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Article *in* Journal of Physics Conference Series · July 2020

DOI: 10.1088/1742-6596/1591/1/012080

3 authors:

Maryam K. Rasheed
University of Baghdad

Fatin A. Hameed

Abdulrahman H Majeed
College of science,Baghdad university

PAPER • **OPEN ACCESS**

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To cite this article: Maryam K. Rasheed *et al* 2020 *J. Phys.: Conf. Ser.* **1591** 012080

On Generalized (α, β) Derivation on Prime Semirings

Maryam K. Rasheed [1], Fatin A. Hameed [2], Abdulrahman. H. Majeed [3]
Mathematics Department, University of Baghdad
maryamkhdhayer@yahoo.com [1]
Mathematics Department, University of Dyala
tonaabd90@gmail.com [2]
Mathematics Department, University of Baghdad
ahmajeed6@yahoo.com [3]
Mathematics Department, University of Baghdad

Abstract: in this paper we introduce generalized (α, β) derivation on Semirings and extend some results of Oznur Golbasi on prime Semiring. Also, we present some results of commutativity of prime Semiring with these derivation.

1. Introduction

Semirings was first introduced in 1934 by vandiver [1]. In 1992 Golan discuss Semirings and their applications and mentioned about the derivation on Semirings [2]. Thereafter, many researchers interested in derivations on Semirings and generalized it in different directions.

Chandramouleeswarn and Thiruveni studied derivations on Semirings, and introduced the notion of (α,β) derivations on semirings, see [3] and [4].

A Semiring is a nonempty set S together with two binary operations (usually denoted by $+$ and $\cdot$) such that $(S,+)$ is commutative Semigroup, $(S,\ \cdot)$ Semigroup and addition distributive with respect to multiplication on S, we say S is commutative Semiring if and only if $x.y = y.x$ for all $x,y \in S$ [2]. A Semiring S is called additively cancellative if $x + y = x + z$ implies $y = z$ for all $x,y,z \in S$, and it is called multiplicatively cancellative if $x.y = x.z$ implies $y = z$ for all $x,y,z \in S$, so S is called cancellative Semiring if and only if it is both additively and multiplicatively cancellative [5]. Moreover, S is called prime if whenever $x\,S\,y = 0$ implies either $x = 0$ or $y = 0$ for all $x,y \in S$.

Let S be any Semiring, an additive map $d: S \longrightarrow S$ is called derivation on S if $d\,(x\,y) = d\,(x)\,y + x\,d\,(y)$ holds for all $x,y \in S$ [6]. Now, if we suppose that α and β are two nonzero automorphisms on S and d is a derivation on S, then d is said to be (α,β) derivation on S if $d\,(x\,y) = \alpha(x)\,d(y) + d(x)\,\beta(y)$ holds for all $x,y \in S$ [6].

In this paper we introduce the notion of generalized (α, β) derivation on Semirings and extend some important results of Oznur Golbasi [7] on prime Semirings and when these Semirings become commutative.

2. Results

Definition 2.1: - Let S be a Semiring and α,β are two automorphisms on S. An additive map $F: S \longrightarrow S$ is called left generalized (α,β) derivation if there exist nonzero left (α,β) derivation $d: S \longrightarrow S$ such that $F(x\,y) = \alpha\,(x)\,F\,(y) + d\,(x)\,\beta\,(y)$ for all $x,y \in S$, and is called right generalized (α,β) derivation if there exist nonzero right (α,β) derivation $d: S \longrightarrow S$ such that $F(x\,y) = \alpha\,(x)\,d\,(y) + F\,(x)\,\beta\,(y)$ for all $x,y \in S$.

If F is both left and right generalized (α, β) derivation then it is called generalized (α, β) derivation that is $F\,(x\,y) \;=\; \alpha\,(x)\,F\,(y) \;+\; d\,(x)\,\beta\,(y) \;=\; \alpha\,(x)\,d\,(y) \;+\; F\,(x)\,\beta\,(y)$ for all x, y ∈ S.

Lemma 2.2: - Let S be a prime Semiring and I be a nonzero ideal of S. If $x\,I\,y = 0$ for all $x, y \in S$, then either $x = 0$ or $y = 0$.

Proof: - Let $x\,I\,y = 0$ for all x, y ∈ S, hence $x\,S\,I\,y = 0$ for all $x, y \in S$.

By primness of S we have either $x = 0$ or $I\,y = 0$.

Now, either $x = 0$ or $I\,S\,y = 0$. By primness of S and since I ≠ 0, we get $y = 0$.

Theorem 2.3: - Let S be a prime Semiring and I be a nonzero ideal of S. Suppose that $F : S \longrightarrow S$ is a generalized (α, β) derivation on S with $\beta\,(I) = I$. If $F\,(I) \subseteq Z\,(S)$ then S is commutative.

Proof: - Let $F\,(I) \subseteq Z\,(S)$, then $F\,(u) \in Z\,(S)$ for all u ∈ I.

Replace u in above relation by s u, where s ∈ S, we get:

$$F\,(s\,u) = \alpha\,(s)\,F\,(u) + d\,(s)\,\beta\,(u) \in Z\,(S).$$

Then,

$$[\alpha\,(s)\,F\,(u) + d\,(s)\,\beta\,(u), \alpha\,(s)] = 0.$$

$$[\alpha\,(s)\,F\,(u), \alpha\,(s)] + [d\,(s)\,\beta\,(u), \alpha\,(s)] = 0.$$

$$\alpha\,(s)[\,F\,(u), \alpha\,(s)] + [\alpha\,(s), \alpha\,(s)]F\,(u) + d\,(s)[\,\beta\,(u)\,, \alpha\,(s)] + [d\,(s), \alpha\,(s)]\,\beta\,(u) = 0$$

Hence,

$$d\,(s)\,[\beta\,(u), \alpha\,(s)] + [d\,(s), \alpha\,(s)]\,\beta\,(u) = 0$$

$$d\,(s)\,\beta\,(u)\,\alpha\,(s) - d\,(s)\,\alpha\,(s)\,\beta\,(u) + d\,(s)\,\alpha\,(s)\,\beta\,(u) - \alpha\,(s)\,d\,(s)\,\beta\,(u) = 0$$

$$d\,(s)\,\beta\,(u)\,\alpha\,(s) - \alpha\,(s)\,d\,(s)\,\beta\,(u) = 0 \qquad \ldots (1)$$

Replace u by u v in (1), where $v \in I$. We obtain,

$$d\,(s)\,\beta\,(u\,v)\,\alpha\,(s) - \alpha\,(s)\,d\,(s)\,\beta\,(u\,v) = 0$$

$$d\,(s)\,\beta\,(u)\,\beta\,(v)\;\alpha\,(s) - \alpha\,(s)\,d\,(s)\,\beta\,(u)\,\beta\,(v) = 0 \qquad \ldots (2)$$

By using (1) we get,

$$d\,(s)\,\beta\,(u)\,\beta\,(v)\,\alpha\,(s) - d\,(s)\,\beta\,(u)\,\alpha\,(s)\,\beta\,(v) = 0.$$

Then, for all u ∈ I implies,

$$d\,(s)\,\beta\,(u)\,[\beta\,(v), \alpha\,(s)] = 0$$

$$d\,(s)\,I\,[\beta\,(v), \alpha\,(s)] = 0.$$

By Lemma 2.2 and since $d \neq 0$ then for all v ∈ I we get,

$$[\beta\,(v), \alpha\,(s)] = 0.$$

$$[I, \alpha\,(s)] = 0.$$

Then, $I \subseteq Z\,(S)$, by [8, Lemma 2.22] we get S is commutative.

Lemma2.4: - Let S be a prime semiring and I be a nonzero ideal of S. Suppose that $F: S \longrightarrow S$ is a nonzero generalized (α, β) derivation and let $x \in S$:

1- If $x. F(u) = 0$ for all $u \in I$ then $x = 0$.
2- If $F(u). x = 0$ for all $u \in I$ then $x = 0$.

Proof: 1- Let $x. F(u) = 0$ for all $u \in I$.

Replace u in above equation by s u, where $s \in S$. Then for all $s \in S$ we have,

$$x. F(s\,u) = 0.$$

$$x. (\alpha(s)\, d(u) + F(s)\, \beta(u)) = x. \alpha(s)\, d(u) + x. F(s)\, \beta(u) = 0$$

Hence,

$$x. \alpha(s)\, d(u) = 0$$

$$\alpha^{-1}(x)\, I\, \alpha^{-1}\big(d(S)\big) = 0.$$

By Lemma 2.2 and since $d \neq 0$ we have, $\alpha^{-1}(x) = 0$. Then, $x = 0$.

Similarly we can prove (2).

Remark 2.5: - Let S be a semiring and α is an automorphism on S. If $\alpha = 0$ on I then $\alpha = 0$ on S

Proof: - Obvious.

Lemma 2.6: - Let S be a prime semiring and I be a nonzero ideal of S. Suppose that $F: S \longrightarrow S$ is a nonzero generalized (α, β) derivation with nonzero automorphisms α and β. If $F = 0$ on I then $d = 0$ on S.

Proof: - Let $F(u) = 0$ for all $u \in I$. Take $s \in S$ then,

$$F(u\,s) = \alpha(u)\, d(s) + F(u)\, \beta(s) = 0.$$

Hence,

$$\alpha(u)\, d(s) = 0.$$

By [8, Lemma 2.27] and since $\alpha \neq 0$ then $d = 0$ on S.

Lemma 2.7: - Let S be a semiring and I be a nonzero ideal of S. Suppose that $F: S \longrightarrow S$ is a generalized (α, β) derivation with nonzero automorphisms α and β. If $F = 0$ on I then $F = 0$ on S.

Proof: - Let $F(u) = 0$ for all $u \in I$. Take $s \in S$ then,

$$F(u\,s) = \alpha(u)\, F(s) + d(u)\, \beta(s) = 0.$$

By Lemma 2.6 we get $\alpha(u)\, F(s) = 0$.

Now, replace u in above equation by u r, where $r \in S$ we get,

$$\alpha(u\,r)\, F(s) = \alpha(u)\, \alpha(r)\, F(s) = 0.$$

Since α is automorphism (onto) Hence, $\alpha(u)\, S\, F(s) = 0$.

By primness and since $\alpha \neq 0$ on S then $F = 0$ on S.

Lemma 2.8: - Let S be a prime semiring and F: S ⟶ S be a generalized (α, β) derivation. Suppose that I is an ideal of S. If $0 \neq r \in S$ with $r \, . \, F(x) = 0$ for all $x \in S$, then $F = 0$ on S.

Proof: - Let $r \, . \, F(x) = 0$ for all $x \in S$. Put $x = x\,y$, where $y \in I$ we get,

$$r \, . \, F(x\,y) = 0$$

$$r \, . \, \alpha(x)\, d(y) + r . F(x)\, \beta(y) = 0$$

Then,

$$r \, . \, \alpha(x)\, d(y) = 0$$

$$r . S\, d(y) = 0.$$

So, by primness of S and since $r \neq 0$ hence, $d(y) = 0$ for all $y \in I$.

That means, $d = 0$ on I. So,

$$F(y\,x) = \alpha(y)\, F(x) + d(y)\, \beta(x) = \alpha(y)\, F(x).$$

Now, $r. F(y\,x) = r\, \alpha(y)\, F(x)$ Implies:

$$r. \alpha(y)\, F(x) = 0$$

$$r. S\, F(x) = 0.$$

By primness of S and since $r \neq 0$ we get, $F = 0$ on S.

Theorem 2.9: - Let S be a prime semiring and I be a nonzero ideal of S. Suppose that F: S⟶S is a nonzero generalized (α, β) derivation such that $d\,F = F\,d$ and $\alpha\, F = F\, \alpha$. If $[F(u), F(v)] = 0$ for all $u, v \in I$, then S is commutative.

Proof: - Let $[F(u), F(v)] = 0$ for all $u, v \in I$.

Replace v in above equation by v s, where $s \in S$ we get,

$$[F(u), F(v\,s)] = [F(u), \alpha(v)\, d(s) + F(v)\, \beta(s)] = 0$$

$$[F(u), \alpha(v)\, d(s)] + [F(u), F(v)\, \beta(s)] = 0$$

$$\alpha(v)\, [F(u), d(s)] + [F(u), \alpha(v)]\, d(s) + F(v)\, [F(u), \beta(s)] + [F(u), F(v)]\, \beta(s) = 0$$

Hence for all $u, v \in I$ we have,

$$F(v)\, [F(u), \beta(s)] = 0$$

By Lemma 2.8 and since $F \neq 0$ on I (Lemma 2.7). So, for all $u \in I$ implies,

$$[F(u), \beta(s)] = 0$$

Therefore, $F(I) \subseteq Z(S)$, and by Theorem 2.3 we have S is commutative.

Theorem 2.10: - Let S be a cancellative prime semiring and I be a nonzero ideal of S. Suppose that F: S⟶S is a generalized (α, β) derivation with nonzero automorphisms α and β. If F acts as homomorphism on S then $d = 0$ on S.

Proof: - Since F acts as homomorphism on S then for all $x, y \in S$,

$$F(x\,y) = F(x)\,F(y) \qquad \dots (1)$$

Since F is generalized (α, β) derivation then for all x, y ∈ S,

$$F(x\,y) = \alpha(x)\,F(y) + d(x)\,\beta(y) \qquad \dots (2)$$

From (1) and (2) we get,

$$F(x)\,F(y) = \alpha(x)\,F(y) + d(x)\,\beta(y) \qquad \dots (3)$$

Replace y by y s in (3), where s ∈ S we obtain,

$$\alpha(x)\,F(y\,s) + d(x)\,\beta(y\,s) = F(x)\,F(y\,s).$$

$$\alpha(x)\,F(y)\,F(s) + d(x)\,\beta(y)\,\beta(s) = F(x)\,F(y)\,F(s).$$

$$= F(x\,y)\,F(s)$$

$$= \alpha(x)F(y)F(s) + d(x)\beta(y)F(s)$$

Since S is cancellative we get, $\beta(s) = F(s)$ for all s ∈ S.

Now, replace s by r s in the above equation, where r ∈ S, w obtain,

$$F(r\,s) = \beta(r\,s)$$

$$\alpha(r)\,d(s) + F(r)\,\beta(s) = \beta(r)\,\beta(s)$$

$$= F(r)\,\beta(s).$$

Since S is cancellative we get, $\alpha(r)\,d(s) = 0$ for all r, s ∈ S.

By [8, Lemma 2.27] and Since $\alpha \neq 0$ on S then $d = 0$ on S.

Theorem 2.11: - Let S be a cancellative prime semiring and I nonzero ideal of S. Suppose that F: S⟶S is a generalized (α, β) derivation with nonzero automorphisms α and β such that $d\,F = F\,d$ and $\alpha\,F = F\,\alpha$. If F acts as anti-homomorphism on S then d = 0 on S.

Proof: - Since F acts as homomorphism on S then for all x, y ∈ S,

$$F(x\,y) = F(y)\,F(x) \qquad \dots (1)$$

Since F is generalized (α, β) derivation then,

$$F(x\,y) = \alpha(x)\,F(y) + d(x)\,\beta(y) \qquad \dots (2)$$

From (1) and (2) we get,

$$F(y)\,F(x) = \alpha(x)\,F(y) + d(x)\,\beta(y) \qquad \dots (3)$$

Replace y by y s in (3), where s ∈ S, we obtain

$$\alpha(x)\,F(y\,s) + d(x)\,\beta(y\,s) = F(x)\,F(y\,s).$$

$$\alpha(x)\,F(s)\,F(y) + d(x)\,\beta(y)\,\beta(s) = F(s)\,F(y)\,F(x).$$

$$= F(s)\,F(x\,y)$$

$$= F(s)\,\alpha(x)\,F(y) + F(s)\,d(x)\,\beta(y).$$

Since $\alpha\,F = F\,\alpha$ and S is cancellative we have,

$$d(x)\,\beta(y)\,\beta(s) = F(s)\,d(x)\,\beta(y)$$

Now, since $d F = F d$ and S is cancellative we have,

$$\beta(s) = F(s) \text{ for all } s \in S.$$

Replace s by r s in the above equation, where r ∈ S, we get

$$F(r s) = \beta(r s)$$

$$\alpha(r)d(s) + F(r)\beta(s) = \beta(r)\beta(s)$$

$$= F(r)\,\beta(s)$$

Since S cancellative then, $\alpha(r)\,d(s) = 0$ for all r, s ∈ S.

By [8, Lemma 2.27] and Since $\alpha \neq 0$ on S then d = 0 on S.

Theorem 2.12: - Let S be a cancellative prime semiring and I be a nonzero ideal of S. Suppose that F: S ⟶ S is a generalized (α, β) derivation with nonzero automorphisms α and β. If F acts as homomorphism on I then $d = 0$ on S.

Proof: - Since F acts as homomorphism on I. Then for all u, v ∈ I,

$$F(u v) = F(u)F(v) \quad \dots (1)$$

Since F is generalized (α, β) derivation then,

$$F(u v) = \alpha(u)\,F(v) + d(u)\,\beta(v) \quad \dots (2)$$

From (1) and (2) we get,

$$F(u)\,F(v) = \alpha(u)\,F(v) + d(u)\,\beta(v) \quad \dots (3)$$

Replace v by v s in (3), where s ∈ S, we obtain

$$\alpha(u)\,F(v s) + d(u)\,\beta(v s) = F(u)\,F(v s).$$

$$\alpha(u)\,F(v)\,F(s) + d(u)\,\beta(v)\,\beta(s) = F(u)\,F(v)\,F(s)$$

$$= F(u v)\,F(s)$$

$$= \alpha(u)F(v)F(s) + d(u)\beta(v)F(s).$$

Since S is cancellative we have, $\beta(s) = F(s)$ for all s ∈ S.

Now, replace s by r s in the above equation, where r ∈ S we get,

$$F(r s) = \beta(r s)$$

$$\alpha(r)\,d(s) + F(r)\,\beta(s) = \beta(r)\,\beta(s)$$

$$= F(r)\beta(s).$$

Since S is cancellative we get, $\alpha(r)\,d(s) = 0$ for all r, s ∈ S.

By [8, Lemma 2.27] and Since α ≠ 0 on S then d = 0 on S.

Notation: - Throughout the following Theorem we use alpha-beta commutator such that $[x, y]_{\alpha,\beta} = \alpha(x)\,y - y\,\beta(x)$.

Theorem 2.13: - Let S be a prime semiring, I nonzero ideal of S and F: S⟶S generalized (α, β) derivation. If α and β commute with d and $F(u v) = F(v u)$ for all u, v ∈ I, then S is commutative.

Proof: - Let u, v ∈ I such that $[u, v]$ is constant element say c with $F(c) = 0$ and $d(c) \neq 0$.

Let z ∈ I hence,

$$F(c z) = \alpha(c) d(z) + F(c) \beta(z)$$

$$= \alpha(z) F(c) + d(z) \beta(c) = F(z c).$$

That gives, $\alpha(c) d(z) = d(z) \beta(c)$ for all z ∈ I.

Since α and β are commute with d then for all z ∈ I yields that,

$$[d(z), c]_{\alpha,\beta} = 0$$

Replace z in the above equation by w z, where w ∈ I we get,

$$[d(w z), c]_{\alpha,\beta} = [d(w)\alpha(z) + \beta(w) d(z), c]_{\alpha,\beta}$$

$$= [d(w)\alpha(z), c)]_{\alpha,\beta} + [\beta(w) d(z), c]_{\alpha,\beta}$$

$$= 0$$

Now, by add and subtract the terms: $d(w)\,\alpha(z)\,\alpha(c)$ and $\beta(w)\,\beta(c)\,d(z)$ we obtain,

$$d(w)\,\alpha(z)\,\alpha(c) - d(w)\,\alpha(z)\,\alpha(c) + d(w)\,\alpha(z)\,\alpha(c) - d(w)\,\beta(c)\,\alpha(z) + d(w)\,\alpha(c)\,\alpha(z) - \beta(c)\,d(w)\,\alpha(z) + \beta(w)\,d(z)\,\alpha(c) - \beta(w)\,\beta(c)\,d(z) + \beta(w)\,\beta(c)\,d(z) - \beta(w)\,\beta(c)\,d(z) + \beta(w)\,\alpha(c)\,d(z) - \beta(c)\,\beta(w)\,d(z) = 0.$$

Hence for all w, z ∈ I,

$$d(w)\,\alpha\,[z, c] + [c, d(w z)]_{\alpha,\beta} + \beta(w)\,[d(z), c]_{\alpha,\beta} + \beta\,[w, c]\,d(z) = d(w)\,\alpha\,[z, c] + \beta\,[w, c]\,d(z) = 0.$$

Replace z in above equation by c then for all w ∈ I we get,

$$\beta\,[w, c]\,d(z) = 0$$

$$[w, c]\,\beta^{-1}(d(c)) = 0$$

Replace w in above equation by s w, where s ∈ S we obtain,

$$\beta\,[s w, c]\,d(z) = 0.$$

Thus, $[s, c]\,w\,\beta^{-1}(d(c)) = 0$ for all w ∈ I. Now, by lemma 2.2 and since $d(c) \neq 0$ we get,

$$[s, c] = 0 \text{ for all } s \in S.$$

Then, I is commutative and by [8, Lemma 2.22] implies S is commutative.

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