), \qquad (4.6)\\
&\leq d_2(J * i^+(\epsilon) - i^+(\epsilon)) + \lambda s^0(i^+)(\epsilon - \sigma\varsigma) - (\mu + \rho)(i^+)(\epsilon) - \sigma (i^+)'(\epsilon),\\
&\leq -\sigma (i^+)'(\epsilon) + d_2 \int_{-\infty}^{+\infty} J(y)\mathrm{e}^{-\Lambda_1 y}dy + \lambda s^0 i^+(\epsilon - \sigma\varsigma) - (\mu + \rho + d_2)i^+(\epsilon),\\
&= d_2 \int_{-\infty}^{+\infty} J(y)\mathrm{e}^{-\Lambda_1 y}dy + \lambda s^0\mathrm{e}^{\Lambda_1(\epsilon - \sigma\varsigma)} - (\mu + \rho + d_2)\mathrm{e}^{\Lambda_1\epsilon} - \sigma\Lambda_1\mathrm{e}^{\Lambda_1\epsilon},\\
&= \mathrm{e}^{\Lambda_1\epsilon}F(\Lambda_1, \sigma),\\
&= 0,
\end{aligned}$$

by the Λ_1 definition.

(iii) Taking $0 < \gamma < \min\left\{\Lambda_1, \frac{\sigma}{d_2}\right\}$. Where $\epsilon \neq \frac{1}{\gamma} ln \frac{1}{M} := \epsilon^*$, and we assert that s^- fulfils

$$-\sigma (s^-)'(\epsilon) + d_1(J * s^-(\epsilon) - s^-(\epsilon)) + \chi - \mu(s^-)(\epsilon) - \lambda s^-(\epsilon)i^+(\epsilon) + \varepsilon i^+(\epsilon) \geq 0.$$

The inequality is directly proved by assuming that $\epsilon > \epsilon^*$, which implies that $s^-(\epsilon) = 0$ in (ϵ^*, ∞). We obtain $s^-(\epsilon) = s^0 - Me^{\gamma\epsilon}$ if $\epsilon < \epsilon^*$. We obtain $\lambda s(\epsilon)i(\epsilon) \leq \lambda s^0 i(\epsilon)$ by the concavity of $L(s(\epsilon), i(\epsilon)$. Next, we have

$$\begin{aligned}
&-\sigma (s^-)'(\epsilon) + d_1(J * s^-(\epsilon) - s^-(\epsilon)) + \chi - \mu(s^-)(\epsilon) - \lambda s^-(\epsilon)i^+(\epsilon) + \rho i^+(\epsilon) \geq 0,\\
&\geq \sigma M\gamma\mathrm{e}^{\gamma\epsilon} + d_1 M\mathrm{e}^{\gamma\epsilon} \int_{-\infty}^{+\infty} J(x)\mathrm{e}^{-\gamma x}dx + \chi - \mu(s^0 - M\mathrm{e}^{\gamma\epsilon}) - \lambda s^0(s^0 - M\mathrm{e}^{\gamma\epsilon}),\\
&= \mathrm{e}^{\gamma\epsilon}\left[\sigma M\gamma\mathrm{e}^{\gamma\epsilon} - d_1 M\mathrm{e}^{\gamma\epsilon} \int_{-\infty}^{+\infty} J(x)\mathrm{e}^{-\gamma x}dx + d_1 M\mathrm{e}^{\gamma\epsilon} - \lambda s^0\left(\frac{s^0}{M}\right)^{\frac{\Lambda-\gamma}{\gamma}}\right].
\end{aligned}$$

Here, we use

$$e^{\gamma\epsilon} < \left(\frac{s^0}{M}\right)^{\frac{\Lambda-\gamma}{\gamma}} \quad \text{for} \quad \epsilon < \epsilon^*.$$

Keeping $\gamma M = 1$ and letting $M \to \infty$ for some $M > s^0$ large enough and γ small enough, we have

$$-\sigma(s^-)'(\epsilon) + d_1(J * s^-(\epsilon) - s^-(\epsilon)) + \chi - \mu(s^-)(\epsilon) - \lambda s^-(\epsilon)i^+(\epsilon) + \gamma i^+(\epsilon) \geq 0.$$

The claim is proved.

(iv) Requiring $L > 0$ to be suitably big and $0 < \eta < \min\{\Lambda_2 - \Lambda_1, \Lambda_1\}$. Consequently, we assert that $i^-(\epsilon)$ satisfies

$$-\sigma(i^-)'(\epsilon) + d_2(J * s(\epsilon) - s(\epsilon)) + \lambda s^-(\epsilon - \sigma\varsigma)i^-(\epsilon - \sigma\varsigma) - (\mu + \rho)i^-(\epsilon) \geq 0, \tag{4.7}$$

with $\epsilon \neq \epsilon_2 := \frac{-lnL}{\eta}$.

We demonstrate this assertion for two distinct scenarios, $\epsilon > \epsilon_2$ and $\epsilon < \epsilon_2$, respectively. $i^-(\epsilon) = 0$ if $\epsilon > \epsilon_2$, indicating that (4.7) is met. $i^-(\epsilon) == e^{\Lambda_1\epsilon}(1 - Le^{\eta\epsilon})$ is obtained if $\epsilon < \epsilon_2$. Here, we demonstrate that (4.7) holds for sufficiently big L, which will be found later. Observe that the following is an expression for Inequality (4.7).

$$\begin{aligned}
&\lambda s^0 i^-(\epsilon - \sigma\varsigma) - \lambda s^-(\epsilon - \sigma\varsigma)i^-(\epsilon - \sigma\varsigma) \\
&\quad \leq -\sigma(i^-)'(\epsilon) + d_2(J * i^-(\epsilon) - i^-(\epsilon)) + \lambda s^0 i^-(\epsilon - \sigma\varsigma) \\
&\qquad - (\mu + \rho)i^-(\epsilon), \\
&\quad \leq -LF(\Lambda_1 + \eta, \sigma)e^{(\Lambda_1+\eta)\epsilon}.
\end{aligned} \tag{4.8}$$

Regarding every $\xi \in (0, \lambda s^0)$. For any $\epsilon < \epsilon_2$, i^- is a bounded function, hence $\delta_0 > 0$ satisfies $0 < i^- < \delta_0$. Since i^- is limited for $\epsilon < \epsilon_2$ and $\lambda s^0 > 0$, we get the existence of $\xi > 0$ very small, the following inequality, $\lambda s^- \geq \lambda s^0 - \xi > 0$, is true for each and every $0 < i^- < \delta_0$. We may exploit that $0 < i^- < \delta_0$ to obtain

$$\begin{aligned}
\lambda s^0 i^-(\epsilon - \sigma\varsigma) - \lambda s^-(\epsilon - \sigma\varsigma)i^-(\epsilon - \sigma\varsigma) &= \Big(\lambda s^0 - \lambda s^-(\epsilon - \sigma\varsigma)\Big)i^-(\epsilon - \sigma\varsigma), \\
&\leq \left(\frac{\lambda s^0 - \lambda s^-(\epsilon-\sigma\varsigma) + i^-(\epsilon-\sigma\varsigma)}{2}\right)^2 \\
&\leq \Big[\lambda s^0 - \big(\lambda s^0 - \xi\big) + i^-(\epsilon - \sigma\varsigma)\Big]^2.
\end{aligned} \tag{4.9}$$

Then, we have

$$\lambda s^0 i^-(\epsilon - \sigma\varsigma) - \lambda s^-(\epsilon - \sigma\varsigma)i^-(\epsilon - \sigma\varsigma) \leq i^{-2}(\epsilon - \sigma\varsigma).$$

So, it is enough to demonstrate that

$$(i^{-2})(\epsilon - \sigma\varsigma) \leq -LF(\Lambda_1 + \eta, c)\mathrm{e}^{(\Lambda_1+\eta)\epsilon}, \tag{4.10}$$

in order to prove Inequality (4.8).

We obtain $(i^{-}(\epsilon - \sigma\varsigma))^2 \leq e^{2\Lambda_1\epsilon}$ where $i^{-} \leq i^{+}$. For the sake of (4.10), we demonstrate that

$$e^{2\Lambda_1\epsilon} \leq -LF(\Lambda_1 + \eta, \sigma)\mathrm{e}^{(\Lambda_1+\eta)\epsilon}. \tag{4.11}$$

Inequality (4.11) holds for M sufficiently big as both of its sides trend to 0 as $\epsilon \to -\infty$ and are limited for any $\epsilon < \epsilon_2$. The proof is finished. □

4.2 Truncated problem

We take into consideration the following bounded set for every $\Upsilon > \max\{|\epsilon^*|, |\epsilon_0|, r\}$.

$$\begin{aligned}\Gamma_\Upsilon(\epsilon) = \Bigg\{ &(\phi(\epsilon)\varphi(\epsilon)) \in C\left([-\Upsilon, \Upsilon], \mathbb{R}^2\right) \Bigg| \phi(-\Upsilon) = s(-\Upsilon), \\ &\varphi - \Upsilon) = i(-\Upsilon), \;\; s^{-}(\epsilon) \leq \phi(\epsilon) \leq s^0, \;\; i^{-}(\epsilon) \leq \varphi(\epsilon) \leq i^{+}(\epsilon), \\ &\epsilon \in [-\Upsilon, \Upsilon]\Bigg\}.\end{aligned}$$

For any $(\phi, \varphi(\epsilon)) \in \Gamma_\Upsilon(\epsilon)$, we define

$$\hat{\phi}(\epsilon) = \begin{cases} \phi(\Upsilon), & \epsilon > \Upsilon, \\ \phi(\epsilon), & |\epsilon| \leq \Upsilon, \\ s^{-}(-\Upsilon), & \epsilon < -\Upsilon, \end{cases}$$

$$\hat{\varphi}(\epsilon) = \begin{cases} \varphi(\Upsilon), & \epsilon > \Upsilon, \\ \varphi(\epsilon), & |\epsilon| \leq \Upsilon, \\ i^{-}(-\Upsilon), & \epsilon < -\Upsilon, \end{cases}$$

$\Gamma_\Upsilon(\epsilon)$ is clearly a closed and convex set. It is satisfied that $\left(\hat{\phi}(\epsilon), \hat{\varphi}(\epsilon)\right)$

$$s^{-}(\epsilon) \leq \hat{\phi}(\epsilon) \leq s^0, \quad i^{-}(\epsilon) \leq \hat{\varphi}(\epsilon) \leq i^{+}(\epsilon), \quad \epsilon \in \mathbb{R}.$$

We omit the truncated problem

$$\begin{cases} \sigma s'(\epsilon) = d_1((J * \hat{\phi})(\epsilon) - s(\epsilon)) + \chi - \mu s(\epsilon) - \beta s(\epsilon)\varphi(\epsilon) + \rho i(\epsilon), \\ \sigma i'(\epsilon) = d_2(\left(J * \hat{\varphi}\right)(\epsilon) - i(\epsilon)) + \lambda\hat{\phi}(\epsilon - \sigma\varsigma)\hat{\varphi}(\epsilon - \sigma\varsigma) - (\mu + \rho)i(\epsilon), \end{cases} \tag{4.12}$$

with

$$s(-\Upsilon) = s^{-}(-\Upsilon), \quad i(-\Upsilon) = i^{-}(-\Upsilon). \tag{4.13}$$

The generic differential equation results guarantee that a unique nonnegative solution $(s_\Upsilon(\epsilon), i_\Upsilon(\epsilon))$ can be found for the starting value problems (4.12) and (4.13) created for

$\epsilon \in [-\Upsilon, \Upsilon]$. Thus, we define the solution map $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$ on $\Gamma_X(\epsilon)$ as follows:

$$\mathcal{F}_1(\phi, \varphi) = s_\Upsilon, \qquad \mathcal{F}_2(\phi, \varphi) = i_\Upsilon.$$

Lemma 4.3 *For all $\Upsilon > \max\{|\epsilon^*|, |\epsilon_0|, r\}$, map $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2) : \Gamma_\Upsilon(\epsilon) \to \Gamma_\Upsilon(\epsilon)$.*

Lemma 4.2 and the comparison principle can be used to infer Lemma 4.3. As an illustration, we can consult [31, Proposition 2.1]

Lemma 4.4 *The map $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2) : \Gamma_\Upsilon(\epsilon) \to \Gamma_\Upsilon(\epsilon)$ is completely continuous.*

Proof We can determine from (4.12) that $(s_\Upsilon(\epsilon), i_\Upsilon(\epsilon)) \in C^1([-\Upsilon, \Upsilon], \mathbb{R}^2)$ for any $(\phi, \varphi) \in \Gamma_\Upsilon(\epsilon)$. Thus, the Arzelà–Ascoli theorem may be used to infer that the map $\mathcal{F}$ is compact. The continuity of $\mathcal{F}$ is then examined.

Suppose that $s_{\Upsilon,k}(\epsilon) = \mathcal{F}_1(\phi_k, \varphi_k)(\epsilon$, for $\epsilon \in [-\Upsilon, \Upsilon]$, $i_{\Upsilon,k}(\epsilon) = \mathcal{F}_2(\phi_k, \varphi_k)(\epsilon)$, where $(\phi_k(\epsilon), \varphi_k(\epsilon)) \in \Gamma_\Upsilon(\epsilon)$ $(k = 1, 2)$. First, we determine if $\mathcal{F}_1$ is continuous. Lemma 4.2 and the comparison principle may be used to infer Lemma 4.3 from the first equation of (4.12). As an illustration, we can consult [31, Proposition 2.1]

$$\begin{aligned} &\sigma(s'_{\Upsilon,1}(\epsilon) - s'_{\Upsilon,2}(\epsilon)) + (d_1 + \mu)(s_{\Upsilon,1}(\epsilon) - s_{\Upsilon,2}(\epsilon)) - \rho(\varphi_2(\epsilon) - \varphi_1(\epsilon)) \\ &= d_1 \int_{\mathbb{R}} J(y)(\hat{\phi}_1(\epsilon - y) - \hat{\phi}_2(\epsilon - y))dy + \lambda s_{\Upsilon,2}(\epsilon)\varphi_2(\epsilon) - \lambda s_{\Upsilon,1}(\epsilon)\varphi_1(\epsilon). \end{aligned} \tag{4.14}$$

Since

$$\int_{\mathbb{R}} J(y)\hat{\phi}(\epsilon - y)dy = \int_{-\infty}^{-\Upsilon} J(\epsilon - y)s(y)dy + \int_{-\Upsilon}^{\Upsilon} J(\epsilon - y)\phi(y)dy + \int_{\Upsilon}^{+\infty} J(\epsilon - y)\phi(\Upsilon)dy,$$

we have

$$\left| \int_{\mathbb{R}} J(y)(\hat{\phi}_1(‘\epsilon - y) - \hat{\phi}_2(\epsilon - y))dy \right| \le 2 \max_{y\in[-\Upsilon,\Upsilon]} |\phi_1(y) - \phi_2(y)|. \tag{4.15}$$

Since $i^+(\epsilon) \le e^{\Lambda_1\epsilon}$ for $\epsilon \in [-\Upsilon, \Upsilon]$, for any $(\phi_1, \varphi_1), (\phi_2, \varphi_2) \in \Gamma_\Upsilon(\epsilon)$, then

$$\left| \lambda\phi_1(\epsilon)\varphi_1(\epsilon) - \lambda\phi_2(\epsilon)\varphi_2(\epsilon) \right| \le M_4 \left[\left|\phi_1(\epsilon) - \phi_2(\epsilon)\right| + \left|\varphi_1(\epsilon) - \varphi_2(\epsilon)\right| \right], \tag{4.16}$$

with $M_4 = \sup\left\{\lambda s^0, \lambda\sigma e^{\Lambda_1\epsilon} : 0 \le \sigma \le s^0\right\}$.

Put $u(\epsilon) = \sigma|s_{\Upsilon,1}(\epsilon) - s_{\Upsilon,2}(\epsilon)|$. Therefore, from (4.14)–(4.16), we get

$$\begin{aligned} u'(\epsilon) = {} & \sigma \,\mathrm{sign}(s_{\Upsilon,1}(\epsilon) - s_{\Upsilon,2}(\epsilon))(s'_{X,1}(\epsilon) - s'_{\Upsilon,2}(\epsilon)), \\ & \le 2d_1 \max_{y\in[-\Upsilon,X]} |\phi_1(y) - \phi_2(y)| - (d_1 + \mu - M_4)|s_{\Upsilon,1}(\epsilon) - s_{\Upsilon,2}(\epsilon)| \\ & \quad + M_4|\varphi_2(\epsilon) - \varphi_1(\epsilon)| - \rho|\varphi_2(\epsilon) - \varphi_1(\epsilon)|, \\ & = \left(\tfrac{d_1+\mu}{\sigma} + \tfrac{M_4}{\sigma}\right)u(\epsilon) + 2d_1 \max_{y\in[-\Upsilon,\Upsilon]} |\phi_1(y) - \phi_2(y)| + (M_4 - \rho)|\varphi_2(\epsilon) - \varphi_1(\epsilon)|. \end{aligned}$$

Thus, for all $\epsilon \in [-\Upsilon, \Upsilon]$, we obtain

$$u(\epsilon) \leq u(-\Upsilon)e^{-\left(\frac{d_1+\mu}{\sigma}+\frac{M_4}{\sigma}\right)(\epsilon+X)} + \int_{-\Upsilon}^{\epsilon}\Bigg[\left(2d_1 \max_{y\in[-\Upsilon,\Upsilon]}|\phi_1(y)-\phi_2(y)|\right) \\ +M_4 \max_{y\in[-\Upsilon,\Upsilon]}|\varphi_1(y)-\varphi_2(y)|\Bigg]e^{-\left(\frac{d_1+\mu}{\sigma}+\frac{M_4-\rho}{\sigma}\right)(\epsilon-\tau)}d\tau. \tag{4.17}$$

From (4.17), we ultimately obtain $\|u(\epsilon)\|_{\Gamma_\Upsilon(\epsilon)} \to 0$ since $u(-\Upsilon) = 0$. This is equivalent to $\|(\phi_2,\varphi_2)-(\phi_1,\varphi_1)\|_{\Gamma_\Upsilon(\epsilon)} \to 0$. $\mathcal{F}_1$ is therefore continuous on $\Gamma_\Upsilon(\epsilon)$. Using a similar logic, we determine that $\mathcal{F}_2$ is continuous. □

Lemmas 4.3 and 4.4, Schauder's fixed point theorem, and the fact that $\Gamma_\Upsilon(\epsilon)$ is closed and convex make the following conclusion true.

Theorem 4.5 *$\mathcal{F}$ admits at least one fixed point $(s^*_\Upsilon(\epsilon), i^*_\Upsilon(\epsilon)) \in \Gamma_\Upsilon(\epsilon)$.*

This is followed by several previous estimations for the fixed point $(s^*_\Upsilon(\epsilon), i^*_\Upsilon(\epsilon))$. The value of $\mathcal{F}$ in $C^{1,1}([-\Upsilon,\Upsilon],\mathbb{R}^2)$, in which

$$C^{1,1}([-\Upsilon,\Upsilon]) = \{u \in C^1([-\Upsilon,\Upsilon],\mathbb{R}^2) : u \text{ and } u' \text{ are Lipschitz continuous}\},$$

endowed with the norm

$$\|u\|_{C^{1,1}([-\Upsilon,\Upsilon])} = \max_{x\in[-\Upsilon,\Upsilon]}|u(x)| + \max_{x\in[-\Upsilon,\Upsilon]}|u'(x)| + \sup_{x,y\in[-`\Upsilon,\Upsilon],x\neq y}\frac{|u'(x)-u'(y)|}{|x-y|}. \tag{4.18}$$

Then, we get the result below.

Lemma 4.6 *Let $(s^*_\Upsilon(\epsilon), i^*_\Upsilon(\epsilon))$ be the fixed point of map $\mathcal{F}$. Therefore, $\|s^*_X(\epsilon)\|_{C^{1,1}([-\Upsilon,\Upsilon])} \leq C$ and $\|i^*_\Upsilon(\epsilon)\|_{C^{1,1}([-\Upsilon,\Upsilon])} \leq C$, $\forall \Upsilon > \max\{|\epsilon^*|, |\epsilon_0|, r\}$.*

Proof Obviously, we have

$$\begin{cases} \sigma s^{*'}_\Upsilon(\epsilon) = d_1 J * s^*_\Upsilon(\epsilon) - d_1 s^*_\Upsilon(\epsilon) + \chi - \mu s^*_\Upsilon(\epsilon) - \lambda s^*_\Upsilon(\epsilon) i^*_\Upsilon(\epsilon) + \rho i^*_\Upsilon(\epsilon), \\ \sigma i^{*'}_\Upsilon(\epsilon) = d_2 J * i^*_\Upsilon(\epsilon) - (d_2+\mu+\rho) i^*_\Upsilon(\epsilon) + \lambda s^*_\Upsilon(\epsilon - \sigma\varsigma) i^*_\Upsilon(\epsilon - \sigma\varsigma), \end{cases} \tag{4.19}$$

for $\epsilon \in [-\Upsilon, \Upsilon]$, where

$$\hat{s}_\Upsilon(\epsilon) = \begin{cases} s^*_\Upsilon(\Upsilon), \epsilon > \Upsilon, \\ s^*_\Upsilon(\epsilon), |\epsilon| \leq \Upsilon, \\ s^-(\Upsilon), \epsilon < -\Upsilon, \end{cases} \quad \hat{i}_\Upsilon(\epsilon) = \begin{cases} i^*_\Upsilon(X), \epsilon > \Upsilon, \\ i^*_\Upsilon(\epsilon), |\epsilon| \leq \Upsilon, \\ i^-(\Upsilon), \epsilon < -\Upsilon. \end{cases}$$

Since $s^*_\Upsilon(\epsilon) \leq s^0$ and $i^*_\Upsilon(\epsilon) \leq \mathrm{e}^{\Lambda_1\epsilon}$ for $\epsilon \in [-\Upsilon, \Upsilon]$, and (4.19), we can obtain

$$|s^{*'}_\Upsilon(\epsilon)| \leq \frac{1}{\sigma}(2d_1 s^0 + \chi + \mu s^0 + \lambda s^0 \mathrm{e}^{\Lambda_1\epsilon} := L_1, \tag{4.20}$$

$$|i^{*'}_\Upsilon(\epsilon)| \leq \frac{1}{\sigma}(2d_2 \mathrm{e}^{\Lambda_1\epsilon} + (\mu+\rho)\mathrm{e}^{\Lambda_1\epsilon} + \lambda s^0 \mathrm{e}^{\Lambda_1\epsilon} := L_2. \tag{4.21}$$

Thus,

$$|s_\Upsilon^*(\epsilon) - s_\Upsilon^*(\eta)| \le L_1|\epsilon - \eta|, \quad |i_\Upsilon^*(\epsilon) - i_\Upsilon^*(\eta)| \le L_2|\epsilon - \eta|. \tag{4.22}$$

By (4.19), (4.20), and (4.21), we further obtain

$$\begin{aligned} &\sigma|s_\Upsilon^{*'}(\epsilon) - s_\Upsilon^{*'}(\eta)| \\ &\quad\le d_1\int_{-\infty}^{+\infty} J(y)(\hat{s}_\Upsilon(\epsilon - y) - \hat{s}_\Upsilon(\eta - y))\,dy + (d_1 + \mu)|s_\Upsilon^*(\epsilon) - s_\Upsilon^*(\eta)| \\ &\qquad + \left|\lambda s_\Upsilon^*(\epsilon)i_\Upsilon^*(\epsilon) - \lambda s_\Upsilon^*(\eta)i_\Upsilon^*(\eta)\right| - \rho\left|i_\Upsilon^*(\epsilon) - i_\Upsilon^*(\eta)\right|, \end{aligned} \tag{4.23}$$

and

$$\begin{aligned} &\sigma|i_\Upsilon^{*'}(\epsilon) - i_\Upsilon^{*'}(\eta)| \\ &\quad\le d_2\int_{-\infty}^{+\infty} J(y)(\hat{i}_\Upsilon(\epsilon - y) - \hat{i}_\Upsilon(\eta - y))\,dy + (d_2 + \mu + \rho)|i_\Upsilon^*(\epsilon) - i_\Upsilon^*(\eta)| \\ &\qquad + \left|\lambda s_\Upsilon^*(\epsilon - \sigma\varsigma)i_\Upsilon^*(\epsilon - \sigma\varsigma) - \lambda s_\Upsilon^*(\eta - \sigma\varsigma)i_\Upsilon^*(\eta - \sigma\varsigma)\right|. \end{aligned} \tag{4.24}$$

Let $[-r, r]$ be $J(x)$'s compact support. Since $J(x)$ is a C^1-function, $J(x) \le L_J$ is verified by the constant $L_J > 0$. $\forall x_1, x_2 \in [-r, r]$, and therefore $|J_n(x_1) - J_n(x_2)| \le L_J|x_1 - x_2|$. Consequently, we deduce that

$$\begin{aligned} \int_{-\infty}^{+\infty} J(y)\hat{s}_\Upsilon(\epsilon - y)\,dy - \int_{-\infty}^{+\infty} J(y)\hat{s}_\Upsilon(\eta - y)\,dy &= \int_{\eta-r}^{\epsilon-r} J(y)s_\Upsilon(y)\,dy + \int_{\epsilon+r}^{\eta+r} J(y)s_\Upsilon(y)\,dy \\ &\quad + \int_{\eta-r}^{\epsilon+r} (J_1(y - \eta) - J(y - \epsilon))s_\Upsilon(y)\,dy \\ &\le 4L_J rs^0|\epsilon - \eta|. \end{aligned}$$

Likewise, we get

$$\int_{-\infty}^{+\infty} J(y)\hat{i}_\Upsilon(\epsilon - y)\,dy - \int_{-\infty}^{+\infty} J(y)\hat{i}_\Upsilon(\eta - y)\,dy \le 4L_J re^{\Lambda_1\epsilon}|\epsilon - \eta|.$$

Then, it follows from (4.16) and (4.22) that

$$\left|\lambda s_\Upsilon^*(\epsilon - \sigma\varsigma)i_\Upsilon^*(\epsilon - \sigma\varsigma) - \lambda s_\Upsilon^*(\eta - \sigma\varsigma)i_\Upsilon^*(\eta - \sigma\varsigma))\right| \le M_4(L_1 + L_2)|\epsilon - \eta|. \tag{4.25}$$

Combining (4.23)–(4.25), we know

$$|s_\Upsilon^{*'}(\epsilon) - s_\Upsilon^{*'}(\eta)| \le C_s|\epsilon - \eta| \quad \text{and} \quad |i_\Upsilon^{*'}(\epsilon) - i_\Upsilon^{*'}(\eta)| \le C_i|\epsilon - \eta|,$$

where

$$\begin{aligned} C_s &= \frac{1}{\sigma}(4d_1L_J rs^0 + (d_1 + \mu)L_1 + M_4(L_1 + L_2)), \\ C_i &= \frac{1}{\sigma}(4d_2L_J re^{\Lambda_1\epsilon} + (d_2 + \mu + \rho)L_2 + M_4(L_1 + L_2)). \end{aligned}$$

Consequently, we have that

$$\|s_X^*(\epsilon)\|_{C^{1,\epsilon}([-X,X])} \le C$$

and

$$\|i_X^*(\epsilon)\|_{C^{1,\epsilon}([-X,X])} \le C_i,$$

where $C = \max\{s^0 + L_1 + C_s, e^{\Lambda_1\epsilon} + L_2 + C_i\}$. □

4.3 Existence of a noncritical TWS

Theorem 4.7 *If $R_0 > 1$ and $\sigma > \sigma^*$, then (2.3) has a solution $(s^*(\epsilon), i^*(\epsilon))$ defined for $\epsilon \in \mathbb{R}$ that satisfies $s^-(\epsilon) \le s^*(\epsilon) \le s^0$, $i^-(\epsilon) \le i^*(\epsilon) \le i^+(\epsilon)$ for $\epsilon \in \mathbb{R}$.*

Proof A series $\{\Upsilon_n\}_{n=1}^{\infty}$ that fulfils $\lim_{n\to\infty} \Upsilon_n = +\infty$ and $\Upsilon_n > \max\{|\epsilon^*|, |\epsilon_0|, r\}$ should be defined. Schauder's fixed point theorem states that for every Υ_n, there is a fixed point $(s^*_{\Upsilon_n}(\epsilon), i^*_{\Upsilon_n}(\epsilon)) \in \Gamma_{\Upsilon_n}(\epsilon)$ of map $\mathcal{F}$. Lemma 4.6 states that $n = 1, 2, \dots,$ and $\|s^*_{\Upsilon_n}(\epsilon)\|_{\mathbb{C}[1-\alpha,\Upsilon_n,\Upsilon_n]} \le C_i$ and $\|i^*_{\Upsilon_n}(\epsilon)\|_{\mathbb{C}[1-\alpha,\Upsilon_n,\Upsilon_n]} \le C$. The uniform boundedness and equicontinuity of $\{(s^{*'}_{\Upsilon_n}(\epsilon), i^{*'}_{\Upsilon_n}(\epsilon))$ and $\{(s^*_{\Upsilon_n}(\epsilon) i^*_{\Upsilon_n}(\epsilon))$ for any integer k are due to the fact that $n \ge k$. Therefore, the Arzelà-Ascoli theorem and the diagonal extraction technique ensure that a subsequence $\{(s^*_{\Upsilon_m}(\epsilon), i^*_{\Upsilon_m}(\epsilon))\}$ fulfilling

$\{(s^{*'}_{\Upsilon_m}(\epsilon), i^{*'}_{\Upsilon_m}(\epsilon))\}$ converge uniformly in each $[-X_k, X_k]$. $(k = 1, 2, \dots)$, $m \to \infty$.

Let $\lim_{m\to\infty}(s^*_{\Upsilon_m}(\epsilon), i^*_{\Upsilon_m}(\epsilon)) = (s^*(\epsilon), i^*(\epsilon))$, then we have

$\lim_{m\to\infty}(s^{*'}_{\Upsilon_m}(\epsilon), i^{*'}_{\Upsilon_m}(\epsilon)) = (s^{*'}(\epsilon), i^{*'}(\epsilon))$. Let r be the supported radius of $J_1(\epsilon)$ and $J_2(\epsilon)$. Since $(s^*_{\Upsilon_m}(\epsilon), i^*_{\Upsilon_m}(\epsilon)) \le (s^{+*}(\epsilon), i^{+*}(\epsilon))$ for $\epsilon \in \mathbb{R}$ and $m = 1, 2, \dots,$ using the Lebesgue dominated convergence theorem, it follows that

$$\lim_{m\to\infty} \int_{\mathbb{R}} J(\epsilon) s^*_{\Upsilon_m}(\epsilon - y) d\epsilon = \lim_{m\to\infty} \int_{-r}^{r} J(\epsilon) s^*_{\Upsilon_m}(\epsilon - y) d\epsilon = J * s^*(\epsilon).$$

Likewise, $\lim_{m\to\infty} i^*_{\Upsilon_m}(\epsilon) = J * i^*(\epsilon)$ may be obtained. Consequently, $(s^*(\epsilon), i^*(\epsilon))$ satisfies (2.3), and for $\epsilon \in \mathbb{R}$, $s^-(\epsilon) \le s^*(\epsilon) \le s^0$ and $i^-(\epsilon) \le i^*(\epsilon) \le i^+(\epsilon)$.

The next step demonstrates that $s^0 > s^*(\epsilon) > 0$ and $i^*(\epsilon) > 0$. Given that $s(-\infty) = s^0 > 0$, if $\epsilon_{00} \in \mathbb{R}$ exists, $\forall \epsilon \in (-\infty, \epsilon_{00})$, then $s'(\epsilon_{00}) \le 0$, confirming that $s(\epsilon_{00}) = 0$ and $s(\epsilon) > 0$. The first equation of (2.3) provides

$$d_1 \int_{-\infty}^{+\infty} J(y) s(\epsilon_{00} - y) dy + \chi \le 0.$$

This is a contradiction. Thus, $s^*(\epsilon) > 0$, $\forall \epsilon \in \mathbb{R}$. Likewise, we obtain $i^*(\epsilon) > 0$, $\forall \epsilon \in \mathbb{R}$. Now, we prove $s^*(\epsilon) < s^0$. Assume that there is $\epsilon_{00} \in \mathbb{R}$ satisfying $s^*(\epsilon_{00}) = s^0$, then, $s^{*'}(\epsilon_{00}) \ge 0$. Together with the first equation of (2.3), it yields

$$d_1 \int_{-\infty}^{+\infty} J(y)(s(\epsilon_{00} - y) - s^0) dy + \chi - \mu s^0 - \lambda s^*(\epsilon_{00}) i^*(\epsilon_{00}) + \rho i^*(\epsilon_{00}) \ge 0,$$

that is,

$$d_1 \int_{-\infty}^{+\infty} J(y)(s(\epsilon_{00} - y) - s^0) dy \lambda s^*(\epsilon_{00}) i^*(\epsilon_{00})) \ge 0,$$

this is a contradiction with $s^*(\epsilon_{00} - y) - s^0 \le 0$ and $L(s^*(\epsilon_{00}), i^*(\epsilon_{00})) > 0$. Thus, $s^*(\epsilon) < s^0$, $\forall \epsilon \in \mathbb{R}$. □

Theorem 4.8 *Let $R_0 > 1$ and $\sigma > \sigma^*$, then (2.3) has a solution $(s^*(\epsilon), i^*(\epsilon))$ defined for $\epsilon \in \mathbb{R}$ satisfying $\lim_{\epsilon\to\infty}(s^*(\epsilon), i^*(\epsilon)) = (s^0, 0)$, $0 < s^*(\epsilon) \le s^0$, and $i^*(\epsilon) > 0$ for $\epsilon \in \mathbb{R}$.*

Proof By Theorem 4.7, there is a solution sequence $\Phi_n(\epsilon) = (s_n^*(\epsilon), i_n^*(\epsilon))$, $n \in \mathbb{N}^*$ and $\epsilon \in \mathbb{R}$, verifying

$$\begin{cases} \sigma s_n^{*'}(\epsilon) = d_1 J * s_n^*(\epsilon) - d_1 s_n^*(\epsilon) + \chi - \mu s_n^*(\epsilon) - \lambda s_n^*(\epsilon) i_n^*(\epsilon) + \rho i_n^*(\epsilon), \\ \sigma i_n^{*'}(\epsilon) = d_2 J * i_n^*(\epsilon) - (d_2 + \mu + \rho) i_n^*(\epsilon) + \lambda s_n^*(\epsilon - \sigma\varsigma) i_n^*(\epsilon - \sigma\varsigma), \end{cases} \tag{4.26}$$

and

$$s(\epsilon) < s_n^*(\epsilon) \le s^0, \quad i(\epsilon) \le i_n^*(\epsilon) \le i(\epsilon), \quad s_n^*(\epsilon) > 0, \quad i_n^*(\epsilon) > 0, \quad \epsilon \in \mathbb{R},$$

because $(e^{\Lambda_1 \epsilon})_n \to +\infty$ in the range $[-1, 1]$. $\{\Phi_n(\epsilon)\}$ is uniformly confined on $[-1, 1]$ as a result. We guarantee that $\{\Phi_n(\epsilon)\}$ and $\{\Phi_n'(\epsilon)\}$, $n > n_1$, are both equicontinuous and uniformly bounded on $[-1, 1]$ by (4.26). According to the Arzelà-Ascoli theorem, $\{\Phi_{1,m}(\epsilon)\}$ is a subsequence of $\{\Phi_n(\epsilon)\}$ that satisfies $\{\Phi_{1,m}(\epsilon)\}$ and $\{\Phi_{1,m}'(\epsilon)\}$, which converges uniformly on $[-1, 1]$ as $m \to \infty$. $i_{1,m}'(\epsilon) \le e^{1+\epsilon}$, $\forall \epsilon \in [-1, 1]$, is also true.

Subsequences $\{\Phi_{k-1,m}(\epsilon)\}$ of $\{\Phi_{k-2,m}(\epsilon)\}$ satisfying $\{\Phi_{k-1,m}(\epsilon)\}$ and $\{\Phi_{k-1,m}'(\epsilon)\}$ converge uniformly on $[-k-1, k-1]$ when $m \to \infty$ are chosen in $[-k-1, k-1]$. $(e^{\Lambda_1 \epsilon})_{k-1,m} \le e^{1+\epsilon}$, $\forall \epsilon \in [-k-1, k-1]$, are also exist. As a result, in $[-k, k]$, we obtain $i_{k,m}^*(\epsilon) \le e^{1+\epsilon}$, $\epsilon \in [-k, k]$, as $(e^{\Lambda_1 \epsilon})_{k-1,m}$ is uniformly confined on $[-k, k]$.

Therefore, $\{\Phi_{k,m}(\epsilon)\}$ is uniformly confined on $[-k, k]$ for $m > m_k$. Both $\{\Phi_{k-1,m}(\epsilon)\}$ and $\{\Phi_{k-1,m}'(\epsilon)\}$ are equicontinuous and uniformly bounded on $[-k, k]$, as demonstrated by the proof of Lemma 4.6. $\{\Phi_{k,m}(\epsilon)\}$ is a subsequence of $\{\Phi_{k-1,m}(\epsilon)\}$ that satisfies $\{\Phi_{k,m}(\epsilon)\}$ and $\{\Phi_{k,m}'(\epsilon)\}$ converges uniformly on $[-k, k]$ as a result for $m \to \infty$. $i_{k,m}^*(\epsilon) \le e^{\Lambda_1 \epsilon}$, $\forall \epsilon \in [-k, k]$. Moreover, the diagonal extraction approach suggests that any $[-k, k] (k = 1, 2, 3, \dots)$ has subsequences $\{\Phi_{m,m}(\epsilon)\}$ and $\{\Phi_{m,m}'(\epsilon)\}$ that converge uniformly. Let $\{\Phi_{m,m}(\epsilon)\} \to (s^*(\epsilon), i^*(\epsilon))$ be $m \to +\infty$. $\{\Phi_{m,m}'(\epsilon)\} \to (s^{*'}(\epsilon), i^{*'}(\epsilon))$ for $m \to +\infty$, as a result. Since for every $m \in \mathbb{N}^*$, we have

$$\begin{cases} \sigma s_{m,m}^{*'}(\epsilon) \\ \quad = d_1 J * s_{m,m}^*(\epsilon) - d_1 s_{m,m}^*(\epsilon) + \chi - \mu s_{m,m}^*(\epsilon) - \lambda s_{m,m}^*(\epsilon) i_{m,m}^*(\epsilon) + \rho i_{m,m}^*(\epsilon), \\ \sigma i_{m,m}^{*'}(\epsilon) \\ \quad = d_2 J * i_{m,m}^*(\epsilon) - (d_2 + \mu + \rho) i_{m,m}^*(\epsilon) + \lambda s_{m,m}^*(\epsilon - \sigma\varsigma) i_{m,m}^*(\epsilon - \sigma\varsigma). \end{cases} \tag{4.27}$$

Taking $m \to +\infty$ and using the continuity of $\lambda s(\epsilon) i(\epsilon)$ function and the dominated convergence theorem gives

$$\begin{cases} \sigma s^{*'}(\epsilon) = d_1 J * s^*(\epsilon) - d_1 s^*(\epsilon) + \chi - \mu s^*(\epsilon) - \lambda s^*(\epsilon) i^*(\epsilon) + \rho i^*(\epsilon), \\ \sigma i^{*'}(\epsilon) = d_2 J * i^*(\epsilon) - (d_2 + \mu + \rho) i^*(\epsilon) + \lambda s^*(\epsilon - \sigma\varsigma) i^*(\epsilon - \sigma\varsigma), \end{cases} \tag{4.28}$$

for each and every $\epsilon \in \mathbb{R}$. In other words, the solution to (2.3) for $\epsilon \in \mathbb{R}$ is $(s^*(\epsilon), i^*(\epsilon))$. $s(\epsilon) < s^*(\epsilon) \le s^0$ and $i(\epsilon) \le i^*(\epsilon)$, $\epsilon \in \mathbb{R}$ are the results of (4.26). We additionally obtain $i^*(\epsilon) \le e^{\Lambda_1 \epsilon}$, $\epsilon \in \mathbb{R}$, $i_{m,m}^*(\epsilon) \le e^{\Lambda_1 \epsilon}$, $\forall \epsilon \in [-k, k]$, and $m \ge k$. $(s^*(\epsilon), i^*(\epsilon))$ confirms

$\lim_{\epsilon\to-\infty}(s^*(\epsilon),i^*(\epsilon))=(s^0,0)$, according to the upper-lower solutions. As in Theorem 4.7, we similarly obtain $0<s^*(\epsilon)<s^0$, $\forall\epsilon\in\mathbb{R}$. Likewise, if $\epsilon'\in\mathbb{R}$, then $i^*(\epsilon')=0$ is verified, and $i^*(\epsilon)>0$, $\forall\epsilon\in(-\infty,\epsilon')$. It is evident that $i^{*'}(\epsilon')\leq 0$ for $\epsilon'>\epsilon_0$. The second equation of (4.28) then gives us

$$\sigma i^{*'}(\epsilon')=d_2J*i^*(\epsilon')-(d_2+\mu+\rho)i^*(\epsilon')+\lambda s^*(\epsilon'-\sigma\varsigma)i^*(\epsilon'-\sigma\varsigma)>0.$$

This is a contradiction. Then, $i^*(\epsilon)>0$, $\forall\epsilon\in\mathbb{R}$. □

The definition of $(s^*(\epsilon),i^*(\epsilon))$ may be found in Theorem 4.8. $(s^*(\epsilon),i^*(\epsilon))\to(s^*,i^*)$ as $\epsilon\to+\infty$ in order to get the asymptotic boundary condition. Using the Lyapunov-LaSalle theorem, we must demonstrate that $(s^*(\epsilon),i^*(\epsilon))\to(s^*,i^*)$ as $\epsilon\to\infty$ in order to establish the existence of noncritical TWS. The following highlights the results that were obtained

Lemma 4.9 $(s^*(\epsilon),i^*(\epsilon))\to(s^*,i^*)$ *uniformly as* $\epsilon\to+\infty$.

Proof We define the Lyapunov functional V by

$$V(\epsilon)=V_1(\epsilon)+V_2(\epsilon), \tag{4.29}$$

where

$$V_1(\epsilon)=\sigma\left(\tfrac{1}{2}(s(\epsilon)-s^*)^2+\tfrac{\mu+\rho}{\lambda}i^*h\left(\tfrac{i(\epsilon)}{i^*}\right)\right)+d_1s^*K_1(\epsilon)+d_2i^*\tfrac{\mu+\rho}{\lambda}K_2(\epsilon),$$
$$V_2(\epsilon)=\tfrac{\mu+\rho}{\lambda}\lambda s^*i^*\int_0^{\sigma\varsigma}h\left(\frac{s(\epsilon-\varepsilon)i(\epsilon-\varepsilon)}{s^*i^*}\right)d\varepsilon.$$

It is evident that $h(x)>0$ for every $x>0$ when $h(x)=x-1-ln(x)$, $x\in\mathbb{R}^+$. With

$$\int_0^{+\infty}a^+h\left(\frac{s(\epsilon-y)}{s^*}\right)dy-\int_{-\infty}^0a^-h\left(\frac{s(\epsilon-y)}{s^*}\right)dy,$$

and

$$K_2(\epsilon)=\int_0^{+\infty}a^+h\left(\frac{i(\epsilon-y)}{i^*}\right)dy-\int_{-\infty}^0a^-h\left(\frac{i(\epsilon-y)}{i^*}\right)dy,$$

$s(\epsilon)>0, i(\epsilon)>0$, and [40, Theorem 1] allow us to conclude that $K_1(\epsilon),K_2(\epsilon)$ are limited from below. Thus, $V(s,i)(\epsilon)$ is bounded from below and properly defined. We have $a^\pm$ satisfying $a^\pm(0)=\frac{1}{2}$, $\frac{da^+(y)}{dy}=J(y)$, and $\frac{da^-(y)}{dy}=-J(y)$.

$$\frac{dK_1(\epsilon)}{d\epsilon}=h\left(\frac{s}{s^*}\right)-\int_{-\infty}^{+\infty}J(y)h\left(\frac{s(\epsilon-y)}{s^*}\right)dy,$$

and

$$\frac{dK_2(\epsilon)}{d\epsilon}=\frac{\mu+\rho}{\lambda}\left(h\left(\frac{i}{i^*}\right)-\int_{-\infty}^{+\infty}J(y)h\left(\frac{i(\epsilon-y)}{i^*}\right)dy\right).$$

Now, we compute $\frac{dV_2(\epsilon)}{d\epsilon}$

$$\begin{aligned}\frac{dV_2(\epsilon)}{d\epsilon} &= \frac{\mu+\rho}{\lambda}\frac{d}{d\epsilon}\left(s^*i^*\right)\int_0^{\sigma\varsigma} h\left(\frac{s(\epsilon-\varepsilon)i(\epsilon-\varepsilon)}{s^*i^*}\right)d\varepsilon,\\ &= -\frac{\mu+\rho}{\lambda}\lambda s^*i^*\Bigg[\left(\frac{s(\epsilon-\sigma\varsigma)i(\epsilon-\sigma\varsigma)}{s^*i^*}\right)-1-ln\left(\frac{s(\epsilon-\sigma\varsigma)i(\epsilon-\sigma\varsigma)}{s^*i^*}\right)-\left(\frac{s(\epsilon)i(\epsilon)}{s^*i^*}\right)\\ &\quad +1+ln\left(\frac{s(\epsilon)i(\epsilon)}{s^*i^*}\right)\Bigg].\end{aligned}$$

Note that (s^*, i^*) satisfies

$$\begin{cases}\chi = \mu s^* + \lambda s^*i^* - \rho i^*,\\ (\mu+\rho)i^* = \lambda s^*i^*.\end{cases}$$

Then, we obtain

$$\begin{aligned}\frac{dV_1(\epsilon)}{d\epsilon} = &\left(s(\epsilon)-s^*\right)\left(d_1(J*s(\epsilon)-s(\epsilon))+\chi-\mu s(\epsilon)-\beta s(\epsilon)i(\epsilon)+\rho i(\epsilon)\right)\\ &+\frac{\mu+\rho}{\lambda}\left(1-\frac{i^*}{i(\epsilon)}\right)\Big(d_2(J*i(\epsilon)-i(\epsilon))+\lambda s(\epsilon-\sigma\varsigma)i(\epsilon-\sigma\varsigma)+\beta s(\epsilon)i(\epsilon)\\ &-\beta s(\epsilon)i(\epsilon)-(\mu+\rho)i(\epsilon)\Big)\\ &+h\left(\frac{s}{s^*}\right)-\int_{-\infty}^{+\infty}J(y)h\left(\frac{s(\epsilon-y)}{s^*}\right)dy+h\left(\frac{i}{i^*}\right)\\ &-\int_{-\infty}^{+\infty}J(y)h\left(\frac{i(\epsilon-y)}{i^*}\right)dy,\\ =\ &Q_1+Q_2,\end{aligned}$$

$$\begin{aligned}Q_1(\epsilon) = s(\epsilon)&\left(1-\frac{s^*}{s(\epsilon)}\right)(d_1(J*s(\epsilon)-s(\epsilon)))+d_2i^*\left(h\left(\frac{s(\epsilon)}{s^*}\right)-\int_{-\infty}^{+\infty}J(y)h\left(\frac{s(\epsilon-y)}{s^*}\right)dy\right)\\ &+\frac{\mu+\rho}{\lambda}\Bigg[\left(1-\frac{i^*}{i(\epsilon)}\right)(d_2(J_2*i(\epsilon)-i(\epsilon)))\\ &+d_2i^*\left(h\left(\frac{i(\epsilon)}{i^*}\right)-\int_{-\infty}^{+\infty}J(y)h\left(\frac{i(\epsilon-y)}{i^*}\right)dy\right)\Bigg].\end{aligned}$$

For Q_1, $ln\left(\frac{s}{s^*}\right)=ln\left(\frac{s(\epsilon-y)}{s^*}\right)-ln\left(\frac{s(\epsilon-y)}{s(\epsilon)}\right)$ and $ln\left(\frac{i}{i^*}\right)=ln\left(\frac{i(\epsilon-y)}{i^*}\right)-ln\left(\frac{i(\epsilon-y)}{i(\epsilon)}\right)$

$$\begin{aligned}&s(\epsilon)\left[\left(1-\frac{s^*}{s(\epsilon)}\right)(d_1(J*s(\epsilon)-s(\epsilon)))+d_1\frac{s^*}{s(\epsilon)}\left(h\left(\frac{s(\epsilon)}{s^*}\right)-\int_{-\infty}^{+\infty}J(y)h\left(\frac{s(\epsilon-y)}{s^*}\right)dy\right)\right],\\ &=-d_1\frac{s^*}{s(\epsilon)}\int_{-\infty}^{+\infty}J(y)h\left(\frac{s(\epsilon-y)}{s(\epsilon)}\right)dy,\end{aligned}$$

and

$$\begin{aligned}&\frac{\mu+\rho}{\lambda}\left[\left(1-\frac{i^*}{i(\epsilon)}\right)(d_2(J*i(\epsilon)-i(\epsilon)))+d_2i^*\left(h\left(\frac{i(\epsilon)}{i^*}\right)-\int_{-\infty}^{+\infty}J_2(y)h\left(\frac{i(\epsilon-y)}{i^*}\right)dy\right)\right],\\ &=\frac{\mu+\rho}{\lambda}\left[d_2i^*\int_{-\infty}^{+\infty}J(y)\left[\frac{i(\epsilon-y)}{i^*}-\frac{i(\epsilon-y)}{i(\epsilon)}-ln(\frac{s(\epsilon)}{s^*})\right]dy-\int_{-\infty}^{+\infty}J_2(y)h\left(\frac{i(\epsilon-y)}{i^*}\right)dy\right],\\ &=\frac{\mu+\rho}{\lambda}\left[d_2i^*\int_{-\infty}^{+\infty}J(y)\left[h\left(\frac{i(\epsilon-y)}{i^*}\right)-h\left(\frac{i(\epsilon-y)}{i(\epsilon)}\right)\right]dy-\int_{-\infty}^{+\infty}J_2(y)h\left(\frac{i(\epsilon-y)}{i^*}\right)dy\right],\\ &=-\frac{\mu+\rho}{\lambda}d_2i^*\int_{-\infty}^{+\infty}J(y)h\left(\frac{i(\epsilon-y)}{i(\epsilon)}\right)dy.\end{aligned}$$

$$
\begin{aligned}
Q_2(\epsilon) = &\left(s(\epsilon)-s^*\right)\left(\chi-\mu s(\epsilon)-\beta s(\epsilon)i(\epsilon)+\rho i(\epsilon)\right)\\
&+\tfrac{\mu+\rho}{\lambda}\left(1-\tfrac{i^*}{i(\epsilon)}\right)\left(\lambda s(\epsilon-\sigma\varsigma)i(\epsilon-\sigma\varsigma)+\beta s(\epsilon)i(\epsilon)-\beta s(\epsilon)i(\epsilon)-(\mu+\rho)i(\epsilon)\right).
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
\tfrac{dV(\epsilon)}{d\epsilon} = &\left(s(\epsilon)-s^*\right)\left(\chi-\mu s(\epsilon)-\beta s(\epsilon)i(\epsilon)+\rho i(\epsilon)\right)\\
&+\tfrac{\mu+\rho}{\lambda}\left(1-\tfrac{i^*}{i(\epsilon)}\right)\left(\lambda s(\epsilon-\sigma\varsigma)i(\epsilon-\sigma\varsigma)+\beta s(\epsilon)i(\epsilon)-\beta s(\epsilon)i(\epsilon)-(\mu+\rho)i(\epsilon)\right)\\
&-\tfrac{\mu+\rho}{\lambda}\lambda s^* i^*\Bigg[\left(\tfrac{s(\epsilon-\sigma\varsigma)i(\epsilon-\sigma\varsigma)}{s^*i^*}\right)-1-ln\left(\tfrac{s(\epsilon-\sigma\varsigma)i(\epsilon-\sigma\varsigma)}{s^*i^*}\right)-\left(\tfrac{s(\epsilon)i(\epsilon)}{s^*i^*}\right)\\
&+1+ln\left(\tfrac{s(\epsilon)i(\epsilon)}{s^*i^*}\right)\Bigg]\\
&-d_1\tfrac{s^*}{s(\epsilon)}\int_{-\infty}^{+\infty}J(y)h\left(\frac{s(\epsilon-y)}{s(\epsilon)}\right)dy-\frac{\mu+\rho}{\lambda}d_2 i^*\int_{-\infty}^{+\infty}J(y)h\left(\frac{i(\epsilon-y)}{i(\epsilon)}\right)dy.
\end{aligned}
$$

By [[41] Sect. 2.1], we get $\frac{dV(\epsilon)}{d\epsilon}\le 0$ and $\frac{dV(\epsilon)}{d\epsilon}=0$ if $s(\epsilon)=s^*$, $i(\epsilon)=i^*$.

In conclusion, we determine that $(s,i)(\infty)=(s^*,i^*)$ and keep in mind that the orbital derivative of V along $\Psi(\epsilon)$ is nonpositive.

Moreover, it is evident that V is continuous and confined below on D. $\Psi(\epsilon)\to(s^*,i^*)$ as $\epsilon\to\infty$, and consequently, $(s,i)\to(s^*,i^*)$ as $\epsilon\to+\infty$, according to this and the Lyapunov-LaSalle theorem. This completes the proof. □

Lemma 4.2 states that the solution of (2.3) satisfies $s^-\le s(\epsilon)\le s^+$, $i^-\le i(\epsilon)\le i^+$, and $(s,i)\to(s^0,0)$ as $\epsilon\to-\infty$. Lemma 4.9 says that $(s,i)\to(s^*,i^*)=\epsilon\to+\infty$. We conclude that the system (2.3) accepts the traveling wave solution of the system (1.1), and this is the sole positive solution that satisfies the (2.4) boundary requirements.

5 Existence of a critical traveling wave solution

This section aims to prove that (2.3) admits a TWS for $R_0>1$ and $\sigma=\sigma^*$.

Lemma 5.1 *If* $R_0>1$, *and* $\sigma=\sigma^*$. *Let*

$$
\begin{aligned}
&s^+(\epsilon)=s^0, && i^+=\mathrm{e}^{\Lambda^*\epsilon},\\
&s^-=\max\left\{s^0-Me^{\gamma\epsilon},0\right\}, && i^-(\epsilon)=\max\{e^{\Lambda^*\epsilon}(1-Je^{\eta\epsilon}),0\},
\end{aligned}
$$

for some positive constants γ, J *and* M, *then* (4.1)–(4.4) *are satisfied.*

Since the proof may be accomplished similarly to the proof of Lemma 4.2, we do not provide it here. By replacing σ by σ^* and Λ_1 by Λ^*, the same process as in Sect. 2 is used to deduce the existence of a TWS for $\sigma=\sigma^*$.

6 Numerical simulation

Initially, we provide a few numerical examples to confirm the TWS of (1.1) see Fig. 1, which links the two equilibria. To achieve this, we consider the basic conditions listed below:

$$
S_0(x)=\begin{cases}5 & \text{if } x\in[-10,0],\\ 2 & \text{if } x\in[0,10],\end{cases}
$$

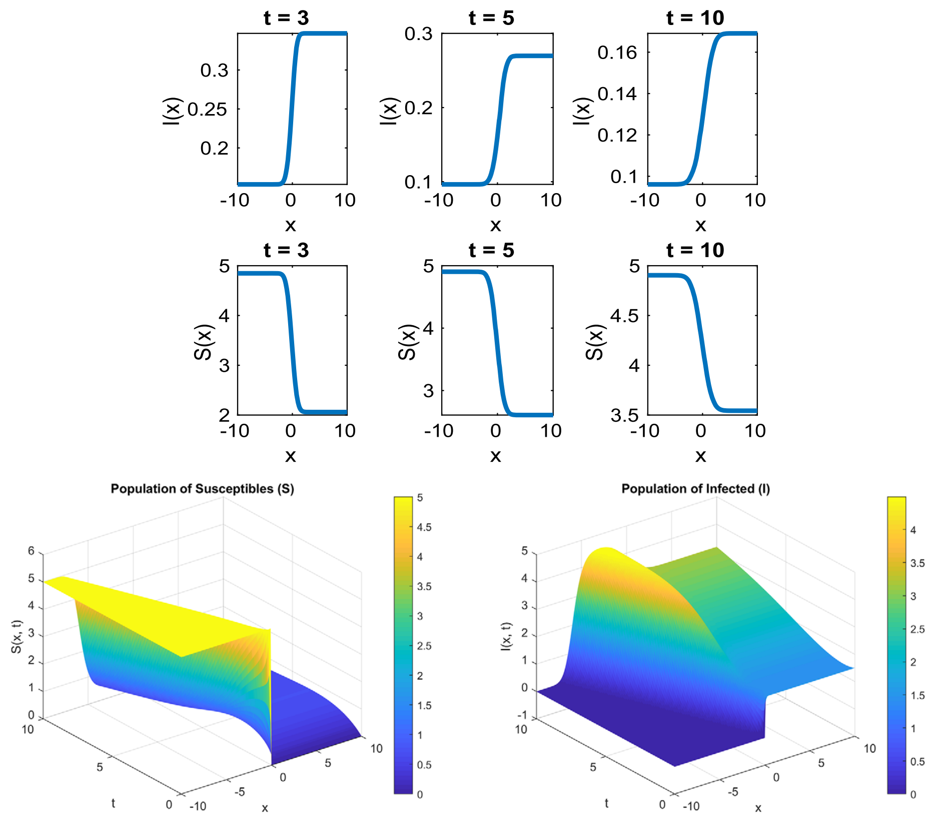

Figure 1 The existence of TWS for the system (2.1), where $\chi = 0.5$, $\lambda = 0.1$, $\mu = 0.1$, $\rho = 0.05$ and $R_0 = 4.45 > 1$

$$I_0(x) = \begin{cases} 0 & \text{if } x \in [-10, 0], \\ 0.35 & \text{if } x \in [0, 10]. \end{cases}$$

We also adopt the kernel function as

$$J(x) = \begin{cases} Ce^{\frac{1}{x^2-1}}, & -0.5 < x < 0.5 \\ 0, & \text{otherwise,} \end{cases}$$

with $C = 0.5$ satisfying $\int_{-0.5}^{0.5} J(x)dx = 1$.

7 Discussion

In this paper, we have analyzed the existence and qualitative properties of traveling wave solutions (TWS) for a time-delayed nonlocal SIS epidemic model. Based on our comprehensive analysis, it is rigorously shown that the occurrence of nontrivial TWS heavily depends on the basic reproduction number R_0 and the wave speed σ. Especially, we have proved that for $R_0 > 1$, traveling wave solutions are valid for all wave speeds $\sigma \geq \sigma^*$ with σ^* being the lowest wave speed, and they are not valid if $\sigma < \sigma^*$. The above facts have been derived through the Schauder fixed point theorem, the construction of upper and lower solutions, and rigorous examination of a truncated problem.

Mathematically, the addition of both nonlocal dispersion and delay is a more realistic configuration to mathematically model the spatial-temporal dispersion of infectious diseases, especially in relation to the current human movement and latency of infection of the disease. The nonlocal term is used to capture long-range interactions, which become increasingly relevant today because of rapid transport and globalization, and the delay is used to produce such phenomena as incubation periods and delayed behavioral response.

In addition, our work generalizes the classical reaction-diffusion models by introducing more complex dynamics, thus shedding new light on how spatial heterogeneity and time delays influence epidemic spreading. The derivation of the minimum wave speed σ^* also has practical implications for estimating thresholds for disease invasion and for planning control measures to prevent or impede the spreading of infectious diseases.

The quantitative solutions provided in Sect. 6 illustrate again the character and behavior of the traveling wave solutions under different parameter settings. These solutions not only affirm the analytical forecast but also illustrate how different parameter adjustments impact the wavefront and rate of disease spread.

While this work provides a good theoretical foundation, future research can explore the specificity and stability of the traveling wave solutions and the solutions' behavior under perturbations. Further studies of the model in multi-dimensional spatial spaces or in heterogeneous media can yield further insights. Incorporation of stochastic effects and empirical epidemiological evidence can also make the model more applicable to practical disease control.

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Author contributions
Rassim Darazirar: Writing- Original draft, Methodology. Rasha M. Yaseen: Investigation, Formal analysis, Writing- Original draft. Ahmed A. Mohsen: Writing-review editing. Aziz Khan: Supervision, review. Thabet Abdeljawad: Software, Supervision.

Data availability
No datasets were generated or analysed during the current study.

Declarations

Competing interests
The authors declare no competing interests.

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