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Fully Extending Modules

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Abstract

Throughout this paper we introduce the concept of quasi closed submodules which is weaker than the concept of closed submodules. By using this concept we define the class of fully extending modules, where an R-module M is called fully extending if every quasi closed submodule of M is a direct summand.This class of modules is stronger than the class of extending modules. Many results about this concept are given, also many relationships with other related concepts are introduced.

Keywords: Closed submodules, Quasi closed submodules, Extending modules, Fully extending modules, Strongly extending modules, Modules has SIP property

1 Introduction

Let R be a commutative ring with unity and let M be a unitary left R-module. A submodule N of M is said to be essential in M, (denoted by $N \leq_e M$), if for any submodule K of M, $N \cap K = 0$ implies that $K = 0$ [12], and a submodule N of M is said to be closed in M if N has no proper essential extension in M; that is if $N \leq_e W < M$ then N=W [12]. An R-module M is called extending (or CS-module), if every submodule of M is essential in a direct summand [6]. It is well known that an R-module M is extending if and only if every closed submodule of M is a direct summand [6].

In this paper we introduce the concept of quasi-closed submodule (briefly qc-submodule), where a submodule N of M is called qc-submodule if for each $x \in M$ with $x \notin N$, there exists a closed submodule L of M containing N and $x \notin L$. it is clear that every closed submodule is a qc-submodule, but not conversely (see Rem and Ex (2.2)(1)). Also we define fully extending module, where an R-module M is called fully extending if every qc-submodule of M is a direct summand

This research consists of three sections. In $S2$ we give a comprehensive study of qc-submodules. Some results are analogous to properties of closed submodules.In $S3$ we study the concept of fully extending module. It is clear that every fully extending module is extending, but not conversely (see Rem and Ex (3.2)(1)). A characterization of fully extending modules is given, so we prove that an R-module M is fully extending module if and only if M is an extending and has SIP(see Th (3.7)), where an R-module M has SIP if the intersection of any two summands of M is a summand of M [15]. Moreover many characterizations of fully extending modules in certain classes of modules are given. Beside that many relationships between fully extending modules and other related concepts are introduced. In $S4$ we show by examples that the direct sum of fully extending modules may not be fully extending module(see Ex (4.1)). However, we give certain conditions under which the direct sum of fully extending modules be fully extending module (see Th (4.2) and Th (4.3)).

2 Quasi-Closed submodules

In this section we introduce the concept of quasi-closed submodules. We investigate the basic properties of this type of submodules, some of these properties are analogous to the properties of closed submodules.

Definition 2.1. *A submodule N of an R-module M is called quasi-closed (briefly qc-submodule), if for each $x \in M$ with $x \notin N$, there exists a closed submodule L of M containing N such that $x \notin L$. An ideal I of a ring R is called qc-ideal if it is qc-submodule of an R-module R.*

Remarks and Examples 2.2.

1. *It is clear that every closed submodule is qc-closed submodule. However, the converse is not true in general, for example: Let* $M = Z \oplus Z_2$ *be the* Z*-module . The submodule* $N = <(2,\bar{0})>$ *is not closed since* N *is essential in* $Z \oplus (\bar{0})$*. On the other hand we can see that* N *is a qc-submodule in* $Z \oplus (\bar{0})$ *as follows: For any* $(n,\bar{0}) \in M$*, where* n *is odd integer),*$(n,\bar{0}) \notin N$*, there exists a submodule* $L = <(1,\bar{1})>$ *of* M *containing* N *such that* $(n,\bar{0}) \notin N$*. Note that* L *is closed submodule in* M *since* $L \oplus ((0) \oplus Z_2) = M$*. Also for any* $(n,\bar{1}) \in M$*,* $(n,\bar{1}) \notin N$*. Take* $L_1 = Z \oplus (\bar{0})$*, it is clear that* L_1 *is a direct summand of* M *hence it is closed submodule in* M *and* $(n,\bar{1}) \notin L_1$*. Thus for each* $x \in M$ *with* $x \notin N$ *there exists a closed submodule* L *of* M *containing* N *such that* $x \notin L$*. Therefore* N *is a qc-submodule in* M*.*

2. *Every direct summand of an* R*-module* M *is a qc-submodule in* M*.*

3. *Let* M *be an* R*-module. if* $A \leq B \leq M$ *such that* A *is a qc-submodule in* B *and* B *is a qc-submodule in* M*, then* A *is a qc-submodule in* M*.*

   proof(3). *Let* $x \in M$ *with* $x \notin A$*, then either* $x \in B$ *or* $x \notin B$*. If* $x \in B$*. Since* A *is a qc-submodule in* B*, so there exists a closed submodule* L *in* B *such that* $A \leq L$ *and* $x \notin L$*. But* L *is closed in* B *and* B *is closed in* M*, hence* L *is closed in* M *[12]. Thus we have a closed submodule* L *in* M *containing* A *and* $x \notin L$*. If* $x \notin B$*, then nothing to prove since* B *is a closed submodule in* M *containing* A *and* $x \notin B$*. Therefore* A *is a qc-submodule in* M*.*

4. *Let* M *be an* R*-module. if* $A \leq B \leq M$ *and* B *is a qc-submodule in* M *then* $\frac{B}{A}$ *is a qc-submodule in* $\frac{M}{A}$*.*

   proof(4). *It is clear.*

The following proposition gives a characterization of *qc*-submodules.

Proposition 2.3. *Let* M *be an* R*-module and let* N *be a submodule of* M*. Then* N *is a qc-submodule in* M *if and only if there exists a collection of submodules* $\{N_\alpha\}_{\alpha\in\Lambda}$ *, where* Λ *is an index set, such that* $\forall\alpha \in \Lambda$ *,* N_α *is a closed submodule in* M *and* $N = \bigcap_{\alpha\in\Lambda} N_\alpha$*.*

proof. $\Rightarrow$) If N is a closed submodule of M then nothing to prove. If N is not closed submodule, then there exists a closed submodule L of M such that N is an essential submodule of L.Assume that $\{N_\alpha\}_{\alpha\in\Lambda}$, (where Λ is an index set) be the collection of all closed submodules of M such that $N \leq_e N_\alpha$ for each $\alpha \in \Lambda$. Hence $N \leq N \bigcap_{\alpha\in\Lambda} N_\alpha$. Now, let $x \in \bigcap_{\alpha\in\Lambda} N_\alpha$ and suppose that $x \notin N$. Since N is a qc-submodule in M, so there exists a closed submodule L of M containing N such that $x \notin L$. Hence $L = N_{\alpha_i}$ for some $\alpha_i \in \Lambda$ and so $x \notin \bigcap_{\alpha\in\Lambda} N_\alpha$ which is a contradiction. $Thus \bigcap_{\alpha\in\Lambda} N_\alpha = N$.

$\Leftarrow$) Suppose that $N = \bigcap_{\alpha\in\Lambda} N_\alpha$, where N_α is a closed submodule in M for each $\alpha \in \Lambda$ and N_α containing N. let $x \in M$ with $x \notin N$, there exists $\alpha_i \in \Lambda$ such that $x \notin N_{\alpha_i}$. But $N \leq N_{\alpha_i}$ and N_{α_i} is a closed submodule in M, hence N is a qc-submodule in M.

Corollary 2.4. *If A and B are qc-submodules in an R-module M, then $A \cap B$ is qc-submodule in M.*

By using Prop(2.3), we can give more examples about qc-submodules.

Examples 2.5.

1. *Consider the Z-module $M = Z_8 \oplus Z_2$.The submodule $N =< (\bar{2},\bar{0}) >$ is not closed submodule in M. Let $N_1 = Z_8 \oplus (\bar{0}) =< (\bar{1},\bar{0}) >$ and $N_2 =< (\bar{1},\bar{1}) >$. It is easy to see that both of N_1 and N_2 are closed submodules in M. Also $N = N_1 \cap N_2$. Thus N is a qc-submodule in M. Similarly if $L =< (\bar{4},\bar{0}) >$, L is not closed in M and $L = N_1 \cap N_3$, where $N_3 =< (\bar{2},\bar{1}) >$, and N_1, N_3 are closed submodules in M.*

2. *Let M be the Z-module $Z_4 \oplus Z_2$. The submodule $N =< (\bar{2},\bar{0}) >$ is not closed in M. However, $N = N_1 \cap N_2$, where $N_1 =< (\bar{1},\bar{0}) >$ and $N_2 =< \bar{1},\bar{1}) >$. But $N_1 \oplus < (\bar{0},\bar{1}) >= M$ and $N_2 \oplus < (\bar{2},\bar{1}) >= M$, hence N_1 and N_2 are closed in M. Thus N is a qc-submodule in M.*

Proposition 2.6. *Let M_1 and M_2 be two R-modules. If A is a qc-submodule in M_1, and B is a qc-submodule in M_2, Then $A \oplus B$ is a qc-submodule in $M = M_1 \oplus M_2$.*

Proof. Let $X = (x_1, x_2) \in M_1 \oplus M_2$ with $X \notin A \oplus B$. then either $x_1 \notin A$ or $x_2 \notin B$. If $x_1 \notin A$. Since A is a qc-submodule in M_1, so there exists a closed submodule L_1 in M_1 such that L_1 containing A and $x_1 \notin L_1$. But L_1 is closed in M_1 implies that $L_1 \oplus M_2$ is closed in M [12], also $L_1 \oplus M_2$ is containing $A \oplus B$ and $X \notin L_1 \oplus M_2$. Similarly if $x_2 \notin B$, then there exists a closed submodule in M containing $A \oplus B$ and does not contain X. Thus $A \oplus B$ is a qc-submodule in M.

The converse of Prop(2.6) is true under certain condition as the following proposition shows.

Proposition 2.7. *Let M_1 and M_2 be R-modules, and let $A \leq M_1$, $B \leq M_2$ such that $ann_R M_1 + ann_R M_2 = R$. If $A \oplus B$ is a qc-submodule in $M = M_1 \oplus M_2$, then A is a qc-submodule in M_1 and B is a qc-submodule in M_2.*

Proof. In order to prove that A is a qc-submodule in M_1, let $x \in M_1$ with $x \notin A$. Then $(x, 0) \notin A \oplus B$, But $A \oplus B$ is a qc-submodule of M, so there exists a closed submodule L in M such that L containing $A \oplus B$ and $(x, 0) \notin L$. Since $ann_R M_1 + ann_R M_2 = R$, so by a part of the proof of ([1], Prop (4.2), P.28), any submodule of $M = M_1 \oplus M_2$ can be written as a direct sum of two submodules of M_1 and M_2 respectively, thus $L = L_1 \oplus L_2$ for some $L_1 \leq M_1$ and $L_2 \leq M_2$. It follows that L_1 is closed in M_1 and L_2 is closed in M_2. Since L containing $A \oplus B$ and $(x, 0) \notin L$, then L_1 containing A and $x \notin L_1$. Therefore A is a qc-submodule in M_1. Similarly, B is a qc-submodule in M_2.

Remark 2.8. *the condition $ann_R M_1 + ann_R M_2 = R$ is necessary in Prop (2.7) as the following example shows: Let M be the Z-module $Z_4 \oplus Z_2$. It is clear that $ann_z Z_4 + ann_z Z_2 = 2Z \neq Z$. Let $N =< (\bar{2}, \bar{0}) >= (\bar{2}) \oplus (\bar{0})$, we see in Ex (2.5)(2), that N is a qc-submodule in M, but $N_1 = (\bar{2})$ is not a qc-submodule in Z_4.*

As analogous statement to the result in ([12], Exc. 17, P.20) we give the following.

Proposition 2.9. *Let M be an R-module, and A, N be submodules of M. If A is a qc-submodule in M and $N \leq_e M$, then $A \cap N$ is a qc-submodule in N.*

Proof. Let $x \in N$ with $x \notin A \cap N$. then $x \in M$ and $x \notin A$. Since A is a qc-submodule in M, so there exists a closed submodule L in M containing A such that $x \notin L$. But L is closed in M and $N \leq_e M$. This implies that $L \cap N$ is closed in N ([12], Exc. 17, P.20). On the other hand, $L \cap N$ containing $A \cap N$, and $x \notin L \cap N$. Thus $A \cap N$ is a qc-submodule in N.

Recall that an R-module M is called multiplication if for each submodule N of M there exists an ideal I of R such that $N = IM$ [5]. Equivalently M is a multiplication if for each submodule N of M, $N = (N : RM)M$ where $(N : RM) = \{r \in R | rM \subseteq M\}$ [7].

Proposition 2.10. *Let M be a faithful finitely generated multiplication R-module, and let N be a submodule of M. Then the following statements are equivalent:*

1. *N is a qc-submodule in M.*

2. $(N : RM)$ *is a qc-ideal in* R.

3. $N = IM$ *for some qc-ideal* I *of* R.

Proof. (1) ⇒(2): Let $a \in R$ with $a \notin (N :_R M)$. Then $aM \nsubseteq N$, so there exists $m \in M$ such that $am \notin N$. But N is a qc-submodule in M, hence there exists a closed submodule L of M containing N such that $am \notin L$. On the other hand, since M is a faithful finitely generated multiplication R-module, then clearly that if L is a closed submodule in M then $(L :_R M)$ is a closed ideal in R. Also $am \notin L$ implies that $a \notin (L :_R M)$. Thus $(L :_R M)$ is a closed ideal in R containing $(N :_R M)$ and $a \notin (N :_R M)$. Therefore $(N :_R M)$ is a qc-ideal in R.
(2)⇒(3): It is clear.
(3)⇒(1): Since I is a qc-ideal in R, so by Prop (2.3), there exists a collection $\{I_\alpha\}_{\alpha\in\Lambda}$, ($\Lambda$ is some index set) of closed ideals in R such that $I = \bigcap_{\alpha\in\Lambda} I_\alpha$. Hence $N = \bigcap_{\alpha\in\Lambda}(I_\alpha M)$, and by ([7], Prop (1.6)), $(\bigcap_{\alpha\in\Lambda} I_\alpha)M = \bigcap_{\alpha\in\Lambda}(I_\alpha M)$. Now, since I_α is closed ideal in R and M is a faithful finitely generated multiplication module, so it is clear that for each $\alpha \in \Lambda$,$I_\alpha M$ is a closed submodule in M. Thus by Prop (2.3) N is a qc-submodule in M.

3 Fully Extending Modules

In this section we introduce and study a class of fully extending modules which is stronger than the class of extending modules.

Definition 3.1. *An* R*-module* M *is called fully extending, if every qc-submodule of* M *is a direct summand of* M*. A ring* R *is called fully extending, if* R *is a fully extending* R*-module.*

Remarks and Examples 3.2.

1. *It is clear that every fully extending module is an extending module. The converse is not true in general as the following example shows: The* Z*-module* $Z \oplus Z_2$ *is an extending module but it is not fully extending, since there exists a qc-submodule* $N =< (2, \bar{0}) >$ *which is not direct summand of* $Z \oplus Z_2$*, (See Rem and Ex (3.2)(1)).*

2. *An* R*-module* M *is an extending if and only if every qc-submodule of* M *is essential in a direct summand.*

3. *Let* M *and* $\acute{M}$ *be two isomorphic* R*-modules. Then* M *is fully extending module if and only if* $\acute{M}$ *is fully extending module.*

4. *Every semisimple module is fully extending module, but the converse is not true in general, for example the Z-module Z is fully extending module and not semisimple module.*

5. *An R-module M is uniform if and only if M is indecomposable and fully extending module.*

Proof(5). *It is straightforward, so it is omitted.*

Proposition 3.3. *A direct summand of fully extending module is fully extending.*

Proof. Let N be a direct summand of M. Then $M = N \oplus W$, for some submodule W of M. Assume that K is a qc-submodule of N. Since N is a direct summand of M, then N is a closed submodule of M, and so by Rem and Ex (2.2)(3), K is a qc-submodule of M. Then by a definition of a fully extending module, K is a direct summand of M. Thus $M = K \oplus T$ for some submodule T of M. Now, $N = M \cap N = (K \oplus T) \cap N$. By a modular law $N = K \oplus (T \cap N)$. Thus K is a direct summand of N, and so N is a fully extending module.

Corollary 3.4. *If an R-module M is fully extending and N is a qc-submodule of M, then $\frac{M}{N}$ is fully extending module.*

Theorem 3.5. *Let M be a faithful finitely generated R-module. Then M is a fully extending module if and only if R is a fully extending ring.*

Proof. $\Rightarrow$) Let I be a qc-ideal in R. Put $N = M$. Since M is a multiplication R-module, then $N = (N : RM)M$ [7], and since M is a finitely generated multiplication, then by Th (2.10), N is a qc-submodule in M and so N is a direct summand of M; that is $M = N \oplus W$ for some submodule W of M. But M is a multiplication module so $W = JM$ for some ideal J of R. Thus $M = IM \oplus JM = (I \oplus J)M$. And by ([7] Th (3.1)), $R = I \oplus J$. Therefore I is a direct summand of R, hence R is a fully extending ring.
$\Leftarrow$) The proof is similarly.

Recall that an R-module M has summand intersection property (briefly SIP), if for each two summand A and B of M, $A \cap B$ is also summand of M [15]. Equivalently, M has SIP if and only if for each decomposition $M = A \oplus B$ and for each R-homomorphism $f : A \to B$, $\ker f$ is a direct summand of M [10].

An R-module M has strongly summand intersection property (briefly $SSIP$), if the intersection of any collection of summands of M is a summand of M [3].

An R-module M is called UC-module if for each submodule N of M, there exists a unique closed submodule W of M such that N is an essential submodule of W [11]. The following lemma appeared in [3], we shall need it in our work.

Lemma 3.6. *Let M be an extending R-module. Then the following statements are equivalent.*

1. *M has SIP property.*
2. *M has $SSIP$ property.*
3. *M is UC-module.*

The following theorem gives a characterization of fully extending modules.

Theorem 3.7. *An R-module M is fully extending if and only if M is an extending and has SIP property.*

Proof $\Rightarrow$) It is clear that M is an extending module. Let N_1 and N_2 be two summand of M, then N_1 and N_2 are closed submodules of M. Hence $N = N_1 \cap N_2$ is a qc-submodule of M. But M is a fully extending module, so N is a direct summand of M. Thus M has SIP property.

$\Leftarrow$) Let N be a qc-submodule in M. By Prop(2.3), $N = \cap_{\alpha\in\Lambda} N_\alpha$, where N_α is a closed submodule in M for each $\alpha \in \Lambda$. Since M is an extending module, so N_α is a direct summand of M for each $\alpha \in \Lambda$. On the other hand by lemma (3.6), M has $SSIP$, therefore $N = \cap_{\alpha\in\Lambda} N_\alpha$ is a direct summand of M. Thus M is a fully extending module.

By using Th (3.7), we can give the following example: Consider the Z-module $M = Z \oplus Z$. Since M has SIP (see Ex 5 in [3]), also M is an extending module, so M is a fully extending module.

Corollary 3.8. *Let M be an R-module. Then the following statements are equivalent.*

1. *M is a fully extending module.*
2. *M an extending module and has SIP property.*
3. *M is an extending module and has $SSIP$ property.*
4. *M is an extending and UC-module.*

Corollary 3.9. *Every extending multiplication module is fully extending module.*

Proof. Since every multiplication module has SIP property ([2] Cor (1.12)), so the result follows from Th (3.7).

S.A.G. Al-Saadi in [4], defined and studied the concept of strongly extending modules, where an R-module M is called strongly extending, if every submodule of M is an essential in a stable direct summand. Equivalently, M is a strongly extending module if and only if every closed submodule of M is a stable direct summand [4]. And a submodule N of an R-module M is called stable, if for each homomorphism $f : N \to M$, $f(N) \subseteq N$ [1].

The class of strongly extending module is contained in the class of fully extending module, as the following proposition shows.

Proposition 3.10. *Every strongly extending module is fully extending.*

Proof. Assume that M is a strongly extending module. By [4], M is an extending module. We depend on Th (3.7), so to prove that M is a fully extending module, it is enough to show that M has SIP property. So suppose that $M = A \oplus B$ where both of A and B are submodules of M, and $f : A \to B \subseteq M$. Since A is a direct summand of M, then A is closed in M. But M is a strongly extending modules, therefore A is stable direct summand of M. It follows that $f(A) \subseteq A$ and so $f(A) \subseteq A \cap B = (0)$. Thus $f(A) = (0)$; that is $ker f = A$ which is a direct summand of M. Thus M has SIP [10] property.

The following example shows that a fully extending module need not be strongly extending module:

The Z-module $M = Z_2 \oplus Z_2$ is a fully extending module since it is a semisimple module, but M itself is not strongly extending. Because if $N = Z_2 \oplus (\bar{0})$ is a submodule of M, then N is a closed submodule in M. However, N is not stable submodule of M.

Recall that an R-module M is called fully stable, if every submodule of M is stable [1].A submodule N of an R-module M is called fully invariant if for each R-homomorphism $f : M \to M$, $f(N) \subseteq N$ [8].

Proposition 3.11. *Let M be a fully stable (or multiplication) R-module. Then the following statements are equivalent.*

1. *M is a fully extending R-module.*
2. *M is a strongly extending R-module.*
3. *M is an extending R-module.*

Proof. If M is a fully stable module, then the result follows directly. Now if M is a multiplication module, then:
(2)$\Rightarrow$(1): It follows from Prop (3.9).
(1)$\Rightarrow$ (3): It is clear.
(3)$\Rightarrow$ (2): Let N be a closed submodule in M. By (3), N is a direct summand of M. Since M is a multiplication R-module, so N is a fully invariant submodule. Thus N is a fully invariant summand of M, which implies that N is a stable summand of M. Thus M is a strongly extending module

Corollary 3.12. *For a ring R the following statements are equivalent.*

1. *R is a fully extending ring.*
2. *R is an extending ring.*
3. *R is strongly extending ring.*

Okado in [14], showed that a ring R is Notherian if and only if every extending R-module is expressed as a direct sum of indecomposable extending (uniform) R-module. In [4], S.A.G. Al-saadi investigated analogous result; a ring R is Notherian if and only if every strongly extending module is expressed as a direct sum of uniform modules. Also he showed that an R-module M is uniform if and only if M is indecomposable and strongly extending module.

So by these results and Rem and Ex (3.2)(5), we have the following result.

Proposition 3.13. *A ring R is Notherian if and only if every fully extending R-module expressed as a direct sum of fully extending (uniform) modules.*

Recall that an R-module M is called injective if for each homomorphism $f : A \to B$, where A and B are R-modules, and for each R-homomorphism $g : A \to M$, there exists an R-homomorphism $h : B \to M$ such that $h \circ f = g$ [9]. An R-module M is called quasi-injective if for each monomorphism $f : A \to M$, where A is a submodule of M, and for each R-homomorphism $g : A \to M$ there exists an R-homomorphism $h : M \to M$ such that $h \circ f = g$ [9].

Theorem 3.14. *Let M be an R-module. Then the following statements are equivalent.*

1. *R is a semisimple ring.*
2. *All R-modules are fully extending modules.*
3. *All quasi-injective R-modules are fully extending modules.*

4. All injective R-modules have SIP property.

Proof. (1)⇒ (2)⇒ (3)⇒ (4): Clear.
(4)⇒ (1): It follows from Prop ((3).b in [15]).

Next we will give some other relations between fully extending modules and other types of modules.

S.A.G. AL-Saadi in [4] defined and studied SS-module, where an R-module M is called SS-module, if every direct summand of M is stable. He proved that every strongly extending module is SS-module ([4], Rem and Ex (2.2.2)(8)), and every SS-module has SIP property ([4] Prop (2.2.6)). Hence every extending SS-module is fully extending module. However, the converse is not true in general, for example:

The Z-module $M = Z_2 \oplus Z_2$ is a fully extending module but it is not SS-module. Also M is not strongly extending module. Note we have the following relations:

Fully extending modules ⇐ Strongly extending modules ⇒ SS-modules

The converses of these relations are not true in general. In fact it is easy to see that under the class of fully stable (or multiplication modules), the concepts: fully extending modules, extending SS-modules, strongly extending and extending modules are equivalent.

Recall that an R-module M is called quasi-Dedekind if $Hom_R(\frac{M}{N}, M) = 0$ for each nonzero submodule N of M. Equivalently, M is a quasi-Dedekind if for each nonzero homomorphism $f : M \to M$, $ker f = 0$ [13]. Also every quasi Dedekind module is indecomposable, therefore every quasi Dedekind module has SIP property ([13],Ch2, Rem (1.3)). Hence we have the following.

Proposition 3.15. *Every extending quasi Dedekind module is fully extending.*

Note that the Z-module Z_{15} is fully extending module, but it is not quasi-Dedekind module; that is the converse of Prop 3.15 is not true in general.

Recall that an R-module M is called nonsingular if $Z_M(M) = 0$, where $Z_M(M)$ is the set of all $m \in M$ such that $ann(m) \leq_e R$ [12], and M is called polyform module, if for each submodule K of M and for each homomorphism $f : K \to M$, $ker f$ is closed submodule in K [6]. Alkan and Harmanci in [3], proved that every extending polyform module has SIP property; that is every extending polyform module is a fully extending module. The converse is not true in general, for example the Z-module Z_{12} is fully extending module but it is not polyform module. However, every nonsingular module is a polyform module. Thus every extending nonsingular module is a fully extending module.

Next recall that an R-module M is called prime if $ann(M) = ann(N)$ for each nonzero submodule N of M. Then we have the following.

Proposition 3.16. *Every injective prime module is fully extending module.*

Proof. Let M be an injective prime module. By ([2], Prop (1.4)), M has SIP. But it is well known that every injective module is an extending module, therefore M is a fully extending module.

4 Direct Sum of Fully Extending Modules

Firstly notice that the direct sum of fully extending modules need not be fully extending modules, as the following examples show.

Examples 4.1.

1. *The Z-module $Z_{(P^\infty)}$ where P is a prime number, is a fully extending module, but $Z_{(P^\infty)} \oplus Z_{(P^\infty)}$ is not fully extending module.*

2. *Each of Z_2 and Z_4 is a fully extending Z-module, but $Z_2 \oplus Z_4$ is not fully extending Z-module.*

3. *Each of Z and Z_2 is fully extending module, but $Z \oplus Z_2$ is not fully extending module (see Rem and Ex 2.2(1)).*

Theorem 4.2. *Let M_1 and M_2 be two R-modules and let $M = M_1 \bigoplus M_2$ such that $ann_R M_1 + ann_R M_2 = R$. Then M is a fully extending module if and only if M_1 and M_2 are fully extending modules.*

Proof. $\Rightarrow$) It follows from prop (3.3).
$\Leftarrow$) Let N be a qc-submodule of M. Since $ann_R M_1 + ann_R M_2 = R$, so by a part of the proof of ([1], Prop (4.2), P.28), any submodule of $M = M_1 \oplus M_2$ can be written as a direct sum of two submodules of M_1 and M_2 respectively. Thus $N = N_1 \oplus N_2$ for some $N_1 \leq M_1$ and $N_2 \leq M_2$. But N is a qc-submodule of M, then by Prop (2.7) both of N_1 and N_2 are qc-submodule of M_1 and M_2 respectively. But M_1 and M_2 are fully extending modules, hence N_1 and N_2 are direct summand of M_1 and M_2 respectively. It follows that $N = N_1 \oplus N_2$ is a direct summand of M. Thus M is a fully extending module.

Theorem 4.3. *Let $M = \bigoplus_{i \in I} M_i$, where M_i is a submodule of M $\forall i \in I$, and suppose that each closed submodule of M is a fully invariant. Then M is a fully extending if and only if M_i is a fully extending module $\forall i \in I$.*

Proof. $\Rightarrow$) It follows from Prop (3.3).

$\Leftarrow$) Since M_i is a fully extending module $\forall i \in I$, so by Th (3.7), M_i is an extending and has SIP property for each $\forall i \in I$. On the other hand, M_i is a direct summand of M $\forall i \in I$, so M_i is closed in M $\forall i \in I$. Thus by hypothesis, , M_i is fully invariant submodule $\forall i \in I$, and by [15], M has SIP property. Now to prove M is an extending module, let S be any closed submodule of M and $\forall i \in I$,$\pi_i : M \to M_i$ be a natural projection. By hypothesis, S is a fully invariant submodule, hence $\pi_i(S) \subseteq (S \cap M_i)$ $\forall i \in I$. It follows that for each $i \in I$, $S = \bigoplus_{i \in I}(S \cap M_i)$; that is $\forall i \in I$,$S \cap M_i$ is a direct summand of M, and hence $\forall i \in I$, $S \cap M_i$ is a closed submodule in S, hence for each $i \in I$, $\cap M_i$ is closed submodule in M [12]. On the other hand $S \cap M_i \leq M_i \leq M$ $\forall i \in I$. So $S \cap M_i$ is a closed submodule in M_i $\forall i \in I$, and since M_i is an extending module, so $S \cap M_i$ is a direct summand of M_i $\forall i \in I$. It follows that S is a direct summand of M. Thus M is an extending module.

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